



Anisotropic spacetimes in $f(T, B)$ theory II: Kantowski–Sachs Universe

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Received: 11 May 2022 / Accepted: 17 July 2022

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Abstract In the context of the modified teleparallel $f(T, B)$ theory of gravity, we consider a homogeneous and anisotropic background geometry described by the Kantowski–Sachs line element. We derive the field equations and investigate the existence of exact solutions. Furthermore, the evolution of the trajectories for the field equations is studied by deriving the stationary points at the finite and infinite regimes. For the $f(T, B) = T + F(B)$ theory, we prove that for a specific limit of the function $F(B)$, the anisotropic Universe has the expanding and isotropic Universe as an attractor with zero spatial curvature. We remark that there are no future attractors where the asymptotic solution describes a Universe with nonzero spatial curvature.

1 Introduction

Modified theories of gravity can be obtained from the Einstein–Hilbert action by introducing geometric invariants, which modify the gravitational field equations [1–4]. That introduce new dynamical degrees of freedom in the field equations which drive the dynamics such that to explain the observational phenomena, see, for instance, [5–18] and references therein. The fundamental invariant in General Relativity is the Ricci scalar R of the Levi-Civita symmetric connection, but that is not the case the Teleparallel Equivalent of General Relativity (TEGR), where the fundamental geometric invariant used for the definition of the gravitational Action Integral is the torsion scalar T . It is defined by the antisymmetric connection of the nonholonomic basis [19–24], the so-called curvature-less Weitzenböck connection [25]. In [26], it was found that in the Teleparallel Equivalence of General Relativity (TEGR) [19–24], the energy distribution using Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum complexes is the same either in general relativity or teleparallel gravity.

However, it is well known that the equivalence of teleparallelism and General Relativity does not hold on the modified theories of gravity based on general functions of the Ricci scalar and that based on general functions of the torsion scalar, for instance, when one compares $f(T)$ -theory and $f(R)$ -theory [5, 27]. Moreover, the two theories have different properties from that of General Relativity [28, 29]. Applications of $f(T)$ theory in the dark energy problem and in the description of the cosmological history are presented in [30–35].

In this research, we are interested in a higher-order teleparallel theory known as $f(T, B)$ where B is the boundary term relating the torsion T with the Ricci scalar R , that is, $B = T + R$ [36, 37]. Because B includes second-order derivatives, the $f(T, B)$ -theory is a fourth-order theory for a nonlinear function f on the variable B . There are some studies dealing with the $f(T, B)$ -theory in cosmology. Exact and analytic cosmological solutions with an isotropic background space were investigated in [38–40], while the reconstruction of the cosmological history in $f(T, B)$ theory was the subject of study in [41–44].

We have devoted a series of papers to obtain conditions under which the $f(T, B)$ -model anisotropic model tends to the homogeneous and isotropic FRW model. A related question is how the parameters and initial conditions of the model influence the isotropization process. The fact is that when using dynamical systems approaches, the existence of one attractor under some conditions means that the attractor solution is attained, irrespective of the initial conditions. Therefore, a solution to the isotropization problem transforms into finding an isotropic late-time attractor, moreover, by taking an anisotropic Kantowski–Sachs Universe as the metric for $f(T, B)$ -model, as some advantages, in the sense that it is not settled from the start the homogeneous and isotropic FRW model, and what is proved when an anisotropic model isotropizes. For example, we know that inflation is the most successful candidate to explain why the observable Universe is currently homogeneous and isotropic with great precision. However, the problem is not completely solved in the literature. That is, one usually assumes from the beginning that the Universe is homogeneous and isotropic, as given by the FLRW metrics, and then examines the evolution of the perturbations, rather than starting with an arbitrary metric and showing that inflation does occur and that the Universe evolves toward homogeneity and isotropy. The complete analysis is complex, even

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using numerical tools [45]. Thus, one should impose another assumption to extract analytical information: consider anisotropic but homogeneous cosmologies. This class of geometries [46] exhibits very interesting cosmological features, both in inflationary and postinflationary epochs [47]. Along these lines, isotropization is a crucial question. Finally, the class of anisotropic geometries has recently gained much interest due to anisotropic anomalies in the Cosmic Microwave Background (CMB) and large-scale structure data. The origin of asymmetry and other statistical anisotropy measures on the largest Universe scales is a long-standing question in cosmology. “Planck Legacy” temperature anisotropy data show strong evidence of a violation of the Cosmological Principle in its isotropic aspect [48, 49].

