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New non-binary quantum codes from skew constacyclic and additive skew constacyclic codes

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Abstract This paper discusses the structure of skew constacyclic codes and their Hermitian dual over finite commutative non-chain ring $\mathfrak{R}_\ell := \mathbb{F}_{q^2}[v_1, v_2, \dots, v_\ell]/\langle v_i^2 - 1, v_i v_j - v_j v_i\rangle_{1\le i,j\le \ell}$, where q is odd prime power. We also extend our study over mixed alphabet $\mathbb{F}_{q^2}\mathfrak{R}_\ell$ codes. First, we find necessary and sufficient conditions for skew constacyclic codes to contain their duals over $\mathfrak{R}_\ell$ and $\mathbb{F}_{q^2}\mathfrak{R}_\ell$. Then, a Gray map $\Psi : \mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}^{2^\ell}$, is defined, and with the help of this map, we also define another Gray map $\Phi : \mathbb{F}_{q^2}\mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}^{2^\ell+1}$ and prove that both maps are $\mathbb{F}_{q^2}$-linear Hermitian dual preserving. Finally, by applying Hermitian construction on dual-containing skew constacyclic codes, we construct many new quantum codes that improve the best-known parameters.

1 Introduction

Although skew polynomial rings were introduced by Ore [1] in 1933, coding with skew polynomial rings has been the center of attention after the significant work of Boucher et al. [2] in 2007. They generalized the notion of cyclic codes in a skew polynomial ring with a non-trivial automorphism and called them *skew cyclic codes*. Along with the algebraic richness, they [2,3] have produced some new codes whose minimum distances are comparatively larger than previously best-known codes. In 2008, skew constacyclic codes were introduced in [4] which are analogous generalizations of constacyclic codes. Later, several skew codes such as skew cyclic, skew constacyclic and skew quasi-cyclic have been studied by many authors in [5–11].

Recently, the construction of quantum error-correcting codes with good parameters has been one of the most active research areas because of their significant role in quantum communication and computation. The first quantum code was constructed by Shor [12] in 1995. These codes have experienced tremendous progress after the seminal work of Calderbank et al. [13] where they discovered a relation between classical and quantum codes. A q-ary quantum code of length n and size K is defined as K-dimensional subspace of the complex Hilbert space $(\mathbb{C}^q)^{\otimes n}$. Let $k = \log_q(K)$. Then, a q-ary quantum code of length n is denoted by $[[n, k, d]]_q$, where n and k represent the number of encoded physical qubits and the number of original information qubits, respectively, whereas d denotes the minimum distance. A quantum code with minimum distance d can correct both bit flip and phase shift type of errors up to $\lfloor \frac{d-1}{2} \rfloor$.

Quantum codes from classical codes have a rich literature, and among these, linear codes (cyclic codes) have a major contribution. Indeed over last few decades by using CSS and Hermitian constructions, researchers have been constructed a significant amount of quantum codes from dual-containing cyclic codes. In this context, along with finite fields, finite rings played an important role to produce good quantum codes. For instance, cyclic codes over finite chain rings such as $\mathbb{F}_4 + u\mathbb{F}_4, u^2 = 0$ in [14], $\mathbb{F}_2 + u\mathbb{F}_2, u^2 = 0$ in [15], and finite non-chain rings such as $\mathbb{F}_p + v\mathbb{F}_p, v^2 = v$ in [16], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ in [17], $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q, v_i^2 = v_i, v_i v_j = v_j v_i = 0$ in [18] are a few well-known studies. Being a generalized class, constacyclic codes also contributed several quantum codes in this direction [19]. It is proved that these codes over $\mathbb{F}_q + u\mathbb{F}_q, u^2 = 1$ in [20], $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q, v^3 = v$ in [21], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ in [22], $\mathbb{F}_q[u, v]/\langle u^2 - 1, v^2 - v, uv - vu\rangle$ in [23], $\mathbb{F}_q[u, v]/\langle u^2 - \gamma u, v^2 - \delta v, uv = vu = 0\rangle$ in [24], $R_{k,m} = \mathbb{F}_{p^m}[u_1, u_2, \dots, u_k]/\langle u_i^2 - 1, u_i u_j - u_j u_i\rangle$ in [25] are indeed a good choice to explore more new quantum codes. In fact, the list of alphabets over which cyclic, constacyclic codes get special attention is long; we refer few of them as [26–30]. Due to rich algebraic structure, along with linear codes, additive codes have been studied for more than five decades. After the introduction of additive codes [31] in 1973, these codes have been enlarged over

