



## Generalized para-Bose states

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Received: 27 June 2022 / Accepted: 24 July 2022

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**Abstract** In this paper, we construct integrals of motion in a para-Bose formulation for a general time-dependent quadratic Hamiltonian, which, in its turn, commutes with the reflection operator. In this context, we obtain generalizations for the squeezed vacuum states (SVS) and coherent states (CS) in terms of the Wigner parameter. Furthermore, we show that there is a completeness relation for the generalized SVS owing to the Wigner parameter. In the study of the transition probability, we found that the displacement parameter acts as a transition parameter by allowing access to odd states, while the Wigner parameter controls the dispersion of the distribution. We show that the Wigner parameter is quantized by imposing that the vacuum state has even parity. We apply the general results to the case of the time-independent para-Bose oscillator and find that the mean values of the coordinate and momentum have an oscillatory behavior similarly to the simple harmonic oscillator, while the standard deviation presents corrections in terms of the squeeze, displacement, and Wigner parameters.

### 1 Introduction

In 1926 Schrödinger obtained quantum states to harmonic oscillator, which allowed for a semiclassical description [1]. In the 1960s, with the advent of the laser, these states were rediscovered and widespread in the works of Glauber-Sudarshan-Klauber [2–4] by showing a semiclassical description of electromagnetic radiation, and where the term “coherent states” (CS) was presented for the first time. These states for the harmonic oscillator can be obtained in three equivalent ways: I - as eigenstates of the annihilation operator; II - by the action of a unitary displacement operator acting in the vacuum state; III - as minimum uncertainty states. The minimization of the uncertainty relation coincides with the value calculated in the vacuum state of the harmonic oscillator, which presents the same value for the standard deviation in both position and momentum. The states satisfying these properties are called canonical CS. In addition, these states have properties of continuity and form an overcomplete basis, which guarantees a prominent role in modern quantum mechanics, with various applications ranging from quantum optics [5, 6], quantum computing [7], and mathematical physics [8].

In its turn, squeezed states (SS) form a class of nonclassical states, which minimize the uncertainty relation [9–11]. At the same time, these states offer the possibility of reducing the standard deviation of a physical quantity to a value less than that calculated in the vacuum state at the expense of increasing the standard deviation of other physical quantity [12, 13]. Thus, these states have great potential for application, for example, in quantum information [14] and for the detectors of gravitational waves [15–17]. Usually, these states are constructed by the action of the unitary squeeze operator on an arbitrary state. In particular, the acting of the squeeze operator on the vacuum state leads to the squeezed vacuum states (SVS) [18].

Generalizing quantum states provides a deeper understanding of corresponding systems since they sometimes bring additional degrees of freedom. In this sense, there is a wide range of publications in the literature that seeks to generalize CS to be able to describe systems in addition to the harmonic oscillator, such as the time-dependent quadratic systems [19, 20], systems with a given Lie group [21, 22], some non-trivial generalizations [23], and squeezed coherent states (SCS) [24, 25]. On the other hand, CS can be generalized by considering deformations in the canonical commutation relation and thus allow us to study a wide range of relevant problems. For instance, the implications of the gravitational effects in quantum mechanics [26–28],  $q$ - and  $f$ -deformed oscillators study which is applied, mainly, to quantum optics [29–31], and to study systems with singularity as the Calogero-like model via Wigner–Heisenberg algebra (WHA) (or  $R$ -deformed Heisenberg algebra) [32, 33]. In particular, we are interested in generalizing CS and SVS via WHA.

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The WHA originated from the quantization proposed by Wigner [34], which generalizes the canonical commutation relation. In particular, this algebra is generated by the creation-annihilation operators and the reflection operator (parity operator), which satisfies commutation and anti-commutation relations [35]. In this formulation, it is possible to obtain a self-adjoint momentum operator on the semi-axis in terms of a parameter that deforms the canonical commutation relation [36, 37]. Furthermore, the WHA obeys the trilinear commutation and anti-commutation relations, which characterize the para-Bose operators [38]. On its turn, this algebra leads to the parastatistical description of physical systems in terms of a deformation parameter, which corresponds to generalization of the Bose-Einstein and Fermi-Dirac statistics [39–41].

According to the order of the deformation parameter, also known as the order of statistics, it is possible to describe para-Bose or para-Fermi particles, see also [42]. Although parastatistics cannot be applied to the particles described by the standard model [43], their distinctive approach has allowed presenting promising proposals, for instance, in the conjecture of candidate particles to explain dark matter [44], in paraquark models description [45], in the study of thermodynamics properties of para-Bose systems [46], and in optical physics by simulation of para-Fermi oscillators [47]. By considering the para-Bose formulation, we will construct integrals of motion, i.e., operators that commute with the Schrödinger’s operator [48] in the form of a Bogoliubov transformation [49].

In the para-Bose formulation [50], the CS were obtained as eigenstates of the annihilation operator that satisfy the characteristic commutation relation of the algebra [51]—see [38, 52, 53]. Recently, some works have paid attention to this topic, for instance, the nonlinear CS [54], the construction of new types of para-Bose states [55–57], and the study of “Schrödinger cat states” [58]. In this context, we will obtain the time-dependent generalized CS and SVS via integrals of motion. These states can be applied to describe Calogero-like models [59]. In its turn, the Calogero model and its generalizations are able to describe several physical phenomena [60], such as the Hall effect [61, 62], anyons [63], and fluctuations in mesoscopic systems [64], for instance. Furthermore, our construction has potential application to describe the recent proposed experimental realization of para-particles [65, 66].

This paper is structured as follows. In Sect. 2, the integrals of motion method in the context of WHA will be formulated. In Sect. 3, the time-dependent generalized SVS in terms of the time-independent para-Bose number states will be constructed. In turn, the completeness relation and transition probability plots will be obtained. In Sect. 4, a generalization of the time-dependent CS in terms of the Wigner, squeeze and displacement parameters will be constructed. Then, the transition probability graph will be shown, and the mean values and the uncertainty relations will be calculated. In Sect. 5, the generalized CS in the coordinate representation will be considered. We apply the general result to the particular case of the para-Bose oscillator in Sect. 6. Then, the concluding remarks are presented in Sect. 7.

## 2 Integrals of motion via WHA

The motion integral method consists of building a time-dependent operator, which commutes with the Schrödinger’s operator. In turn, the eigenstates of this operator are obtained and imposed to satisfy the Schrödinger’s equation. This technique introduced by Lewis and Reisenfeld to study the time-dependent harmonic oscillator [67] proved to be useful in several other problems, e.g., in the study of time-dependent quadratic Hamiltonians in one dimension [68–70] and for multidimensional systems [71], Dirac equation [72], and supersymmetric quantum mechanics [73]. Here, our contribution will be to reformulate the integrals of motion method according to the WHA. This algebraic formulation allows studying time-dependent systems with a singularity at the origin characterized by centrifugal potentials.

The WHA is composed by the annihilation  $\hat{a}$ , creation  $\hat{a}^\dagger$  and reflection  $\hat{R}$  operators, satisfying commutation  $([\hat{b}, \hat{c}] = \hat{b}\hat{c} - \hat{c}\hat{b})$  and anti-commutation  $(\{\hat{b}, \hat{c}\} = \hat{b}\hat{c} + \hat{c}\hat{b})$  relations,

$$[\hat{a}, \hat{a}^\dagger] = 1 + \nu\hat{R}, \quad \{\hat{R}, \hat{a}\} = 0 = \{\hat{R}, \hat{a}^\dagger\}, \quad \hat{R}^2 = 1, \tag{1}$$

where  $\nu$  is the Wigner parameter (deformation parameter) related to the fundamental energy level  $\varepsilon$  of the para-Bose oscillator given by

$$\nu = 2\varepsilon - 1. \tag{2}$$

Notice that by setting  $\nu = 0 \Rightarrow \varepsilon = 1/2$  we recover the canonical commutation relation.

The operators  $\hat{a}$  and  $\hat{a}^\dagger$  also obey the trilinear commutation and anti-commutation relation

$$[\{\hat{a}, \hat{a}^\dagger\}, \hat{a}] = -2\hat{a}, \quad [\{\hat{a}, \hat{a}^\dagger\}, \hat{a}^\dagger] = 2\hat{a}^\dagger, \tag{3}$$

which characterize the para-Bose operators [35, 38].

For the sake of simplicity, we will consider a Hamiltonian which commutes with the reflection operator, and in that way, the eigenstates of  $\hat{H}$  can be even or odd. In this sense, the most general form for a one-dimensional time-dependent quadratic Hamiltonian is given by

$$\hat{H} = \frac{1}{2}\hbar(\alpha^* \hat{a}^2 + \alpha \hat{a}^{\dagger 2}) + \frac{1}{2}\hbar\beta(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) + \hbar\delta, \tag{4}$$

where  $\alpha = \alpha(t)$ ,  $\beta = \beta(t)$  and  $\delta = \delta(t)$  are time-dependent functions and the signs  $\dagger$  and  $*$  denote Hermitian and complex conjugation, respectively. From the hermiticity condition  $\hat{H} = \hat{H}^\dagger$ , we have that  $\beta$  and  $\delta$  must be real functions. By assuming the condition  $\beta > |\alpha|$ , we get that the Hamiltonian (4) is positive definite, i.e., it can be written in the form of an oscillator-type Hamiltonian.

In its turn, the quantum states  $|\Psi\rangle$  which describe the time evolution of the system should satisfy the Schrödinger’s equation

$$\begin{aligned} \hat{\Lambda}|\Psi\rangle &= 0, \\ \hat{\Lambda} &= \frac{i}{\hbar}\hat{H} + \partial_t, \quad \partial_t = \frac{\partial}{\partial t}, \end{aligned} \tag{5}$$

where  $\hat{\Lambda}$  is the Schrödinger’s operator.

Let us consider a time-dependent operator  $\hat{A} = \hat{A}(t)$ , as a linear combination of the  $\hat{a}$  and  $\hat{a}^\dagger$  operators, in the form

$$\hat{A} = f\hat{a} + g\hat{a}^\dagger + \varphi, \tag{6}$$

where  $f = f(t)$ ,  $g = g(t)$  and  $\varphi = \varphi(t)$  are time-dependent complex functions. Eq. (6) is the well-known Bogoliubov’s transformation; those coefficients have been studied in Ref. [49].