Paper [50] is the first part of a series of studies on analyzing the $f(T, B)$ -theory considering some anisotropic spacetimes. The previous analysis of $f(T, B)$ -gravity was reviewed, and the global dynamics of a locally rotational Bianchi I background geometry were investigated. Considering $f(T, B) = T + F(B)$ theory, a criterion for solving the homogeneity problem in the $f(T, B)$ -theory was deduced. Finally, the integrability properties for the field equations were investigated by applying the Painlevé analysis, obtaining an analytic solution in terms of a right Painlevé expansion.

The present study extends the previous analysis presented in [50]. Specifically, we construct a family of exact anisotropic solutions while investigating the evolution of the field equations using dynamical system analysis in $f(T, B) = T + F(B)$ theory of gravity. From the analysis, it follows that in this theory, with initial conditions of Kantowski–Sachs geometry, the future attractor of the Universe can be a spatially flat spacetime which describes acceleration. The de Sitter spacetime exists for a specific value of the free parameter.

The third paper [51] in the sequel is devoted to the study of the asymptotic dynamics of $f(T, B)$ -theory in an anisotropic Bianchi III background geometry. We show that an attractor always exists for the field equations, which depends on a free parameter provided by the specific $f(T, B)$ functional form. The attractor is an accelerated spatially flat FLRW or a non-accelerated LRS Bianchi III geometry. Consequently, the $f(T, B)$ -theory provides a spatially flat and isotropic accelerated Universe. The study continued in [52] continued by applying the Noether symmetries for constructing conservation laws.

Kantowski–Sachs Universes [53] are four-dimensional anisotropic and homogeneous spacetimes with a topology $R \times S^2$ and admit a four-dimensional Lie algebra as a Killing vector field. The Killing fields act on spacelike hypersurfaces, which possess a three-parameter subgroup whose orbits are the two-dimensional sphere. Thus, Kantowski–Sachs geometries possess translational symmetry and the spherical symmetry [54]. Kantowski–Sachs metric differs from Bianchi I and Bianchi III according to their spacelike surface topology. These metrics are the natural extensions of closed FLRW, flat FLRW and open FLRW geometries, respectively. Therefore, there are expected substantial differences in the cosmological behavior according to whether the spacelike surface has closed, flat, or open topology (see discussion in Sect. 3). For that reason, we present the three cases in separate works.

There exist related works on cosmological analyses in teleparallel $f(T)$ [27] theory for the anisotropic Kantowski–Sachs universe [55–57]. However, they have not used a proper frame for vierbein fields, and their results are invalid since the limit of GR is not recovered in their frame. In [58] was given the prescription for a proper frame selection.

The plan of the paper is as follows.

In Sect. 2, we briefly present the basic definitions of $f(T, B) = T + F(B)$ theory. In Sect. 3, we define the proper frame for the vierbein fields, and we show that the field equations in $f(T, B) = T + F(B)$ have a minisuperspace description, that is, there exists a point-like Lagrangian which provides the field equations under a variation, while the higher-order degrees of freedom can be attributed to a scalar field. An exact anisotropic solution is determined in Sect. 4. The main material of this study, i.e., the dynamical analysis, is presented in Sect. 5. Finally, in Sect. 6, we summarize the results and draw our conclusions.

2 $f(T, B)$ gravity

We consider the higher-order teleparallel theory of gravity known as $f(T, B)$ -theory, in which T is the torsion scalar for the Weitzenböck connection [25] and $B = 2e^{-1}\partial_\nu(eT_\rho^{\rho\nu})$ is the boundary term relating the Ricciscalar to the torsion T as follows $B = T + R$.

The gravitational Action Integral is considered to be [36]

$$S_{f(T,B)} = \frac{1}{16\pi G} \int d^4x e f(T, B) \tag{1}$$

which provides the higher-order gravitational field equations

$$0 = e f_{,T} G_a^\lambda + \left[\frac{1}{4} (T f_{,T} - f) e h_a^\lambda + e (f_{,T})_{,\mu} S_a^{\mu\lambda} \right] + \left[e (f_{,B})_{,\mu} S_a^{\mu\lambda} - \frac{1}{2} e \left(h_a^\sigma (f_{,B})_{,\sigma}{}^{;\lambda} - h_a^\lambda (f_{,B})^{;\mu\nu} g_{\mu\nu} \right) + \frac{1}{4} e B h_a^\lambda f_{,B} \right]. \tag{2}$$

In this study, we are interested in the $f(T, B) = T + F(B)$ theory where the field equations are written in the simpler form [50]

$$e G_a^\lambda + T_a^{(B)\lambda} = 0 \tag{3}$$

in which now $T_a^{(B)\lambda}$ describes the geometrodynamical parts of the field equations which correspond to the boundary term, that is,

$$T_a^{(B)\lambda} = e\phi_{,\mu} S_a^{\mu\lambda} - \frac{1}{2}e(h_a^\sigma \phi_{;\sigma}{}^{;\lambda} - h_a^\lambda \phi^{;\mu\nu} g_{\mu\nu}) + \frac{1}{4}eh_a^\lambda V(\phi) + \frac{1}{4}eh_a^\lambda f. \tag{4}$$

The scalar field ϕ has been introduced to attribute the higher-order derivatives, that is, $\phi = F_{,B}$ and $V(\phi) = F - BF_{,B}$. An important characteristic of $f(T, B) = T + F(B)$ theory is that a minisuperspace description exists [59].