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mixed alphabets. Among many advantages, the flexibility of parameters is one of the prime reasons to investigate such codes. There are a large number of published articles to address mixed codes in a different setting, and we refer [32–36]. Moreover, quantum codes in mixed alphabets are investigated in [24,37–40]. It is well known that a skew polynomial ring possesses more polynomial factorization than a commutative ring. Thus, uses of such rings always help us to obtain new and better codes. The quantum codes from skew codes have appeared in very few articles [10,40–43], and hence, there is still enough scope to study further. It is noted that like commutative cases, first, we derive the dual-containing skew constacyclic codes and then use them to construct quantum codes.

With a strong and enough motivation, we extend our previous study of constacyclic codes [25] to skew constacyclic codes over a class of finite commutative non-chain rings $\mathfrak{R}_\ell := \mathbb{F}_{q^2}[v_1, v_2, \dots, v_\ell]/\langle v_i^2 - 1, v_i v_j - v_j v_i\rangle_{1\le i,j\le \ell}$. Note that earlier in [25], we used Euclidean inner product, whereas in this article we use Hermitian inner product to construct quantum codes. Further, we extend our study to mixed alphabets skew constacyclic codes and then obtain quantum codes from these codes. It is worth mentioning that our study produces several new quantum codes which are better in terms of parameters than the codes obtained over commutative structures.

2 Background

Let $\mathbb{F}_{q^2}$ be the finite field with characteristic p and size q^2 where $q = p^m$ and p is an odd prime. Throughout this paper, we use $\mathfrak{R}_\ell := \mathbb{F}_{q^2}[v_1, v_2, \dots, v_\ell]/\langle v_i^2 - 1, v_i v_j - v_j v_i\rangle_{1\le i,j\le \ell}$. If $\mathfrak{r} \in \mathfrak{R}_\ell$, then we can write as $\mathfrak{r} = r_0 + \sum_{1\le i_1\le \ell} r_{i_1} v_{i_1} + \sum_{1\le i_1<i_2\le \ell} r_{i_1,i_2} v_{i_1} v_{i_2} + \cdots + \sum_{1\le i_1<i_2<\cdots<i_\ell\le \ell} r_{i_1,i_2,\dots,i_\ell} v_{i_1} v_{i_2} \dots v_{i_\ell}$, where $r_0, r_{i_1,i_2,\dots,i_j} \in \mathbb{F}_{q^2}$, for all $1 \le i_j \le \ell$. From [25], it is known that $\mathfrak{R}_\ell$ is a finite commutative non-chain ring with characteristic p.

Now, we define a map $\mu : \mathfrak{R}_\ell \longrightarrow \mathfrak{R}_\ell$ by

$$\mu\left(r_0 + \sum_{1\le i_1\le \ell} r_{i_1} v_{i_1} + \sum_{1\le i_1<i_2\le \ell} r_{i_1,i_2} v_{i_1} v_{i_2} + \cdots + \sum_{1\le i_1<i_2<\cdots<i_\ell\le \ell} r_{i_1,i_2,\dots,i_\ell} v_{i_1} v_{i_2} \dots v_{i_\ell}\right)$$
$$= r_0^q + \sum_{1\le i_1\le \ell} r_{i_1}^q v_{i_1} + \sum_{1\le i_1<i_2\le \ell} r_{i_1,i_2}^q v_{i_1} v_{i_2} + \cdots + \sum_{1\le i_1<i_2<\cdots<i_\ell\le \ell} r_{i_1,i_2,\dots,i_\ell}^q v_{i_1} v_{i_2} \dots v_{i_\ell}.$$