In order to  $\hat{A}$  to be an integral of motion, it has to commute with the Schrödinger’s operator (5), that is

$$\dot{\hat{A}} = [\hat{\Lambda}, \hat{A}] = 0, \quad \dot{\hat{A}} = \frac{d\hat{A}}{dt}. \tag{7}$$

Substituting Eqs. (4), (5) and (6) into (7), we obtain the following equations for the functions  $f$ ,  $g$  and  $\varphi$ :

$$\dot{f} = i(\beta f - \alpha^* g), \quad \dot{g} = i(f\alpha - \beta g), \quad \dot{\varphi} = 0, \tag{8}$$

where  $\varphi(t) = \varphi_0$  is a constant function for any instant of time.

The commutator between  $\hat{A}$  and  $\hat{A}^\dagger$  reads

$$[\hat{A}, \hat{A}^\dagger] = \mu(1 + \nu\hat{R}), \quad \mu = |f|^2 - |g|^2 = |f_0|^2 - |g_0|^2, \quad \text{for arbitrary } t, \tag{9}$$

where  $f_0 = f(0)$  and  $g_0 = g(0)$  are initial conditions.

It follows from (6) and (9), that

$$\hat{a} = \frac{f^*\hat{A} - g\hat{A}^\dagger + u}{\mu}, \quad u = g\varphi_0^* - f^*\varphi_0. \tag{10}$$

Notice that  $|f| \neq |g|$  for all time ensures that  $\hat{a}$  is well-defined for all  $t$ . It is worth highlighting that the  $\mu$ -parameter will be useful to study special cases in which the Hamiltonian presents linear terms. In addition, the  $\mu$ -parameter will be useful to replace the functions  $f$  and  $g$  with the squeeze and displacement parameters in the constructed states.

### 3 Time-dependent generalized SVS

The SVS are pure states known to be one of the most useful nonclassical states, see for instance, [10]. In addition, these states make it possible to obtain the standard deviation of a physical quantity less than its value in the vacuum state. In this section, we will obtain the time-dependent generalized SVS via para-Bose formulation. Thus, assuming the condition

$$\varphi_0 = 0 \implies u = 0, \tag{11}$$

one can obtain the generalized SVS following the nonunitary approach, as described in Ref. [48]. In what follows we will apply this condition.

#### 3.1 Para-Bose number states

In this subsection, we will recall some properties of the para-Bose number states, since the generalized SVS can be expanded in these states. It is well-known that the para-Bose number states  $|n, \varepsilon\rangle$  form a complete set and are orthonormal [38, 51], i.e.,

$$\sum_{n=0}^{\infty} |n, \varepsilon\rangle\langle\varepsilon, n| = 1, \quad \langle\varepsilon, n|m, \varepsilon\rangle = \delta_{n,m}, \tag{12}$$

where  $\delta_{n,m}$  is the Kronecker delta.

Taking into account that the vacuum state has even parity ( $\hat{R}|0, \varepsilon\rangle = |0, \varepsilon\rangle$ ), the action of the generators of the WHA in  $|n, \varepsilon\rangle$ , reads

$$\begin{aligned} \hat{a}|2n, \varepsilon\rangle &= \sqrt{2n}|2n - 1, \varepsilon\rangle, \quad \hat{a}^\dagger|2n, \varepsilon\rangle = \sqrt{2(n + \varepsilon)}|2n + 1, \varepsilon\rangle, \\ \hat{a}|2n + 1, \varepsilon\rangle &= \sqrt{2(n + \varepsilon)}|2n, \varepsilon\rangle, \quad \hat{a}^\dagger|2n + 1, \varepsilon\rangle = \sqrt{2(n + 1)}|2n + 2, \varepsilon\rangle, \\ \hat{n}|n, \varepsilon\rangle &= \left(\frac{1}{2}\{\hat{a}, \hat{a}^\dagger\} - \varepsilon\right)|n, \varepsilon\rangle = n|n, \varepsilon\rangle, \quad \hat{R}|n, \varepsilon\rangle = (-1)^n|n, \varepsilon\rangle, \quad n = 0, 1, 2, 3, \dots \end{aligned} \tag{13}$$

From a recurrence relation, one can write the number states  $|n, \varepsilon\rangle$  in terms of the vacuum state  $|0, \varepsilon\rangle$ , in the form

$$\begin{aligned} |2n, \varepsilon\rangle &= \sqrt{\frac{\Gamma(\varepsilon)}{2^{2n}n!\Gamma(n + \varepsilon)}}(\hat{a}^\dagger)^{2n}|0, \varepsilon\rangle, \\ |2n + 1, \varepsilon\rangle &= \sqrt{\frac{\Gamma(\varepsilon)}{2^{2n+1}n!\Gamma(n + \varepsilon + 1)}}(\hat{a}^\dagger)^{2n+1}|0, \varepsilon\rangle. \end{aligned} \tag{14}$$

### 3.2 Time-dependent generalized SVS via para-Bose number states

In what follows, we aim to apply the nonunitary approach, as in the recent publication [48], to construct the generalized SVS. From condition (11), one can write the nonunitary operator  $\hat{S}$ , in the form

$$\hat{S} = \exp\left(\frac{1}{2}\zeta\hat{a}^{\dagger 2}\right), \quad \zeta = \frac{g}{f}, \quad \dot{\zeta} = i\alpha^*\zeta^2 - 2i\beta\zeta + i\alpha, \tag{15}$$

such that the commutators from Baker-Campbell-Hausdorff relation of the second order onwards become null. Here,  $\zeta = \zeta(t)$  represents the squeeze parameter for the generalized SVS. The form of  $\hat{S}$  allows us to introduce the most convenient squeeze parameter, as well as relate the motion integral  $\hat{A}$  directly with the annihilation operator  $\hat{a}$ . Furthermore, this approach guarantees that there is a direct relationship between the eigenstates of the motion integral with the para-Bose number states, as we will see next.

Applying the Baker–Campbell–Hausdorff theorem, we can express the operator  $\hat{a}$  in terms of the integrals of motion  $\hat{A}$ , as follows

$$\hat{a} = \frac{1}{f}\hat{S}\hat{A}\hat{S}^{-1}. \tag{16}$$

The application from (16) on the vacuum state  $|0, \varepsilon\rangle$ , which satisfies the annihilation condition  $\hat{a}|0, \varepsilon\rangle = 0$ , yields:

$$\hat{A}|\zeta\rangle = 0, \tag{17}$$

whose general solution is given by

$$|\zeta\rangle = \Phi \exp\left(-\frac{1}{2}\zeta\hat{a}^{\dagger 2}\right)|0, \varepsilon\rangle, \tag{18}$$

where  $\Phi = \Phi(t)$  is an arbitrary function, which will be determined such that the states  $|\zeta\rangle$  satisfy the Schrödinger’s equation (5).

Substituting the states (18) into (5), we obtain the following equation for  $\Phi$ :

$$\frac{\dot{\Phi}}{\Phi} = \frac{1}{2} \frac{\langle \zeta | \hat{a}^{\dagger 2} | \zeta \rangle}{\langle \zeta | \zeta \rangle} \dot{\zeta} - \frac{i}{\hbar} \frac{\langle \zeta | \hat{H} | \zeta \rangle}{\langle \zeta | \zeta \rangle}. \tag{19}$$

Using the representation (10) together with the condition (17), one can easily calculate the mean values in (19), with  $\dot{\zeta}$  given in (15). Thus, the general solution from (19), reads

$$\Phi = \frac{C}{f^\varepsilon} \exp\left(-i \int \delta dt\right), \tag{20}$$

where  $C$  is a real normalization constant. Taking into account the normalization condition, we find that:

$$\langle \zeta | \zeta \rangle = 1 \Rightarrow C = \mu^{\varepsilon/2}. \tag{21}$$

Then, the normalized states  $|\zeta\rangle$  that satisfy the Schrödinger’s equation are given by

$$\begin{aligned}
 |\zeta\rangle &= \frac{\sqrt{\mu^\varepsilon}}{f^\varepsilon} \exp\left(-i \int \delta dt\right) \exp\left(-\frac{1}{2}\zeta \hat{a}^{\dagger 2}\right) |0, \varepsilon\rangle \\
 &= \frac{\sqrt{\mu^\varepsilon}}{f^\varepsilon} \exp\left(-i \int \delta dt\right) \sum_{n=0}^\infty (-\zeta)^n \sqrt{\frac{\Gamma(n+\varepsilon)}{n! \Gamma(\varepsilon)}} |2n, \varepsilon\rangle.
 \end{aligned}
 \tag{22}$$

The above sum converges as long as the condition  $|\zeta| < 1$  is satisfied.

On the other hand, we can express  $f$  and  $\mu$  in terms of squeeze parameter in the form:

$$f = f_0 \exp\left[-i \int (\alpha^* \zeta - \beta) dt\right], \quad \mu = |f|^2 (1 - |\zeta|^2).
 \tag{23}$$

From here, the states  $|\zeta\rangle$  take the form

$$|\zeta\rangle = (1 - |\zeta|^2)^{\frac{\varepsilon}{2}} e^{i\vartheta} \sum_{n=0}^\infty (-\zeta)^n \sqrt{\frac{\Gamma(n+\varepsilon)}{n! \Gamma(\varepsilon)}} |2n, \varepsilon\rangle,
 \tag{24}$$

where the phase  $\vartheta$  is given by

$$\vartheta = \int [\varepsilon \operatorname{Re}(\alpha \zeta^*) - \varepsilon \beta - \delta] dt.
 \tag{25}$$

In what follows, we call the time-dependent states (24) generalized SVS. Notice that, if we assume  $\varepsilon = 1/2$  and  $\zeta = e^{i\theta} \tanh(r)$  ( $r \Rightarrow \mu = 1$ , Eq. (24) takes the form:

$$|\zeta\rangle = \frac{1}{\sqrt{\cosh(r)}} e^{i\vartheta} \sum_{n=0}^\infty \frac{\sqrt{(2n)!}}{2^n n!} \left(-e^{i\theta} \tanh(r)\right)^n \left|2n, \frac{1}{2}\right\rangle,
 \tag{26}$$

where  $\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi} 4^{-n} (2n)!}{n!}$ ,  $\Gamma(1/2) = \sqrt{\pi}$ . So, we can conclude that the states obtained in Eq. (26) reproduce the usual case [74]. In this case, the Eq. (24) corresponds to a generalization of the SVS, differing by a time-dependent phase factor due to the time evolution that was included in our analysis. We must highlight that when identifying  $\varepsilon = 2k$ , the states (24) coincide with the time-independent CS of the  $SU(1, 1)$  group constructed by Peremolov [21].