3 Kantowski–Sachs Universe

Kantowski–Sachs geometry can be seen as the anisotropic extension of the closed Friedmann–Lemaître–Robertson–Walker (FLRW) geometry. The metric depends on two essential scale factors in the spacelike hypersurface. With the use of Misner-like variables [60, 61], the one scale factor describes the radius of the spacelike hypersurface while the second scale factor describes the anisotropy. When the anisotropy becomes constant, the Kantowski–Sachs spacetime possesses a six-dimensional Lie algebra as Killing vector fields, and the limit of the closed FLRW geometry is achieved [62]. The Kantowski–Sachs space is directly related to the Bianchi spacetimes [60]. Indeed, the Kantowski–Sachs geometry follows under a Lie contraction in the locally rotational spacetime (LRS) Bianchi type IX.

Because of the importance of the Kantowski–Sachs geometries, there are various studies in the literature. The effects of the cosmological constant in the context of General Relativity were the subject of study in [63, 64], while the case of a perfect fluid obeys the barotropic equation of state was investigated in [65]. In the presence of the cosmological constant, the Kantowski–Sachs Universe leads to the de Sitter Universe as a future attractor [63]; thus, the Kantowski–Sachs geometry can be used to describe the pre-inflationary era [64]. In [65], it was found that the models with a fluid source admit a past asymptotic behavior leading to a big-bang singularity and as a future attractor, which provides a big crunch. The effects of the electromagnetic field were investigated in [66–68]. The inflationary scenario in Kantowski–Sachs models with scalar fields or modified theories of gravity was the subject of study in a series of studies, see, for instance, [69–78] and references therein.

The importance of the Kantowski–Sachs geometry is also evident in inhomogeneous models, specifically, in the case of Silent Universes [79]. Moreover, for the Szekeres spacetimes [80] the field equations of the Kantowski–Sachs constrain the dynamical variables of the field equations system. That means there exists a family of silent inhomogeneous and anisotropic universes where in the limit of homogenization, the spacetime reduces to the anisotropic Kantowski–Sachs geometry [81].

The line element for the Kantowski–Sachs spacetime is

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left(e^{2\beta(t)}dx^2 + e^{-\beta(t)}(dy^2 + \sin^2(y) dz^2)\right) \tag{5}$$

where $N(t)$ is the lapse function, $\alpha(t)$ is the scale factor for the three-dimensional hypersurface and $\beta(t)$ is the anisotropic parameter. For $\beta(t) \rightarrow 0$, the line element (5) reduces to the closed FLRW geometry.

We assume the vierbein fields [58]

$$e^1 = dt \tag{6}$$

$$e^2 = e^{a+\beta} \cos z \sin y \, dx + e^{a-\frac{\beta}{2}} (\cos y \cos z \, dy - \sin y \sin z \, dz) \tag{7}$$

$$e^3 = e^{a+\beta} \sin y \sin z \, dx + e^{a-\frac{\beta}{2}} (\cos y \sin z \, dy - \sin y \cos z \, dz) \tag{8}$$

$$e^4 = e^{a+\beta} \cos y \, dx - e^{a-\frac{\beta}{2}} \sin y \, dy \tag{9}$$

which provide

$$T = \frac{1}{N^2} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 \right) - 2e^{-2\alpha+\beta}, \tag{10}$$

such thatTEGR is recovered.

Moreover, the boundary term is calculated

$$B = \frac{6}{N^2} \left(\ddot{\alpha} - \dot{\alpha} \frac{\dot{N}}{N} + 3\dot{\alpha}^2 \right). \tag{11}$$

Therefore, the point-like Lagrangian is determined to be

$$\mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, \phi, \dot{\phi}) = \frac{1}{N} \left(e^{3\alpha} \left(6\dot{\alpha}^2 - \frac{3}{2}\dot{\beta}^2 \right) - 6e^{3\alpha} \dot{\alpha} \dot{\phi} \right) + Ne^{3\alpha} V(\phi) - 2Ne^{\alpha+\beta}. \tag{12}$$

where for $N = 1$ and $H = \dot{\alpha}$, the cosmological field equations for the anisotropic model are

$$0 = 6H^2 - \frac{3}{2}\dot{\beta}^2 - 6H\dot{\phi} + V(\phi) - 2e^{-2\alpha+\beta}, \tag{13}$$

$$0 = \dot{H} + 3H^2 + \frac{1}{6}V_{,\phi}, \quad (14)$$

$$0 = \ddot{\beta} + 3H\dot{\beta} - \frac{2}{3}e^{-2\alpha+\beta}, \quad (15)$$

and

$$0 = \ddot{\phi} + 3H^2 + \frac{1}{2}V(\phi) + \frac{1}{3}V_{,\phi} + \frac{3}{4}\dot{\beta}^2 - \frac{1}{3}e^{-2\alpha+\beta}. \quad (16)$$

We proceed with our analysis with the study of the existence of exact solutions of a particular interest for the field equations.