It is easy to check that μ is an automorphism on $\mathfrak{R}_\ell$ of order 2 and $\mu\mid_{\mathbb{F}_{q^2}} = \sigma$ is an automorphism on $\mathbb{F}_{q^2}$ given by $a \longmapsto a^q$ for all $a \in \mathbb{F}_{q^2}$. Moreover, the fixed subring under μ is $\mathbb{F}_q[v_1, v_2, \dots, v_\ell]/\langle v_i^2 - 1, v_i v_j - v_j v_i\rangle_{1\le i,j\le \ell}$. We denote it by $\mathfrak{R}_\ell^\mu$. Let us consider the set

$$\mathcal{R} := \mathfrak{R}_\ell[x; \mu] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_j \in \mathfrak{R}_\ell\ \forall\ j, n \in \mathbb{N}\}.$$

Now, we define *addition* on $\mathcal{R}$ as the usual addition of polynomials and *multiplication* as the multiplication of polynomials under the condition $(\alpha x^i)(\beta x^j) = \alpha\mu^i(\beta)x^{i+j}$. It is easy to verify that the set $\mathcal{R}$ forms a ring under above defined binary operations. Clearly, $(\alpha x^i)(\beta x^j) \ne (\beta x^j)(\alpha x^i)$ in general unless μ is the identity automorphism. Thus, $\mathfrak{R}_\ell$ is a non-commutative ring and known as *skew polynomial ring*. In particular, if μ is the identity automorphism, then $\mathfrak{R}_\ell[x; \mu] \equiv \mathfrak{R}_\ell[x]$, where $\mathfrak{R}_\ell[x]$ is a commutative polynomial ring with coefficient from $\mathfrak{R}_\ell$. Moreover, an element $f(x) \in \mathcal{R}$ is in the center of $\mathcal{R}$ if and only if $f(x)g(x) = g(x)f(x)$ for all $g(x) \in \mathcal{R}$. We denote the center by $Z(\mathcal{R}) = \mathfrak{R}_\ell^\mu[x^2]$. By following the same line of proof of the result [[9], Proposition 2.2], where they considered ring $\mathcal{R}$ as a chain ring, we prove the result for non-chain ring.

Theorem 1 *Let $\mathcal{R} = \mathfrak{R}_\ell[x; \mu]$ be a skew polynomial ring, $\lambda \in \mathfrak{R}_\ell$ be unit in $\mathfrak{R}_\ell$ and n, a positive integer. Then, the following are equivalent:*

1. $x^n - \lambda \in Z(\mathcal{R})$.
2. $\langle x^n - \lambda\rangle$ *is a two-sided ideal.*
3. $\mathcal{R}$ *is a principal one-sided ideal ring.*
4. n *is even, and* λ *is fixed by* μ.

A linear code $\mathfrak{C}$ of length n over $\mathfrak{R}_\ell$ is an $\mathfrak{R}_\ell$-submodule of $\mathfrak{R}_\ell^n$, and the Hermitian dual $\mathfrak{C}^{\perp_H}$ of $\mathfrak{C}$ is defined as

$$\mathfrak{C}^{\perp_H} = \{\mathfrak{r} \in \mathfrak{R}_\ell^n \mid <\mathfrak{r}, \mathfrak{r}'>_H = 0 \text{ for all } \mathfrak{r}' \in \mathfrak{C}\}.$$

Here, $<\mathfrak{r}, \mathfrak{r}'>_H = \sum_{i=0}^{n-1} \mathfrak{r}_i \mu(\mathfrak{r}'_i)$ is the Hermitian inner product of vectors $\mathfrak{r} = (\mathfrak{r}_0, \mathfrak{r}_1, \cdots, \mathfrak{r}_{n-1})$ and $\mathfrak{r}' = (\mathfrak{r}'_0, \mathfrak{r}'_1, \dots, \mathfrak{r}'_{n-1})$ in $\mathfrak{R}_\ell^n$. Let $J = \{i_1, i_2, \dots, i_k\}$ be a subset of $\Lambda = \{1, 2, \dots, \ell\}$ where $i_1 < i_2 < \cdots < i_k$ and $\nu \in \mathbb{F}_{q^2}$ such that $2^\ell \nu \equiv 1 \pmod{p}$. Suppose

$$v_J = \prod_{i\in J} v_i, \quad \text{and for } J = \phi, \quad v_\phi = 1;$$

$$\text{and} \quad \gamma_J = \nu \prod_{i_j \in J}(1 - \upsilon_{i_j}) \prod_{i_j \notin J}(1 + \upsilon_{i_j}),$$