The overlap of two generalized SVS with different  $\zeta$ , for example  $\langle \zeta_1 | \zeta_2 \rangle$  reads

$$\langle \zeta_1 | \zeta_2 \rangle = \frac{(1 - |\zeta_1|^2)^{\frac{\varepsilon}{2}} (1 - |\zeta_2|^2)^{\frac{\varepsilon}{2}}}{(1 - \zeta_1^* \zeta_2)^\varepsilon} \exp\left\{i\varepsilon \int \operatorname{Re}[\alpha(\zeta_2^* - \zeta_1^*)] dt\right\}.
 \tag{27}$$

In turn, the transition probability  $P_{2n}(\zeta, \varepsilon) = |\langle \varepsilon, 2n | \zeta \rangle|^2$  is given by

$$P_{2n}(\zeta, \varepsilon) = \frac{(1 - |\zeta|^2)^\varepsilon}{\Gamma(\varepsilon)} \frac{\Gamma(n + \varepsilon) |\zeta|^{2n}}{n!}.
 \tag{28}$$

The transition probability (28) has been plotted in Fig. 1. As we see, the increase in the value of the  $\varepsilon$  parameter implies a larger dispersion of the transition probability.

### 3.3 Completeness relation on squeeze parameter

Considerations on the completeness relation for the SVS and squeezed odd number states, in the context of canonical algebra, can be seen in Refs. [74, 75]. Here, our contribution is to obtain the completeness relation in the context of the WHA, and as will be seen, its existence depends on the  $\varepsilon$ -parameter.

Let us consider a weight function  $w(\zeta)$  such that the states  $|\zeta\rangle$  lead to the following closure relation

$$\int_{\mathbb{C}} |\zeta\rangle \langle \zeta | w(\zeta) d^2\zeta = 1.
 \tag{29}$$

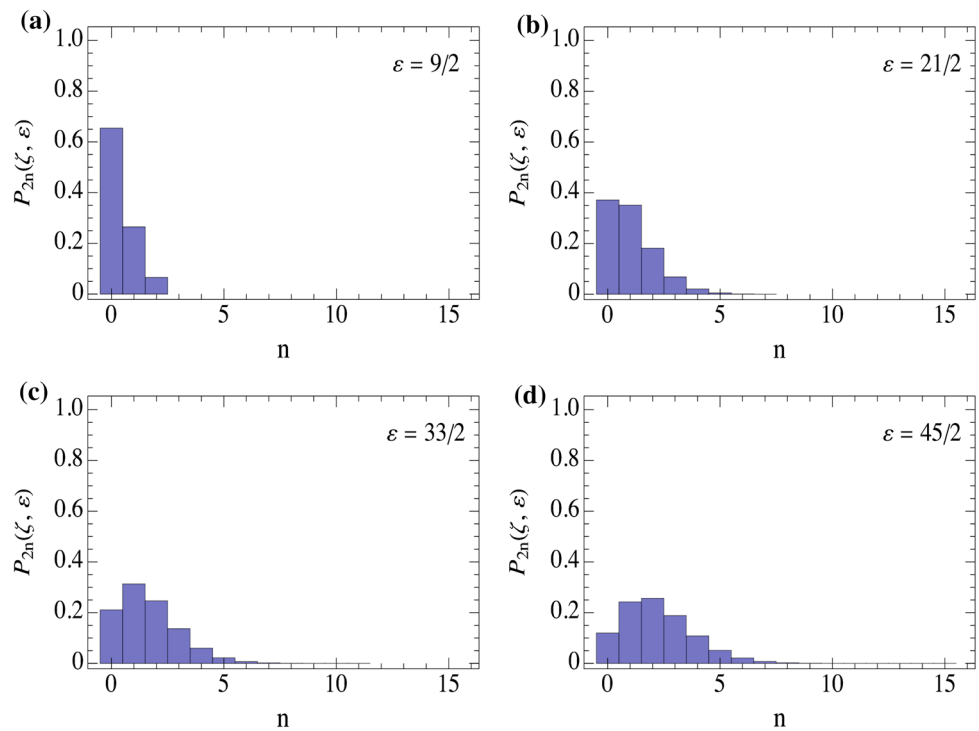
In turn, substituting the states (24) into Eq. (29), we find

$$\sum_{n,m=0}^\infty |2n, \varepsilon\rangle \langle \varepsilon, 2m| \sqrt{\frac{\Gamma(n+\varepsilon)}{n! \Gamma(\varepsilon)} \frac{\Gamma(m+\varepsilon)}{m! \Gamma(\varepsilon)}} \int_{\mathbb{C}} (1 - |\zeta|^2)^\varepsilon (-\zeta)^n (-\zeta^*)^m w(\zeta) d^2\zeta = 1.
 \tag{30}$$

Now, by using polar coordinates in the above relation,

$$\zeta = r_\zeta e^{i\theta_\zeta}, \quad d^2\zeta = r_\zeta dr_\zeta d\theta_\zeta, \quad 0 \leq r_\zeta < 1, \quad 0 \leq \theta_\zeta \leq 2\pi,
 \tag{31}$$

**Fig. 1** Transition probability of the generalized SVS by considering fixed value for  $|\zeta| = 0.3$



we get

$$\sum_{n,m=0}^{\infty} |2n, \epsilon\rangle \langle \epsilon, 2m| \sqrt{\frac{\Gamma(n + \epsilon)}{n! \Gamma(\epsilon)} \frac{\Gamma(m + \epsilon)}{m! \Gamma(\epsilon)}} \int_0^1 \int_0^{2\pi} (1 - r_\zeta^2)^\epsilon (-r_\zeta)^n (-r_\zeta)^m w(\zeta) \exp[i(n - m)\theta_\zeta] r_\zeta dr_\zeta d\theta_\zeta = 1. \tag{32}$$

On the other hand, we may readily show that  $w(\zeta) = w(r_\zeta)$ , which implies that the integral on  $\theta_\zeta$  becomes

$$\int_0^{2\pi} d\theta_\zeta \exp[i(n - m)\theta_\zeta] = 2\pi \delta_{n,m}. \tag{33}$$

From here, we can write (32) in the form

$$2\pi \sum_{n=0}^{\infty} |2n, \epsilon\rangle \langle \epsilon, 2n| \frac{\Gamma(n + \epsilon)}{n! \Gamma(\epsilon)} \int_0^1 (1 - r_\zeta^2)^\epsilon r_\zeta^{2n+1} w(r_\zeta) dr_\zeta = 1. \tag{34}$$

By using the following relationship among gamma functions

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = 2 \int_0^1 (1 - t^2)^{y-1} t^{2x-1} dt, \quad \text{Re}(x) > 0, \quad \text{Re}(y) > 0, \tag{35}$$

one can see that the weight function  $w(r_\zeta)$  must have the form

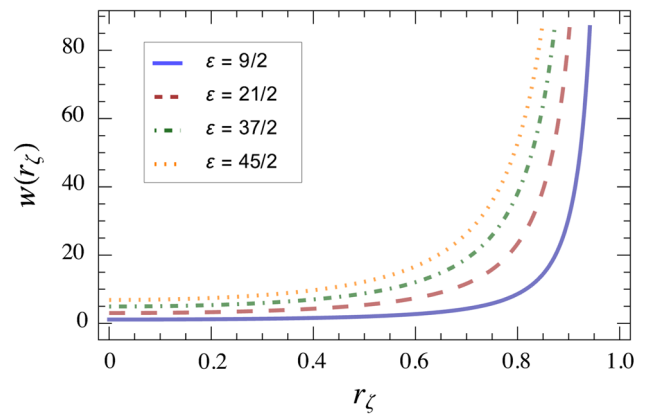
$$w(r_\zeta) = \frac{\Gamma(\epsilon)}{\pi \Gamma(\epsilon - 1) (1 - r_\zeta^2)^2} = \frac{\epsilon - 1}{\pi (1 - r_\zeta^2)^2}, \quad \epsilon > 1, \tag{36}$$

in order to ensure that the completeness relation (29) is satisfied. It is important to highlight that the condition  $\epsilon > 1$  leads to a positive weight function  $w(r_\zeta)$ . At the same time, this condition shows that the canonical algebra ( $\epsilon = 1/2$ ) does not allow to obtain a completeness relation for the generalized SVS. It follows from (27) and (29) that these states form an overcomplete set of states on the Hilbert space. The plot of weight function can be seen in Fig. 2.

#### 4 Time-dependent generalized CS

In Ref. [48], the SCS has been built through the nonunitary approach by considering the canonical algebra. In another way, by considering the WHA, it is not possible to obtain the SCS since, from Baker–Campbell–Hausdorff theorem, we cannot establish

**Fig. 2** Graphics of  $w(r_\zeta)$



a direct relationship between the para-Bose operators and integral of motion, due the deformed commutation relation. However, instead of following the nonunitary approach, we will construct the eigenstates of the integral of motion  $\hat{A}$ , in the form

$$\hat{A}|z, t\rangle = z|z, t\rangle, \tag{37}$$

where  $z$  is a complex constant. Eq. (37) allows us to assume

$$\varphi_0 = 0 \implies u = 0, \tag{38}$$

without loss of generality.