4 Exact solutions

It is important to study if the field Eqs. (13)–(16) admit exact solutions. For the case of a spatially flat FLRW universe, it was found that the $T + F(B)$ theory can reproduce any scale factor. That is not true in the presence of curvature or when new degrees of freedom are introduced, such as the anisotropic parameter.

Consider that the Hubble function is constant, that is, $H = H_0$. Then, from Eq. (14) it follows

$$V(\phi) = -18H_0^2\phi + V_0, \quad (17)$$

while (16) provides the closed-form solution $\beta = \beta_0 t$, with $\beta_0 = 2H_0$ and $H_0^2 = \frac{1}{9}$. Consequently, Eq. (16) is a linear equation in terms of ϕ and it can be explicitly solved. The closed-form solution is

$$\phi(t) = e^{3H_0 t} \phi_1 + \phi_0 \quad (18)$$

with $V_0 = 2$ and $\phi_1 = 18H_0^2$.

The Kantowski–Sachs spacetime reads

$$ds^2 = -dt^2 + e^{2H_0 t} \left(e^{4H_0 t} dx^2 + e^{-2H_0 t} (dy^2 + \sin^2(y) dz^2) \right), \quad H_0^2 = \frac{1}{9}. \quad (19)$$

Similarly, for $\alpha(t) = p \ln t$ and $H = \frac{p}{t}$, for the field equations we find the closed-form solution

$$\beta(t) = 2(p-1) \ln t, \quad (20)$$

$$V(\phi) = -\frac{2(2(1-3p) + 3pt\dot{\phi})}{t^2}, \quad (21)$$

and

$$\phi(t) = \frac{\phi_1}{1+3p} t^{1+3p} + \phi_0 - \frac{2(8-21p+9p^2)}{3(1+3p)} \ln((1+3p)t), \quad (22)$$

with $6p(3p-4) + 4 = 0$.

Thus, for $\phi_1 = 0$, the scalar field potential reads

$$V(\phi) = 4(1+3p)(p(8+3p(3p-4)) - 1) \exp\left(\frac{3(1+3p)}{8+3p(3p-7)}(\phi - \phi_0)\right). \quad (23)$$

In the following section, we continue with the analysis of the asymptotic dynamics for Eqs. (13)–(16).

5 Asymptotic dynamics

We define the dimensionless variables in the H -normalization approach [82]

$$\Sigma = \frac{\dot{\beta}}{2H}, \quad x = \frac{\dot{\phi}}{H}, \quad y = \frac{V(\phi)}{H^2}, \quad \Omega_R = \frac{e^{-2\alpha+\beta}}{3H^2}, \quad \lambda = -\frac{V_{,\phi}}{V} \quad (24)$$

where for the scalar field potential we assume the exponential function $V(\phi) = V_0 e^{-\lambda\phi}$, which correspond to the $F(B) = -\frac{1}{\lambda} B \ln B$ model.

The selection of the exponential potential function is twofold. In terms of dynamics, for such a potential function, the dimension of the dynamical system is reduced by one; however, this can provide the stationary points and for other potential functions in the limit where $\lambda = \text{const}$, see, for instance, discussion in [83]. Additionally, the exponential potential is of particular interest in terms of an isotropic universe. In previous studies, [84, 85] it was found that such potential is cosmological viable and can explain the

principal epochs of the cosmological evolution. Finally, the exponential potential has been found to provide integrable cosmological equations in the case of isotropic and spatially flat universe [83].

Thus, in the new variables (Σ, x, y, η) the field equations are written as the following system of algebraic–differential equations

$$\frac{d\Sigma}{d\tau} = -\lambda y \Sigma + \Omega_R, \tag{25}$$

$$\frac{dx}{d\tau} = 3(\Sigma^2 - 1) + (2\lambda - 3)y + x(3 - \lambda y) + \Omega_R, \tag{26}$$

$$\frac{dy}{d\tau} = -y(\lambda(x + 2y) - 6), \tag{27}$$

and

$$\frac{d\Omega_R}{d\tau} = 2(2 - \lambda y + \Sigma)\Omega_R, \tag{28}$$

with algebraic equation

$$1 - x - y - \Sigma^2 + \Omega_R = 0, \tag{29}$$

in which the new independent variable τ is defined as $d\tau = H dt$.

Furthermore, in the new variables the deceleration parameter $q = -1 - \frac{\ddot{H}}{H^2}$ is expressed as follows

$$q(\Sigma, x, y, \eta) = 2 - \lambda y. \tag{30}$$

In order to understand the asymptotic dynamics for the field equations, we shall determine the stationary points for the dynamical system (25)–(29). Analyzing the stationary points’ stability properties is essential to understand the dynamics’ general evolution and reconstructing the cosmological history.

Because of the constraint Eq. (29), the dimensional of the dynamical system can be reduced by one, and without loss of generality, we select to replace $y = 1 - x - \Sigma^2 + \Omega_R$ in the field equations, and we end with the system (25), (26) and (28).

By definition, for the asymptomatic solution to describe a real solution, it follows $\Omega_R \geq 0$. However, the rest of the variables are not constrained, which means that we should investigate the existence of stationary points for the dynamical system (25), (26) and (28) at the finite and infinite regimes.

5.1 Analysis at the finite regime

We replace y from (29) in Eqs. (25), (26) and (28) and we end with the system

$$\frac{d\Sigma}{d\tau} = -\lambda y \Sigma + \Omega_R, \tag{31}$$

$$\frac{dx}{d\tau} = 3(\Sigma^2 - 1) + (2\lambda - 3)(1 - x - \Sigma^2 + \Omega_R) + x(3 - \lambda(1 - x - \Sigma^2 + \Omega_R)) + \Omega_R, \tag{32}$$

$$\frac{d\Omega_R}{d\tau} = 2(2 - \lambda(1 - x - \Sigma^2 + \Omega_R) + \Sigma)\Omega_R, \tag{33}$$

where now the deceleration parameter becomes

$$q(\Sigma, x, \eta) = 2 - \lambda(1 - x - \Sigma^2 + \Omega_R). \tag{34}$$

The stationary points $A = (\Sigma(A), x(A), \Omega_R(A))$ of the latter dynamical system are

$$\begin{aligned} A_1 &= (\Sigma, (1 - \Sigma^2), 0), \\ A_2 &= \left(0, \left(2 - \frac{6}{\lambda}\right), 0\right), \\ A_3 &= \left(\frac{4 - \lambda}{1 + 2\lambda}, \frac{6(\lambda - 1)}{\lambda(1 + 2\lambda)}, \frac{3(4 - \lambda)(\lambda + 2)}{(1 + 2\lambda)^2}\right). \end{aligned}$$

The family of points A_1 describe anisotropic Bianchi I exact solutions, where the deceleration parameter is $q(A_1) = 2$. The eigenvalues of the linearized system around the stationary points are

$$e_1(A_1) = 0, e_2(A_1) = 2(\Sigma + 2), e_3(A_1^\pm) = (6 + \lambda \Sigma^2 - 1).$$

Because of the zero eigenvalues, the center manifold theorem should be applied in order to make inferences about the stability properties of the point. However, as we did in the previous analysis [50], the stability properties may be numerically investigated. However, on the surface $\Omega_R = 0$, the dynamical system reduces to that of Bianchi I spacetime, where we found before [50] that this