The states  $|z, t\rangle$  can be expanded in terms of the para-Bose number states, as follows

$$|z, t\rangle = \sum_{n=0}^{\infty} c_n |n, \varepsilon\rangle = \sum_{n=0}^{\infty} (c_{2n} |2n, \varepsilon\rangle + c_{2n+1} |2n + 1, \varepsilon\rangle), \tag{39}$$

where  $c_n = c_n(t)$  are time-dependent coefficients and will be determined such that Eqs. (37) and (5) will be satisfied. Substituting (6), (13) and (39) into (37), we find the following equations for the coefficients

$$\begin{aligned} c_{2n} &= \sqrt{\frac{n! \Gamma(\varepsilon)}{\Gamma(n + \varepsilon)}} \left(-\frac{g}{f}\right)^n L_n^{\varepsilon-1}\left(\frac{z^2}{2gf}\right) c_0, \\ c_{2n+1} &= \frac{z}{f} \sqrt{\frac{n! \Gamma(\varepsilon)}{2\Gamma(n + \varepsilon + 1)}} \left(-\frac{g}{f}\right)^n L_n^{\varepsilon}\left(\frac{z^2}{2gf}\right) c_0, \end{aligned} \tag{40}$$

where  $L_n^\alpha(x)$  are the associated Laguerre polynomials and  $c_0$  is a time-dependent function, which will be determined such that the states  $|z, t\rangle$  satisfy the Schrödinger's Eq. (5).

Here, it is convenient to introduce the squeeze  $\zeta = g/f$  and displacement  $\xi = z/f$  parameters to rewrite the states  $|z, t\rangle \longrightarrow |\zeta, \xi\rangle$ , in the form

$$|\zeta, \xi\rangle = \sqrt{\Gamma(\varepsilon)} c_0 \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[ \frac{L_n^{\varepsilon-1}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{\Gamma(n + \varepsilon)}} |2n, \varepsilon\rangle + \frac{\xi L_n^{\varepsilon}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{2\Gamma(n + \varepsilon + 1)}} |2n + 1, \varepsilon\rangle \right], \tag{41}$$

with  $\zeta$  and  $\xi$  satisfy the following differential equations:

$$\dot{\zeta} = i\alpha^* \zeta^2 - 2i\beta\zeta + i\alpha, \quad \dot{\xi} = i(\alpha^* \zeta - \beta)\xi. \tag{42}$$

From normalization condition

$$\langle \xi, \zeta | \zeta, \xi \rangle = 1, \tag{43}$$

we found the following form for the function  $c_0$ ,

$$c_0 = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \frac{\sqrt{1-|\zeta|^2} \exp\left\{\frac{\zeta^* \xi^2}{2(1-|\zeta|^2)} + i \int [\text{Re}(\alpha\zeta^*) - \beta] dt + i\phi\right\}}{\Gamma(\varepsilon) \sqrt{I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) + I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right)}}, \tag{44}$$

where  $I_k(Z)$  is the modified Bessel function of the first kind, and  $\phi$  is a time-dependent real function. Thus, the normalized states  $|\zeta, \xi\rangle$  take the form

$$|\zeta, \xi\rangle = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \sqrt{\frac{1-|\zeta|^2}{I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) + I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right)}} \exp\left\{\frac{\zeta^*\xi^2}{2(1-|\zeta|^2)} + i \int [\text{Re}(\alpha\zeta^*) - \beta] dt + i\phi\right\} \\ \times \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[ \frac{L_n^{\varepsilon-1}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{\Gamma(n+\varepsilon)}} |2n, \varepsilon\rangle + \frac{\xi L_n^{\varepsilon}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{2\Gamma(n+\varepsilon+1)}} |2n+1, \varepsilon\rangle \right], \tag{45}$$

Substituting  $|\zeta, \xi\rangle$  into Schrödinger’s equation, we find the following expression for  $\phi$ ,

$$\phi = - \int \delta dt. \tag{46}$$

Therefore, the normalized states  $|\zeta, \xi\rangle$  that satisfy the Schrödinger’s equation are given by

$$|\zeta, \xi\rangle = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \sqrt{\frac{1-|\zeta|^2}{I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) + I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right)}} \exp\left[\frac{\zeta^*\xi^2}{2(1-|\zeta|^2)} + i\tilde{\vartheta}\right] \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[ \frac{L_n^{\varepsilon-1}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{\Gamma(n+\varepsilon)}} |2n, \varepsilon\rangle + \frac{\xi L_n^{\varepsilon}\left(\frac{\xi^2}{2\zeta}\right)}{\sqrt{2\Gamma(n+\varepsilon+1)}} |2n+1, \varepsilon\rangle \right], \tag{47}$$

where

$$\tilde{\vartheta} = \int [\text{Re}(\alpha\zeta^*) - \beta - \delta] dt. \tag{48}$$

In the following, we call the time-dependent states (47) generalized para-Bose CS.

It’s worth mentioning that the states (47) present a set of information that, to our knowledge, has not been previously derived, see, e.g., [38, 51–58, 65, 66]. Such information consists of the explicit form of the squeeze, displacement, and deformation parameters (Wigner-parameter). These parameters are present explicitly in the standard deviation and transition probability. Furthermore, the time-dependent states (47) have been expanded in terms of the time-independent para-Bose number states. Thus, the states (47) lead to a deeper understanding of the physical systems described by the Hamiltonian (4). As we will see from Sect. 5, the Hamiltonian can be seen as a generalization of Calogero model [59].

In particular, taking into account the condition  $\zeta = 0 \implies \alpha = 0$ , the states (47) take the form

$$|\xi\rangle = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \frac{\exp[-i \int (\beta + \delta) dt]}{\sqrt{I_{\varepsilon-1}(|\xi|^2) + I_{\varepsilon}(|\xi|^2)}} \sum_{n=0}^{\infty} \left(\frac{\xi^2}{2}\right)^n \left[ \frac{|2n, \varepsilon\rangle}{\sqrt{n! \Gamma(n+\varepsilon)}} + \frac{\xi |2n+1, \varepsilon\rangle}{\sqrt{2n! \Gamma(n+\varepsilon+1)}} \right] \\ = \sqrt{\Gamma(\varepsilon)} \exp\left[-i \int (\beta + \delta) dt\right] \left(\frac{\hat{a}^\dagger}{\sqrt{2}}\right)^{1-\varepsilon} \frac{I_{\varepsilon-1}(\xi \hat{a}^\dagger) + I_{\varepsilon}(\xi \hat{a}^\dagger)}{\sqrt{I_{\varepsilon-1}(|\xi|^2) + I_{\varepsilon}(|\xi|^2)}} |0, \varepsilon\rangle, \tag{49}$$

with  $\dot{\xi} = -i\beta\xi$  and  $|\xi\rangle = |0, \xi\rangle$ . Except for time evolution, these states correspond to the para-Bose CS obtained in Ref. [51]. In turn, we have that the condition  $\varepsilon = 1/2$  reduces the states (49) to the form:

$$|\xi\rangle = \exp\left(\int \frac{\beta + 2\delta}{2i} dt - \frac{|\xi|^2}{2}\right) \exp(\xi \hat{a}^\dagger) |0\rangle, \quad |0\rangle = |0, \varepsilon = 1/2\rangle, \tag{50}$$

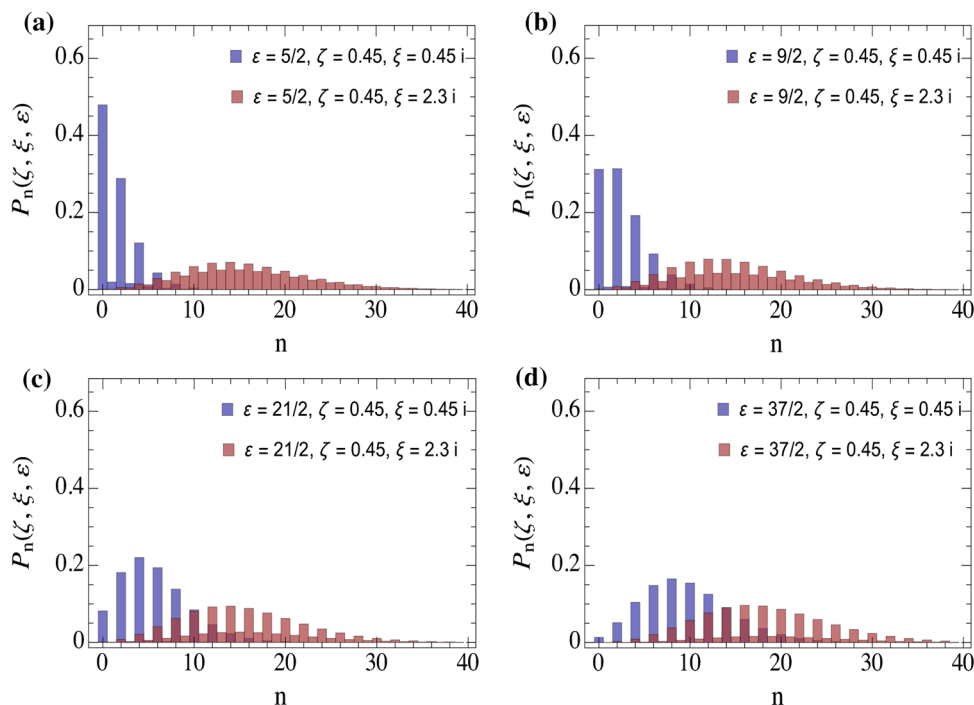
which are the time-dependent canonical CS [76, 77].

The overlap of the states  $|\zeta, \xi\rangle$  for different squeeze  $\zeta$  and displacement  $\xi$  parameters is given by

$$\langle \xi_1, \zeta_1 | \zeta, \xi \rangle = \left(\frac{\xi \xi_1^*}{2}\right)^{\varepsilon-1} \sqrt{\frac{(1-|\zeta|^2)(1-|\zeta_1|^2)}{\left[ I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) + I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) \right] \left[ I_{\varepsilon-1}\left(\frac{|\xi_1|^2}{1-|\zeta_1|^2}\right) + I_{\varepsilon}\left(\frac{|\xi_1|^2}{1-|\zeta_1|^2}\right) \right]}} \\ \times \exp\left[\frac{\zeta^*\xi^2(1-|\zeta_1|^2) + \zeta_1 \xi_1^{*2}(1-|\zeta|^2)}{2(1-|\zeta|^2)(1-|\zeta_1|^2)} + i \int \text{Re}(\alpha\zeta^* - \alpha\zeta_1^*) dt\right] \\ \times \sum_{n=0}^{\infty} (\zeta \zeta_1^*)^n n! \left[ \frac{L_n^{\varepsilon-1}\left(\frac{\xi^2}{2\zeta}\right) L_n^{\varepsilon-1}\left(\frac{\xi_1^{*2}}{2\zeta_1^*}\right)}{\Gamma(n+\varepsilon)} + \frac{\xi \xi_1^* L_n^{\varepsilon}\left(\frac{\xi^2}{2\zeta}\right) L_n^{\varepsilon}\left(\frac{\xi_1^{*2}}{2\zeta_1^*}\right)}{2 \Gamma(n+\varepsilon+1)} \right]. \tag{51}$$



**Fig. 3** Transition probability of the generalized CS



Here, we can analyze the transition probability  $P_n(\zeta, \xi, \epsilon) = |\langle \epsilon, n | \zeta, \xi \rangle|^2$  of the number states  $|n, \epsilon\rangle$  to the CS  $|\zeta, \xi\rangle$ :

$$P_n(\zeta, \xi, \epsilon) = \left(\frac{|\xi|^2}{2}\right)^{\epsilon-1} \frac{(1 - |\zeta|^2) \exp\left[\frac{\text{Re}(\zeta^* \xi^2)}{1 - |\zeta|^2}\right]}{I_{\epsilon-1}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right) + I_{\epsilon}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right)} m! |\zeta|^{2m} \left[ \frac{|L_m^{\epsilon-1}\left(\frac{\xi^2}{2\zeta}\right)|^2 \delta_{n,2m}}{\Gamma(m + \epsilon)} + \frac{|\xi|^2 |L_m^{\epsilon}\left(\frac{\xi^2}{2\zeta}\right)|^2 \delta_{n,2m+1}}{2\Gamma(m + \epsilon + 1)} \right] \tag{52}$$

The transition probability  $P_n(\zeta, \xi, \epsilon)$  has been shown in Fig. 3. As we can see, the  $\xi$ -parameter allows access to odd states while the  $\epsilon$  controls the dispersion of the probability density. Furthermore, higher values of the  $\epsilon$ -parameter allow access to states with higher principal quantum number  $n$ , and attenuate the access to the odd states.

4.1 Mean values and uncertainty relations

Let us now consider the mean value of the momentum  $\hat{P}$  and position  $\hat{x}$  operators, which satisfy the commutation relation

$$[\hat{x}, \hat{P}] = i\hbar [1 + (2\epsilon - 1)\hat{R}], \tag{53}$$

in the generalized CS (47). For this, we will rewrite these operators in terms of the integrals of motion, as seen below

$$\begin{aligned} \hat{a} &= \frac{\hat{A}_f - \zeta \hat{A}_{f^*}^\dagger}{1 - |\zeta|^2}, \quad \hat{A}_f \equiv \frac{1}{f} \hat{A}, \\ \hat{x} &= \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} l = \frac{l}{\sqrt{2}} \frac{(1 - \zeta^*) \hat{A}_f + (1 - \zeta) \hat{A}_{f^*}^\dagger}{1 - |\zeta|^2}, \\ \hat{P} &= \hbar \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}l} = \frac{\hbar}{i\sqrt{2}l} \frac{(1 + \zeta^*) \hat{A}_f - (1 + \zeta) \hat{A}_{f^*}^\dagger}{1 - |\zeta|^2}, \end{aligned} \tag{54}$$

where  $l$  is a length-dimensional parameter which is related to the initial standard deviation [48]. The new operator  $\hat{A}_f$  acts on states  $|\zeta, \xi\rangle$  as follows

$$\hat{A}_f |\zeta, \xi\rangle = \xi |\zeta, \xi\rangle. \tag{55}$$

Using the relations (54) and (55), we can easily calculate the mean values of the operators  $\hat{x}$  and  $\hat{P}$ ,

$$\bar{x} = \bar{x}(t) = \langle \xi, \zeta | \hat{x} | \zeta, \xi \rangle = \sqrt{2}l \frac{\text{Re}[(1 - \zeta^*)\xi]}{1 - |\zeta|^2},$$

$$\bar{P} = \overline{P}(t) = \langle \xi, \zeta | \hat{P} | \zeta, \xi \rangle = \frac{\sqrt{2}\hbar \operatorname{Im}[(1 + \zeta^*)\xi]}{l(1 - |\zeta|^2)}. \tag{56}$$

From here, there is a correspondence between the squeeze  $\zeta$  and displacement  $\xi$  parameters with the mean values of  $\bar{x}$  and  $\bar{P}$ ,

$$\xi = \frac{1 + \zeta}{\sqrt{2}l} \bar{x} + \frac{il(1 - \zeta)}{\hbar\sqrt{2}} \bar{P}. \tag{57}$$

Taking the square of the operators  $\hat{x}$  and  $\hat{P}$ , we have

$$\begin{aligned} \hat{x}^2 &= \frac{l^2(1 - \zeta^*)^2 \hat{A}_f^2 + (1 - \zeta)^2 \hat{A}_{f*}^{\dagger 2} + 2|1 - \zeta|^2 \hat{A}_{f*}^{\dagger} \hat{A}_f}{(1 - |\zeta|^2)^2} + l^2 \frac{|1 + \zeta|^2 (2\varepsilon - 1)\hat{R} + 1}{1 - |\zeta|^2}, \\ \hat{P}^2 &= -\frac{\hbar^2(1 + \zeta^*)^2 \hat{A}_f^2 + (1 + \zeta)^2 \hat{A}_{f*}^{\dagger 2} - 2|1 + \zeta|^2 \hat{A}_{f*}^{\dagger} \hat{A}_f}{(1 - |\zeta|^2)^2} + \frac{\hbar^2 |1 + \zeta|^2 (2\varepsilon - 1)\hat{R} + 1}{l^2(1 - |\zeta|^2)}. \end{aligned} \tag{58}$$

So, from these results, we can calculate the mean values of  $\overline{x^2}$  and  $\overline{P^2}$ , as follow

$$\begin{aligned} \overline{x^2} = \overline{x^2}(t) &= \langle \xi, \zeta | \hat{x}^2 | \zeta, \xi \rangle = \frac{2l^2 \operatorname{Re}^2[(1 - \zeta^*)\xi]}{(1 - |\zeta|^2)^2} + l^2 \frac{|1 + \zeta|^2 (2\varepsilon - 1)\hat{R} + 1}{1 - |\zeta|^2}, \\ \overline{P^2} = \overline{P^2}(t) &= \langle \xi, \zeta | \hat{P}^2 | \zeta, \xi \rangle = \frac{2\hbar^2 \operatorname{Im}^2[(1 + \zeta^*)\xi]}{l^2(1 - |\zeta|^2)^2} + \frac{\hbar^2 |1 + \zeta|^2 (2\varepsilon - 1)\hat{R} + 1}{l^2(1 - |\zeta|^2)}, \end{aligned} \tag{59}$$

where

$$\bar{R} = \langle \xi, \zeta | \hat{R} | \zeta, \xi \rangle = \frac{I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) - I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right)}{I_{\varepsilon-1}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right) + I_{\varepsilon}\left(\frac{|\xi|^2}{1-|\zeta|^2}\right)}. \tag{60}$$

Note that  $\xi = 0$  leads to a well-defined even parity of the states (47), once  $\bar{R} = 1$ .

In what follows, we calculate the standard deviation, i.e.,

$$\begin{aligned} \sigma_x = \sigma_x(t) &= \sqrt{\overline{x^2} - \bar{x}^2} = l|1 - \zeta| \sqrt{\frac{1 + (2\varepsilon - 1)\bar{R}}{2(1 - |\zeta|^2)}}, \\ \sigma_P = \sigma_P(t) &= \sqrt{\overline{P^2} - \bar{P}^2} = \frac{\hbar}{l}|1 + \zeta| \sqrt{\frac{1 + (2\varepsilon - 1)\bar{R}}{2(1 - |\zeta|^2)}}. \end{aligned} \tag{61}$$

From Eq. (61), it is easy to see that standard deviations in position and momentum present the squeezing property.

Finally, we aim to find the Heisenberg uncertainty relation in the generalized states (47) by considering the WHA. Therefore, in the hold of previous results, we can explicitly calculate the product  $\sigma_x \sigma_P$ , as shown below:

$$\sigma_x \sigma_P = \hbar \frac{|1 - \zeta||1 + \zeta|}{1 - |\zeta|^2} \frac{1 + (2\varepsilon - 1)\bar{R}}{2} = \hbar \sqrt{1 + \frac{4\operatorname{Im}^2(\zeta)}{(1 - |\zeta|^2)^2} \frac{1 + (2\varepsilon - 1)\bar{R}}{2}}. \tag{62}$$

In particular, if we consider that the squeeze parameter assumes real values, this implies that  $\sigma_x \sigma_P = \hbar/2(1 + (2\varepsilon - 1)\bar{R})$ . In turn, for this particular choice, we should note that the Heisenberg uncertainty assumes the minimum value predicted by the relation

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|. \tag{63}$$

It must be highlighted that in Ref. [38], a detailed analysis has been applied to show how the para-Bose uncertainty relation leads to uncertainty relation computed from canonical commutation relation.

Taking into account the covariance  $\sigma_{xP}$ ,

$$\sigma_{xP} = \frac{\langle \xi, \zeta | \hat{P}\hat{x} | \zeta, \xi \rangle + \langle \xi, \zeta | \hat{x}\hat{P} | \zeta, \xi \rangle}{2} - \bar{x}(t)\bar{P}(t) = -\hbar \operatorname{Im}(\zeta) \frac{1 + (2\varepsilon - 1)\bar{R}}{1 - |\zeta|^2}, \tag{64}$$

we can calculate the Schrödinger-Robertson uncertainty relation [78],

$$\sigma_x^2 \sigma_P^2 - \sigma_{xP}^2 = \frac{\hbar^2}{4} [1 + (2\varepsilon - 1)\bar{R}]^2, \tag{65}$$

which, as can we see, is minimized.