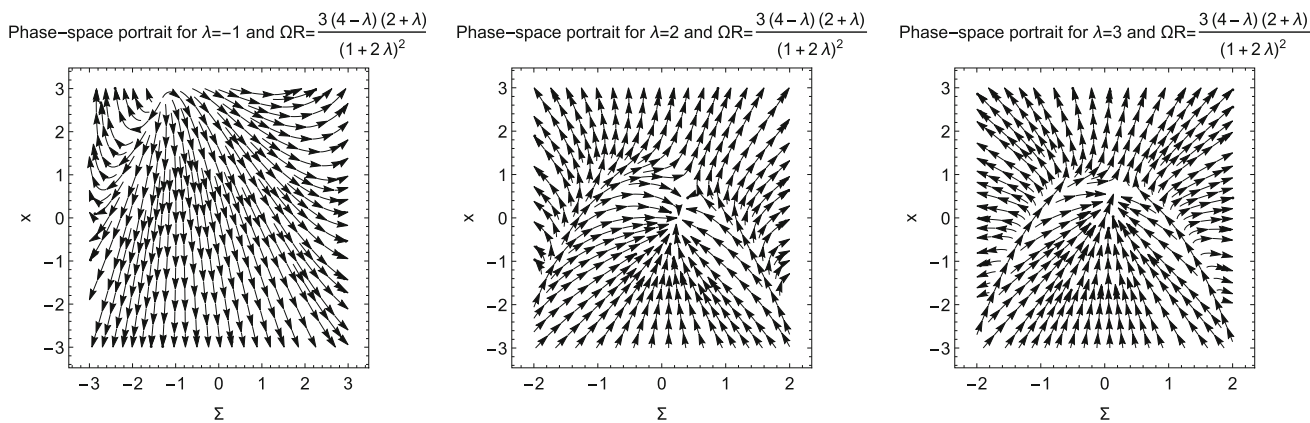


Fig. 1 Phase-space portrait for the dynamical system on three-dimensional surface (Σ, x) for $\Omega_R = \frac{3(4-\lambda)(\lambda+2)}{(1+2\lambda)^2}$ from where we observe that the Kantowski–Sachs solution described by A_3 is always a saddle point

Table 1 Stationary points at the finite regime

Point	Existence	Spacetime	$q < 0$	Stable?
A_1	Always	Bianchi I	No	No
A_2	$\lambda \neq 0$	FLRW (flat)	$\lambda < 4$	$\lambda < 4$
A_3	$-2 \leq \lambda \leq 4$	Kantowski–Sachs	$-\frac{1}{2} \leq \lambda \leq 4$	No

family of solutions is always unstable, and the corresponding points are saddle points. We can easily infer that points A_1 are saddle points.

Point A_2 describes spatially flat FLRW universe with deceleration parameter $q = \lambda - 4$. Thus, for $\lambda < 4$, the asymptotic solution describes an inflationary Universe, and for $\lambda = 3$, the de Sitter Universe is recovered. The eigenvalues of the linearized system around the stationary point are

$$e_1(A_2) = (\lambda - 6), e_2(A_2) = (\lambda - 6), e_3(A_2) = 2(\lambda - 4).$$

Hence, for $\lambda < 4$ point A_2 is an attractor.

Point A_3 is of special interest because it describes an asymptotic solution where the spatial curvature contributes to the cosmological fluid. This means that the background space at the point A_3 corresponds to a Kantowski–Sachs geometry. The point exists when $\lambda(1 + 2\lambda) \neq 0$, and it is physically accepted when $(4 - \lambda)(\lambda + 2) \geq 0$, that is, $-2 \leq \lambda \leq 4$. The deceleration parameter is derived $q(A_3) = \frac{\lambda-4}{2\lambda+1}$, from where we infer that $q(A_3) > 0$ in when $-\frac{1}{2} \leq \lambda \leq 4$. The limits $\lambda = 4$ and $\lambda = -2$ are special cases where the background space reduces to that of Bianchi I spacetime. The eigenvalues of the linearized system are derived

$$e_1(A_3) = -\frac{3(2+\lambda)}{1+2\lambda}, e_{2,3}(A_3) = \frac{-3\lambda - 6 \pm i\sqrt{3(2+\lambda)(\lambda(16\lambda - 59) - 38)}}{2(1+2\lambda)},$$

Therefore, point A_3 is always a saddle point. In Fig. 1, we present phase-space portraits for the dynamical system on the surface (Σ, x) and $\Omega_R = \frac{3(4-\lambda)(\lambda+2)}{(1+2\lambda)^2}$.

The results are summarized in Table 1.

5.2 Analysis at the infinity

In order to perform the analysis at the infinity, we define the Poincaré variables

$$x = \frac{\rho}{\sqrt{1-\rho^2}} \cos \Theta, \Sigma = \frac{\rho}{\sqrt{1-\rho^2}} \sin \Theta \cos \Psi, \tag{35}$$

$$\Omega_R = \frac{\rho^2}{1-\rho^2} \sin^2 \Theta \sin^2 \Psi, d\sigma = \sqrt{1-\rho^2} d\tau. \tag{36}$$

where now the field equations are written as follows

$$\begin{aligned} 16 \frac{d\rho}{d\sigma} &= -4\sqrt{1-\rho^2} \rho^3 (-8\lambda + \lambda \cos(2(\Theta + \Psi)) + (\lambda + 1) \cos(2(\Theta - \Psi))) \\ &+ (10 - 4\lambda) \cos(2\Theta) + \cos(2(\Theta + \Psi)) - 2\lambda \cos(2\Psi) - 2 \cos(2\Psi) + 14 \\ &+ 4\sqrt{1-\rho^2} \rho (-8\lambda + (10 - 4\lambda) \cos(2\Theta) + \cos(2(\Theta - \Psi)) + \cos(2(\Theta + \Psi)) - 2 \cos(2\Psi) + 14) \end{aligned}$$

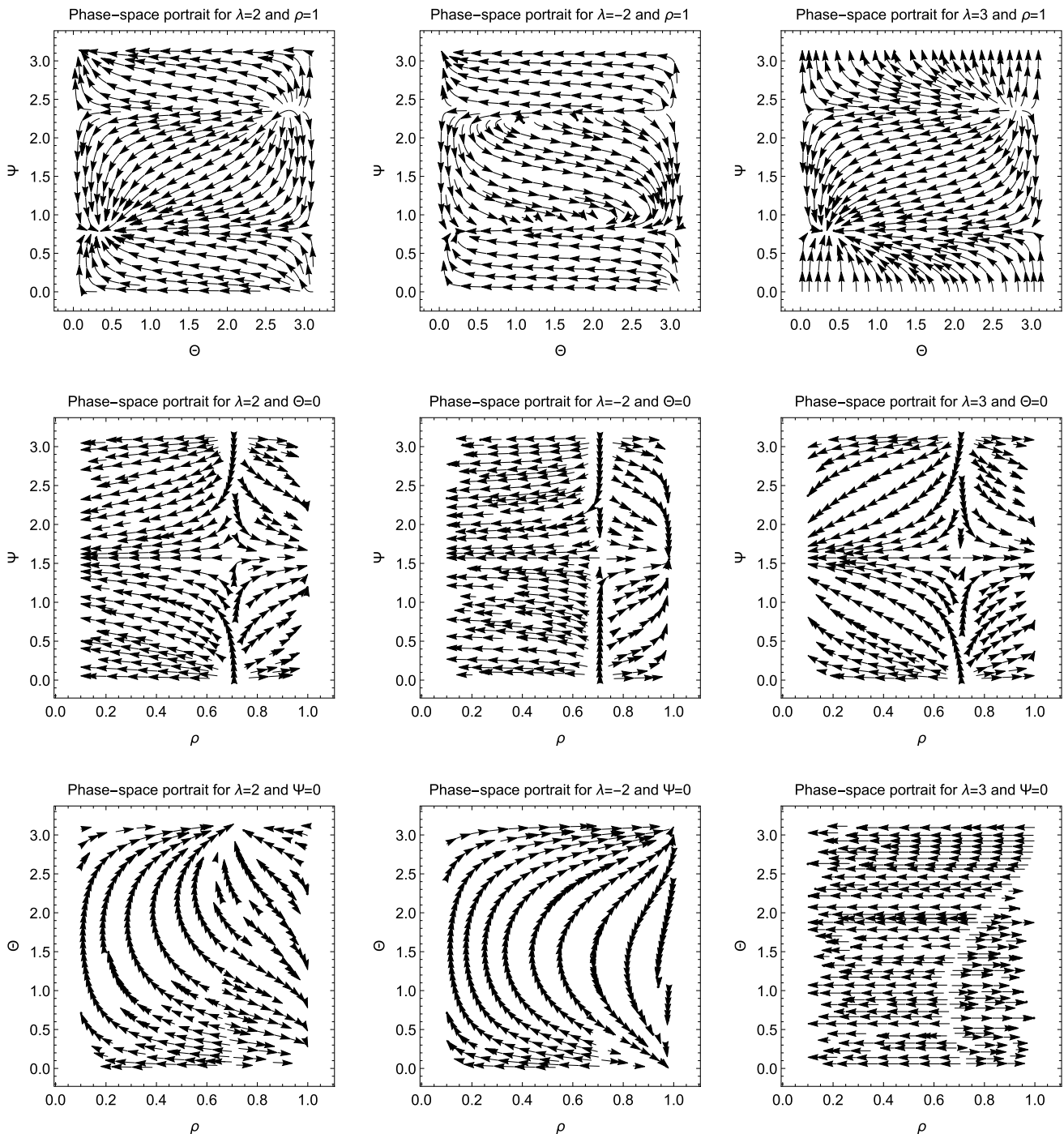


Fig. 2 Phase-space portrait for the dynamical system on the Poincaré variables. From the plot, it is clear that the stationary points at the infinity always describe unstable solutions

$$\begin{aligned}
 &+ 8\rho^4(\cos(\Theta)(2\lambda + 4(\lambda - 2)\sin^2(\Theta)\cos(2\Psi) - 13) - 4\sin^3(\Theta)\sin^2(\Psi)\cos(\Psi) + \cos(3\Theta)) + \\
 &\rho^2(-8\cos(\Theta)(6\lambda + 4(\lambda - 2)\sin^2(\Theta)\cos(2\Psi) - 25) - 8(\cos(3\Theta) - 4\sin^3(\Theta)\sin^2(\Psi)\cos(\Psi))) \\
 &+ 32(\lambda - 3)\cos(\Theta) , \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\rho}{\sin(\Theta)} \frac{d\Theta}{d\sigma} &= 6 - 2\lambda - \sqrt{1 - \rho^2}\rho\cos(\Theta)(-2\lambda + \cos(2\Psi) + 5) \\
 &+ \rho^2(-2(\lambda - 1)\sin^2(\Theta)\sin^4(\Psi) + 2(\lambda - 3)\sin^2(\Theta)\cos^4(\Psi) - \sin^2(\Theta)\sin^2(2\Psi)) \\
 &+ \rho^2(\sin(2\Theta)\sin^2(\Psi)\cos(\Psi) + 2(\lambda - 3)\sin^2(\Psi) + 2(\lambda - 3)\cos^2(\Psi)) , \tag{38}
 \end{aligned}$$