### 5 Coordinate representation of the generalized CS

It is well-known that the phase space in the context of quantum mechanics encounters difficulties due to the uncertainty principle. Considering the states  $|\zeta, \xi\rangle$  in a coordinate representation allow us to introduce the quasiprobability Wigner distribution, which plays an analogous role to the classical distributions [79–81].

According to WHA,  $\hat{P}$  is a self-adjoint operator on semi-axis ( $x \geq 0$ ), see [36], and have the following form:

$$\hat{P} = -i\hbar\partial_x + \frac{i\hbar}{2x}(2\varepsilon - 1)\hat{R}. \tag{66}$$

In this case, the annihilation operator takes the form

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{l} + \frac{i l}{\hbar}\hat{P}\right) = \frac{l}{\sqrt{2}}\left(\partial_x - \frac{2\varepsilon - 1}{2x}\hat{R} + \frac{x}{l^2}\right). \tag{67}$$

Applying the annihilation condition

$$\hat{a}\Psi_{0,\varepsilon}(x) = 0, \quad \Psi_{0,\varepsilon}(x) = \langle x|0, \varepsilon\rangle, \tag{68}$$

we can obtain a differential equation for the vacuum state, as follows:

$$\left(\partial_x + \frac{x}{l^2} - \frac{2\varepsilon - 1}{2x}\right)\Psi_{0,\varepsilon}(x) = 0, \quad \hat{R}\Psi_{0,\varepsilon}(x) = \Psi_{0,\varepsilon}(x). \tag{69}$$

The general solution reads

$$\Psi_{0,\varepsilon}(x) = Cx^{\varepsilon - \frac{1}{2}} \exp\left(-\frac{x^2}{2l^2}\right), \tag{70}$$

with  $C$  being a real constant, which will be determined through the normalization condition, as seen below,

$$2C^2 \int_0^\infty x^{2\varepsilon - 1} \exp\left(-\frac{x^2}{l^2}\right) dx = 1 \Rightarrow C = \frac{1}{l^\varepsilon \sqrt{\Gamma(\varepsilon)}}. \tag{71}$$

It should be noted that the condition (69) leads to the following quantization condition:

$$\varepsilon = 2\ell + \frac{1}{2}, \quad \ell = 0, 1, 2, \dots, \tag{72}$$

where  $\ell$  is analogous to the angular momentum. Such quantization was obtained in [37] to ensure that the eigenfunctions of  $\hat{P}$  are differentiable at the origin.

Thus, the vacuum state with even parity  $\Psi_{0,\varepsilon}(x) \equiv \Psi_{0,\ell}^e(x)$  takes the form

$$\Psi_{0,\ell}^e(x) = \frac{1}{l^{2\ell + \frac{1}{2}} \sqrt{\Gamma(2\ell + \frac{1}{2})}} x^{2\ell} \exp\left(-\frac{x^2}{2l^2}\right). \tag{73}$$

Taking into account that  $\Psi_{\zeta,\xi}^\ell(x, t) = \langle x|\zeta, \xi\rangle$  and replacing the relations (14) in the states (47), we obtain

$$\Psi_{\zeta,\xi}^\ell(x, t) = \xi^{2\ell - \frac{1}{2}} \frac{\sqrt{(1 - |\zeta|^2)\Gamma(2\ell + \frac{1}{2})} \exp\left[\frac{\zeta^*\xi^2}{2(1 - |\zeta|^2)} + i\tilde{\vartheta}\right]}{2^{\frac{4\ell - 1}{4}} \sqrt{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right)}} \sum_{n=0}^\infty \left(-\frac{\zeta\hat{a}^{\dagger 2}}{2}\right)^n \left[\frac{L_n^{2\ell - \frac{1}{2}}\left(\frac{\xi^2}{2\zeta}\right)}{\Gamma(n + 2\ell + \frac{1}{2})} + \frac{L_n^{2\ell + \frac{1}{2}}\left(\frac{\xi^2}{2\zeta}\right)}{\Gamma(n + 2\ell + \frac{3}{2})} \frac{\xi\hat{a}^\dagger}{2}\right] \Psi_{0,\ell}^e(x). \tag{74}$$

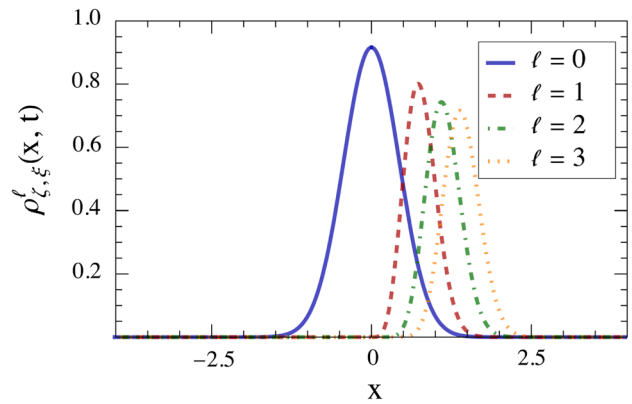
Using the results below

$$\begin{aligned} (\hat{a}^\dagger)^{2n} \Psi_{0,\ell}^e(x) &= (-1)^n 2^n n! L_n^{2\ell - \frac{1}{2}}\left(\frac{x^2}{l^2}\right) \Psi_{0,\ell}^e(x), \\ (\hat{a}^\dagger)^{2n+1} \Psi_{0,\ell}^e(x) &= \frac{\sqrt{2}x}{l} (-1)^n 2^n n! L_n^{2\ell + \frac{1}{2}}\left(\frac{x^2}{l^2}\right) \Psi_{0,\ell}^e(x), \end{aligned} \tag{75}$$

we can write (74), as follows

$$\Psi_{\zeta,\xi}^\ell(x, t) = \langle x|\zeta, \xi\rangle = \frac{\sqrt{1 - |\zeta|^2} \sqrt{x}}{1 - \zeta} \frac{I_{2\ell - \frac{1}{2}}\left(\frac{\sqrt{2}\xi x}{1 - \zeta l}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{\sqrt{2}\xi x}{1 - \zeta l}\right)}{\sqrt{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{|\xi|^2}{1 - |\zeta|^2}\right)}} \exp\left[-\frac{1 + \zeta}{1 - \zeta} \frac{x^2}{2l^2} - \frac{(1 - \zeta^*)\xi^2}{2(1 - \zeta)(1 - |\zeta|^2)} + i\tilde{\vartheta}\right]. \tag{76}$$

**Fig. 4** Displayed is the probability density by assuming fixed values  $l = 1, \zeta = 0.45$  and  $\xi = i$ . For  $\ell = 0$ , we have the SCS, while for  $\ell > 0$ , we obtained the generalized para-Bose CS



In particular, if  $\ell = 0$  we obtain the following result:

$$\Psi_{\zeta,\xi}^0(x,t) = \frac{(1-|\zeta|^2)^{1/4}}{\sqrt{\sqrt{\pi}l(1-\zeta)}} \exp \left[ -\frac{1}{2l^2} \frac{1+\zeta}{1-\zeta} \left( x - \frac{l\sqrt{2}\xi}{1+\zeta} \right)^2 + \frac{(1+\zeta^*)}{(1+\zeta)(1-|\zeta|^2)} \frac{\xi^2}{2} - \frac{1}{2} \frac{|\xi|^2}{1-|\zeta|^2} + i\varrho \right],$$

$$\varrho = \frac{1}{2} \int [\text{Re}(\alpha\zeta^* - \beta) - 2\delta] dt. \tag{77}$$

Performing the following identifications  $\zeta = g/f, \xi = z/f = -\varphi/f$  and  $\mu = |f|^2(1-|\zeta|^2) = 1$  leads to the results of the recent publication [48].

The probability density that corresponds to the generalized CS is given by

$$\rho_{\zeta,\xi}^\ell(x,t) = \left| \Psi_{\zeta,\xi}^\ell(x,t) \right|^2 = \frac{1-|\zeta|^2}{|1-\zeta|^2} \frac{x}{l^2} \frac{\left| I_{2\ell-\frac{1}{2}} \left( \frac{\sqrt{2}\xi x}{1-\zeta} \right) + I_{2\ell+\frac{1}{2}} \left( \frac{\sqrt{2}\xi x}{1-\zeta} \right) \right|^2}{\left| I_{2\ell-\frac{1}{2}} \left( \frac{|\xi|^2}{1-|\zeta|^2} \right) + I_{2\ell+\frac{1}{2}} \left( \frac{|\xi|^2}{1-|\zeta|^2} \right) \right|^2} \exp \left[ -\frac{1-|\zeta|^2}{|1-\zeta|^2} \frac{x^2}{l^2} - \frac{1}{1-|\zeta|^2} \text{Re} \left( \frac{1-\zeta^*}{1-\zeta} \xi^2 \right) \right]. \tag{78}$$

In Fig. 4, we have obtained some plots of the probability density. As we can see, the probability density of the generalized CS has a shape of a Gaussian distribution, which moves in space as  $\ell$  increases.