and

$$\frac{d\Psi}{d\sigma} = \sin(\Psi) \left(\rho \sin(\Theta) \cos(2\Psi) + 2\sqrt{1-\rho^2} \cos(\Psi) \right). \quad (39)$$

We focus on the brunch $\rho > 0$. Infinity is reached at the limit $\rho \rightarrow 1$, and the stationary points are of the form $B = (\Theta, \Psi)$. Variables Θ, Ψ take values in the region $\Theta \in [0, \pi]$ and $\Psi \in [0, \pi]$.

Thus, the stationary points for the latter dynamical system at the infinity are

$$B_1 = (0, \Psi), \quad B_2 = (\pi, \Psi), \quad (40)$$

while for $\lambda = 3$, the stationary points are

$$C_1 = (\Theta, 0) \text{ and } C_2 = (\Theta, \pi) \quad (41)$$

Hence, the stationary points at the infinity are a family of points with $\Omega_R = 0$, which means that the asymptotic solutions are that of Bianchi I. In particular, for the points B_1 and B_2 the asymptotic solutions describe spatially flat FLRW geometries with a stiff fluid, that is, $q(B_1) = 2$ and $q(B_2) = 2$. On the other hand, points C_1, C_2 correspond to families of anisotropic Bianchi I solutions with $q(C_1) = 2$ and $q(C_2) = 2$.

The eigenvalues of the linearized system around the stationary points B_1 and B_2 at the infinity are

$$\begin{aligned} e_1(B_1) &= 0, \quad e_2(B_1) = 0, \quad e_3(B_1) = -2\lambda, \\ e_1(B_2) &= 0, \quad e_2(B_2) = 0, \quad e_3(B_2) = 2\lambda. \end{aligned}$$

Consequently, the center manifold theorem may be applied to make inferences about the stability properties. However, such an analysis does not contribute to the physical discussion of this work, and we omit it. Therefore, we prefer to work numerically.

In Fig. 2, we present two-dimensional phase portraits for the dynamical system on the Poincare variables. From Fig. 2, we can easily infer that the stationary points at the infinity always describe unstable solutions.

6 Conclusions

We study the evolution of the cosmological field equations in modified teleparallel $f(T, B)$ theory of gravity for the anisotropic and homogeneous Kantowski–Sachs geometry. $f(T, B)$ is a fourth-order theory of gravity, and we have used a Lagrange multiplier in order to introduce a scalar field such that we can write the field equations as second-order equations by increasing the number of the dependent variables.

We focused on the case of $f(T, B) = T + F(B)$, with $F(B) = -\frac{1}{\lambda} B \ln B$. For this specific theory, the field equations admit a minisuperspace description. Moreover, when we use dimensionless variables, the equivalent dynamical system which describes the field equations has the minimum degrees of freedom. Indeed, the dimension of the space in which the physical variables lie is three.

There is a critical value for the parameter λ , which affects the trajectories' behavior and the physical space's evolution. For $\lambda < 4$, the trajectories of the field equations have a future attractor point A_2 which describes an accelerated FLRW geometry spatially. Hence, according to Fig. 2, the trajectories have various origins in the finite and infinity regimes. However, for $\lambda > 4$, there is no attractor, which means that the trajectories move through the lines connecting the saddle points. Another essential characteristic is that the dynamical system does not provide any isotropic closed FLRW if we start from anisotropic initial conditions. Last, the $\lambda = 3$ is an exciting value because the de Sitter Universe is recovered.

We conclude that in for order the homogeneity and the flatness problems to be solved in $f(T, B) = T + F(B)$ theory, for initial conditions which describe a Kantowski–Sachs geometry, the parameter λ should take values $\lambda < 4$.

That is the second part of a series of studies on the $f(T, B)$ theory in anisotropic spacetimes. In [50], we investigated the case where the background space is that of Bianchi I. We found that the homogeneity is achieved for $\lambda < 6$, while for $\lambda < 4$, the future attractor describes acceleration for the Universe.

Acknowledgements The research of Genly Leon is funded by Vicerrectoría de Investigación y Desarrollo Tecnológico at Universidad Católica del Norte.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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