Finally, substituting (54) into (4), the Hamiltonian takes the form

$$\begin{aligned} \hat{H} &= \frac{1}{2} \hbar(\alpha^* \hat{a}^2 + \alpha \hat{a}^{\dagger 2}) + \frac{1}{2} \hbar\beta(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) + \hbar\delta \\ &= \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 + \frac{1}{2} \Omega(\hat{P} \hat{x} + \hat{x} \hat{P}) + \mathcal{E}, \\ &= -\frac{\hbar^2}{2m} \partial_x^2 - i\hbar\Omega x \partial_x + \frac{\hbar^2}{2mx^2} 2\ell(2\ell - \hat{R}) + \frac{1}{2} m\omega^2 x^2 + \mathcal{E} - \frac{i\hbar\Omega}{2}, \end{aligned} \tag{79}$$

where the time-dependent quantities  $m, \omega, \Omega$  and  $\mathcal{E}$  reads

$$\begin{aligned} \frac{1}{m} &= \frac{l^2}{\hbar} \text{Re}(\beta - \alpha), \quad m\omega^2 = \frac{\hbar}{l^2} \text{Re}(\beta + \alpha), \quad \Omega = \text{Im}(\alpha), \quad \mathcal{E} = \hbar\delta, \\ \omega^2 &= \beta^2 - \text{Re}^2(\alpha). \end{aligned} \tag{80}$$

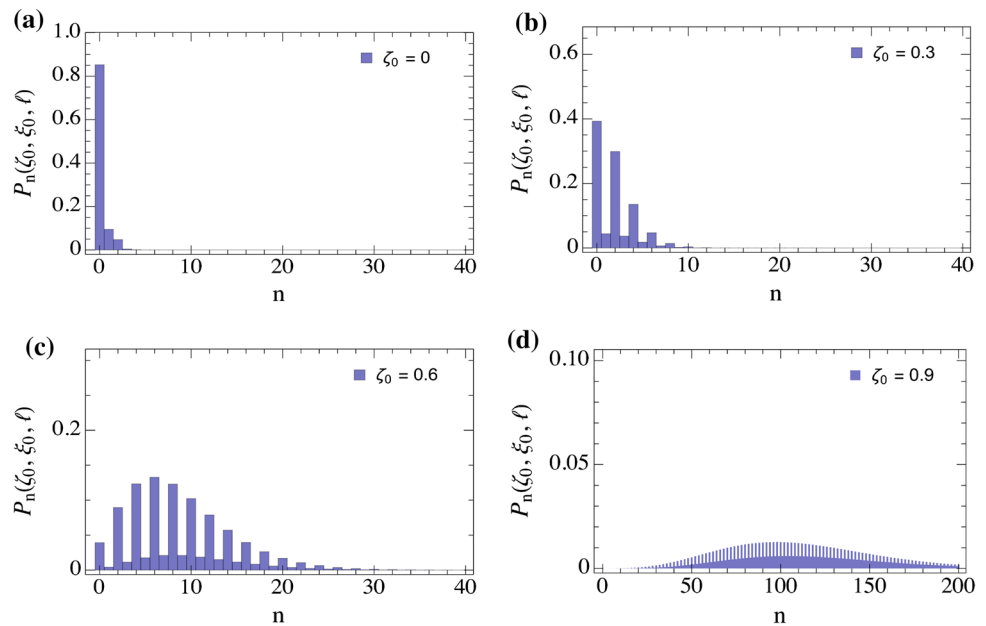
From Hamiltonian (79), we can describe the physical systems such as harmonic oscillators, which are described by confining potential  $V_{HO} \sim x^2$ . We also can establish an analogy of the potential  $V_C = \frac{\hbar^2}{2mx^2} 2\ell(2\ell - \hat{R})$  with the centrifugal potential. On the other hand, one may identify the potential  $V_C$  with the conformal sector of quantum conformal mechanics, except by the  $\hat{R}$  operator [82]. Since the reflection operator acting on even parity states leads to  $\hat{R}\psi_e = \psi_e$ , it must be noticed that the potential  $V_C$  has a negative correction term owing to the reflection operator. Furthermore, the repulsiveness of the  $V_C$  potential is weakened by the correction arising from the reflection operator.

Finally, we can relate the action of the reflection operator on a particular state to the reduction of the dimensionality of space. Let us analyze the centrifugal term of the  $d$ -dimensional Laplacian, which can be written in the form [37]:

$$\Delta^{(d)} = \partial_r^2 + \frac{d-1}{r} \partial_r + \frac{L(L+d-2)}{r^2}. \tag{81}$$

At this point, we can establish a direct relationship between the action of the reflection operator on even parity states and the reduction of the space from three to one dimension ( $d = 1$ ), from the perspective of the centrifugal term by considering that  $L = 2\ell$ .

**Fig. 5** The transition probability is showed by considering fixed values  $\ell = 2$ ,  $|\xi_0| = 1$ , and  $\theta_\xi = \pi/2$ . These conditions imply that  $\bar{x}_0 = 0$  and  $\bar{P}_0 = \sqrt{2}m_0\omega_0 |\xi_0|(1 + |\zeta_0|)/(1 - |\zeta_0|^2)$ . In Figure **a** we recover the para-Bose CS. Meanwhile, in Figures **b**, **c** and **d** we have the generalized para-Bose CS



### 6 Time-independent para-Bose Oscillator

Since the results obtained in this work can be applied to both time-dependent and time-independent physical systems, for the sake of simplicity, we apply this approach to the study of the time-independent para-Bose oscillator. First, we obtain the Hamiltonian of the time-independent para-Bose oscillator applying the following conditions  $\alpha = \delta = \Omega = 0$  and  $\beta = \beta_0$  on Eq. (4), which can be written in the form:

$$\hat{H} = \frac{1}{2} \hbar \beta_0 (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \frac{\hbar^2}{2m_0} \left[ -\partial_x^2 + \frac{2\ell(2\ell - \hat{R})}{x^2} + \frac{m_0^2 \omega_0^2}{\hbar^2} x^2 \right],$$

$$\omega_0 = \beta_0, \quad m_0 = \frac{\hbar}{l^2 \beta_0}, \tag{82}$$

where the subindex labels the initial time, and from (72) we can write the para-Bose number states  $|n, \varepsilon\rangle \Rightarrow |n, \ell\rangle$ , such that

$$\hat{H}|n, \ell\rangle = \hbar\omega_0 \left( n + 2\ell + \frac{1}{2} \right) |n, \ell\rangle. \tag{83}$$

Notice that, taking  $\ell = 0$  lead to the standard harmonic oscillator.

In this case, the equation system (8), take the form

$$\dot{f} = i\omega_0 f, \quad \dot{g} = -i\omega_0 g, \quad \dot{\varphi} = 0, \tag{84}$$

whose solution is given by

$$f = f_0 e^{i\omega_0 t}, \quad g = g_0 e^{-i\omega_0 t}, \quad \varphi \equiv 0. \tag{85}$$

From here, we can write the squeeze and displacement parameters in the form

$$\zeta = \frac{g}{f} = \zeta_0 e^{-2i\omega_0 t}, \quad \xi = \frac{z}{f} = \xi_0 e^{-i\omega_0 t},$$

$$\zeta_0 = |\zeta_0| e^{i\theta_\zeta} = \frac{g_0}{f_0}, \quad \xi_0 = |\xi_0| e^{i\theta_\xi} = \frac{z}{f_0}. \tag{86}$$

In the following, we can rewrite the operators (54):

$$\hat{a} = \frac{\hat{A}_f - \zeta_0 e^{-2i\omega_0 t} \hat{A}_{f^*}^\dagger}{1 - |\zeta_0|^2}, \quad \hat{A}_f \equiv \frac{1}{f_0 e^{i\omega_0 t}} \hat{A},$$

$$\hat{x} = \frac{l}{\sqrt{2}} \frac{(1 - \zeta_0^* e^{2i\omega_0 t}) \hat{A}_f + (1 - \zeta_0 e^{-2i\omega_0 t}) \hat{A}_{f^*}^\dagger}{1 - |\zeta_0|^2},$$

$$\hat{P} = \frac{\hbar}{i\sqrt{2}l} \frac{(1 + \zeta_0^* e^{2i\omega_0 t}) \hat{A}_f - (1 + \zeta_0 e^{-2i\omega_0 t}) \hat{A}_{f^*}^\dagger}{1 - |\zeta_0|^2}, \tag{87}$$

and the generalized CS (47), as follows

$$|\zeta, \xi\rangle = \left(\frac{|\xi_0| e^{i\theta_\xi}}{\sqrt{2}}\right)^{2\ell - \frac{1}{2}} \sqrt{\frac{1 - |\zeta_0|^2}{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right)}} \exp\left[\frac{|\zeta_0| |\xi_0|^2 e^{2i\theta_\xi - i\theta_\zeta}}{2(1 - |\zeta_0|^2)}\right] \\ \times \sum_{n=0}^\infty (-\zeta_0)^n \sqrt{n!} e^{-2i\omega_0(n+\ell)t} \left[ \frac{L_n^{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2 e^{2i\theta_\xi}}{2|\zeta_0| e^{i\theta_\zeta}}\right) e^{-\frac{i\omega_0 t}{2}}}{\sqrt{\Gamma(n + 2\ell + \frac{1}{2})}} |2n, \ell\rangle + \frac{|\xi_0| e^{i\theta_\xi} L_n^{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2 e^{2i\theta_\xi}}{2|\zeta_0| e^{i\theta_\zeta}}\right) e^{-\frac{3i\omega_0 t}{2}}}{\sqrt{2\Gamma(n + 2\ell + \frac{3}{2})}} |2n + 1, \ell\rangle \right]. \tag{88}$$

From Eq. (56), we get  $\bar{x}$  and  $\bar{P}$  in the form

$$\bar{x} = \bar{x}_0 \cos(\omega_0 t) + \frac{\bar{P}_0}{m_0 \omega_0} \sin(\omega_0 t), \quad \bar{P} = \bar{P}_0 \cos(\omega_0 t) - m_0 \omega_0 \bar{x}_0 \sin(\omega_0 t), \\ \bar{x}_0 = \sqrt{2}l|\xi_0| \frac{\cos(\theta_\xi) - |\zeta_0| \cos(\theta_\xi - \theta_\zeta)}{1 - |\zeta_0|^2}, \quad \bar{P}_0 = \sqrt{2}lm_0\omega_0|\xi_0| \frac{\sin(\theta_\xi) + |\zeta_0| \sin(\theta_\xi - \theta_\zeta)}{1 - |\zeta_0|^2}. \tag{89}$$

Notice that taking  $\xi_0 = 0$ , the mean values of  $\bar{x}$  and  $\bar{P}$  are equal to zero, which corresponds to the mean values evaluated in the generalized SVS.

The uncertainty relation (62) and (65) becomes

$$\sigma_x \sigma_P = \hbar \sqrt{1 + \frac{4|\zeta_0|^2 \sin^2(\theta_\zeta - 2\omega_0 t)}{(1 - |\zeta_0|^2)^2} \frac{1 + 4\ell\bar{R}}{2}}, \\ \sigma_x^2 \sigma_P^2 - \sigma_{xP}^2 = \frac{\hbar^2}{4} (1 + 4\ell\bar{R})^2, \tag{90}$$

with the mean value of  $\hat{R}$  is given by Eq. (60),

$$\bar{R} = \frac{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right) - I_{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right)}{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - |\zeta_0|^2}\right)}. \tag{91}$$

One can see that the uncertainty relation has an oscillatory behavior, reaching minimum values at specific points given by

$$\sin(\theta_\zeta - 2\omega_0 t_k) = 0, \quad t_k = \frac{\theta_\zeta - k\pi}{2\omega_0}, \tag{92}$$

where  $k = 0, \pm 1, \pm 2, \dots$ . Here  $t_k$  corresponds to the values on the time for which the uncertainty relation is minimal. The condition  $\theta_\zeta = 0 \implies \zeta_0 = \zeta_0^*$  leads to minimum uncertainty at  $t = 0$ . From Eqs. (61), (72) and (90), we can write the  $l$ -parameter in term of the quantities  $\sigma_x(0) = \sigma_{x_0}$ ,  $\zeta_0$ ,  $\xi_0$  and  $\ell$ , as follows

$$l = \sqrt{\frac{1 + \zeta_0}{1 - \zeta_0} \frac{2}{1 + 4\ell\bar{R}}} \sigma_{x_0}, \quad \sigma_{x_0} = \hbar \frac{1 + 4\ell\bar{R}}{2\sigma_{P_0}}. \tag{93}$$

In what the following, let us consider  $\zeta_0$  as being a real parameter.

It must be highlighted that the para-Bose number states are eigenstates of the Hamiltonian (82), and therefore the standard deviation for this operator is null when evaluated on this basis. Since  $\hat{H}$  is time-independent, we have that  $|n, \ell\rangle$  are eigenstates of  $\hat{H}$  with well-defined energy. Furthermore, it is interesting to verify the transition probability from the states with well-defined energy to generalized CS (88). Therefore, it follows from (52), (72) and (86) that

$$P_n(\zeta_0, \xi_0, \ell) = \left(\frac{|\xi_0|^2}{2}\right)^{2\ell - \frac{1}{2}} \frac{(1 - \zeta_0^2) \exp\left[\frac{\zeta_0 |\xi_0|^2}{1 - \zeta_0^2} \cos(2\theta_\xi)\right]}{I_{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - \zeta_0^2}\right) + I_{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2}{1 - \zeta_0^2}\right)} n! \zeta_0^{2n} \left[ \frac{\left|L_n^{2\ell - \frac{1}{2}}\left(\frac{|\xi_0|^2 e^{2i\theta_\xi}}{2\zeta_0}\right)\right|^2}{\Gamma(n + 2\ell + \frac{1}{2})} + \frac{|\xi_0|^2 \left|L_n^{2\ell + \frac{1}{2}}\left(\frac{|\xi_0|^2 e^{2i\theta_\xi}}{2\zeta_0}\right)\right|^2}{2\Gamma(n + 2\ell + \frac{3}{2})} \right]. \tag{94}$$

Notice that the transition probability is time-independent. As we saw in Figs. 1 and 3, the transition probability has its shape significantly altered by the displacement and Wigner parameters. In Fig. 5 we will make an analysis considering some values for the squeeze parameter  $\zeta_0$ , keeping the other parameters fixed; namely,  $\xi_0$  and  $\ell$ .

From Eq. (89), we have that the  $\zeta_0, \xi_0$ -parameters are directly related to the initial conditions that lead to the temporal evolution of the mean values in the position and momentum. As we saw, the standard deviation and the uncertainty relations are expressed in

terms of special functions, whose analysis is not straightforward. Therefore, it is interesting to investigate the asymptotic form of Bessel functions of the first kind present in uncertainty relations as follows:

(a) Asymptotic form for small arguments  $Z$  and fixed  $\kappa$  (see Eq. 9.6.7, page 375 in Ref. [83]).

$$I_\kappa(Z) \sim \frac{1}{\Gamma(\kappa + 1)} \left(\frac{Z}{2}\right)^\kappa. \tag{95}$$

This limit implies that  $|\xi_0| \rightarrow 0$  and  $\zeta_0 < 1$ . In its turn, the mean value of the reflection operator  $\bar{R}$  can be rewritten:

$$\bar{R} \sim \frac{(4\ell + 1)(1 - \zeta_0^2) - |\xi_0|^2}{(4\ell + 1)(1 - \zeta_0^2) + |\xi_0|^2}, \tag{96}$$

and the uncertainty relations (90) become

$$\begin{aligned} \sigma_x \sigma_P &\sim \frac{\hbar}{2} \sqrt{1 + \frac{4\zeta_0^2 \sin^2(2\omega_0 t)}{(1 - \zeta_0^2)^2} \frac{(4\ell + 1)^2(1 - \zeta_0^2) - (4\ell - 1)|\xi_0|^2}{(4\ell + 1)(1 - \zeta_0^2) + |\xi_0|^2}}, \\ \sigma_x^2 \sigma_P^2 - \sigma_{xP}^2 &\sim \frac{\hbar^2}{4} \left[ \frac{(4\ell + 1)^2(1 - \zeta_0^2) - (4\ell - 1)|\xi_0|^2}{(4\ell + 1)(1 - \zeta_0^2) + |\xi_0|^2} \right]^2. \end{aligned} \tag{97}$$

From (96), it is easy to see that the parity of the states is even when we take  $|\xi_0| = 0$ . On the other hand, the value of the uncertainties, Eq. (97), increases as  $\ell$  increases, while for  $\zeta_0 = \ell = 0$  is minimized.

b) Asymptotic form for large arguments  $Z$  and fixed  $\kappa$  (see Eq. 9.7.1, page 377 in Ref. [83]),

$$I_\kappa(Z) \sim \frac{e^Z}{\sqrt{2\pi Z}} \left[ 1 - \frac{1}{2Z} \left( \kappa^2 - \frac{1}{4} \right) \right]. \tag{98}$$

This limit can be obtained in two different ways, first it is  $|\xi_0| \rightarrow \infty$  with  $\zeta_0 < 1$

and second  $\zeta_0 \rightarrow 1$  with  $|\xi_0| < \infty$ . In this second case, we have a high degree of squeeze in  $\sigma_x$ , which becomes smaller compared to  $\sigma_P$ , as  $\zeta_0$  approaches 1. From Eq. (98), we can write the mean value of  $\bar{R}$  as follows

$$\bar{R} \sim \frac{\ell(1 - \zeta_0^2)}{|\xi_0|^2 - 2\ell^2(1 - \zeta_0^2)}. \tag{99}$$

Note that the range of values  $|\xi_0|^2 < 2\ell^2(1 - \zeta_0^2)$  takes  $\bar{R} < 0$ , indicating that only odd parity states stay at this limit. The uncertainty relations can be rewritten in the form

$$\begin{aligned} \sigma_x \sigma_P &\sim \frac{\hbar}{2} \sqrt{1 + \frac{4\zeta_0^2 \sin^2(2\omega_0 t)}{(1 - \zeta_0^2)^2} \frac{|\xi_0|^2 + 2\ell^2(1 - \zeta_0^2)}{|\xi_0|^2 - 2\ell^2(1 - \zeta_0^2)}}, \\ \sigma_x^2 \sigma_P^2 - \sigma_{xP}^2 &\sim \frac{\hbar^2}{4} \left[ \frac{|\xi_0|^2 + 2\ell^2(1 - \zeta_0^2)}{|\xi_0|^2 - 2\ell^2(1 - \zeta_0^2)} \right]^2. \end{aligned} \tag{100}$$

It is easy to see that  $\ell = 0$  lead the uncertainty relations to the value obtained when evaluated in terms of the canonical commutation relation ( $[\hat{x}, \hat{p}] = i\hbar, \hat{p} = -i\hbar\partial_x$ ). Besides, it can be seen that the squeeze parameter  $\zeta_0$  ensures that the Heisenberg uncertainty relation oscillates over time.

This example illustrates how this general procedure can be applied to a range of problems by simply determining the parameters of the Hamiltonian, then finding the functions  $f$  and  $g$ . From these functions, we obtain the squeeze and displacement parameters that, in its turn, modify the uncertainty relation and transition probability.

### 7 Concluding remarks

In this article, we study the integrals of motion method in a para-Bose formulation. This approach generalizes the usual canonical commutation relation. In turn, we obtain a generalization of the usual SVS, which admits a completeness relation in terms of the Wigner parameter. This relation depends on a range of values for the Wigner parameter, which does not include that corresponding to canonical algebra. We also obtain a generalization of the CS in terms of the even and odd time-independent para-Bose number states. These states are thoroughly determined in terms of the time-dependent squeeze and displacement parameters, as well as the Wigner parameter. In the study of the transition probability, we saw that the displacement parameter has an additional role, which is a kind of transition parameter by allowing access to the odd states of the system. Meanwhile, the Wigner parameter has the role of controlling the “dispersion,” and attenuating the access to the odd states. We show that the minimization of the Heisenberg uncertainty relation is easily obtained by taking the real value of the squeeze parameter and that the squeezing properties can be

seen from the standard deviation of the position and momentum. Taking the coordinate representation of the generalized CS, we found a quantization condition on the Wigner parameter, analogous to the quantization of the angular momentum, which arises by imposing that the parity of the vacuum state is even. This quantization condition also ensures that the eigenstates of the momentum operator are differentiable at the origin. Finally, the para-Bose oscillator has been discussed in this framework.

**Acknowledgements** We would like to thank CNPq, CAPES and CNPq/PRONEX/FAPESQ-PB (Grant no. 165/2018), for partial financial support. ASL and FAB acknowledge support from CNPq (Grant nos. 150601/2021-2, 312104/2018-9). ASP thanks the support of the Instituto Federal do Pará.

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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