and if $J = \phi$, then $\gamma_\phi = \nu \prod_{i_j=1}^{\ell}(1 + \upsilon_{i_j})$. Again, from [25], we have

$$\gamma_J \gamma_{J'} = \begin{cases} \gamma_J, & \text{if } J = J' \\ 0, & \text{if } J \neq J' \end{cases}$$

and $\sum_{J \subseteq \Lambda} \gamma_J = 1$ in $\mathfrak{R}_\ell$. Thus, the collection $\{\gamma_J\}_{J \subseteq \Lambda}$ is a set of primitive orthogonal idempotent elements in $\mathfrak{R}_\ell$. Now, by using decomposition theorem [[44], Ch. VI], we decompose $\mathfrak{R}_\ell$ as

$$\mathfrak{R}_\ell = \bigoplus_{J \subseteq \Lambda} \gamma_J \mathfrak{R}_\ell \cong \bigoplus_{J \subseteq \Lambda} \gamma_J \mathbb{F}_{q^2}.$$

Then, every element $\mathfrak{r} = \sum_{J \subseteq \Lambda} \alpha_J \upsilon_J \in \mathfrak{R}_\ell$ can be uniquely expressed as

$$\begin{aligned} \mathfrak{r} &= \alpha_0 \gamma_0 + \sum_{1 \le i_1 \le \ell} \alpha_{i_1} \gamma_{i_1} + \sum_{1 \le i_1 < i_2 \le \ell} \alpha_{i_1, i_2} \gamma_{i_1, i_2} + \dots \\ &\quad + \sum_{1 \le i_1 < i_2 < \dots < i_\ell \le \ell} \alpha_{i_1, i_2, \dots, i_\ell} \gamma_{i_1, i_2, \dots, i_\ell} \\ &= \sum_{J \subseteq \Lambda} \alpha_J \gamma_J, \quad \text{where } \alpha_J \in \mathbb{F}_{q^2} \text{ for all } J \subseteq \Lambda. \end{aligned}$$

Now, we define a map

$$\Psi : \mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}^{2^\ell}$$

by

$$\begin{aligned} \mathfrak{r} &= \alpha_0 \gamma_0 + \sum_{1 \le i_1 \le \ell} \alpha_{i_1} \gamma_{i_1} + \sum_{1 \le i_1 < i_2 \le \ell} \alpha_{i_1, i_2} \gamma_{i_1, i_2} + \dots + \sum_{1 \le i_1 < i_2 < \dots < i_\ell \le \ell} \alpha_{i_1, i_2, \dots, i_\ell} \gamma_{i_1, i_2, \dots, i_\ell} \\ &\longmapsto (\alpha_0, \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_\ell}, \alpha_{i_1, i_2}, \alpha_{i_1, i_3}, \dots, \alpha_{i_{\ell-1}, i_\ell}, \dots, \alpha_{i_1, i_2 \dots i_\ell}) M \\ &= (\alpha_1, \alpha_2, \dots, \alpha_{2^\ell}) M \\ &= \boldsymbol{r} M, \end{aligned}$$

where $M \in GL_{2^\ell}(\mathbb{F}_{q^2})$ such that $MM^T = \kappa I$. Here, $\kappa \in \mathbb{F}_{q^2}^*$, M^T is the transpose of M and I is the identity matrix in $GL_{2^\ell}(\mathbb{F}_{q^2})$. We use $\boldsymbol{r} = (\alpha_1, \alpha_2, \dots, \alpha_{2^\ell})$ to enumerate the vector $(\alpha_0, \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_\ell}, \alpha_{i_1, i_2}, \alpha_{i_1, i_3}, \dots, \alpha_{i_{\ell-1}, i_\ell}, \dots, \alpha_{i_1, i_2 \dots i_\ell})$. The map Ψ can be extended from $\mathfrak{R}_\ell^n$ to $\mathbb{F}_{q^2}^{2^\ell n}$ componentwise. The Hamming weight of a codeword $\mathfrak{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathfrak{C}$ is denoted by $wt_H(\mathfrak{c})$ and defined as the number of nonzero components in $\mathfrak{c}$. The Hamming distance for the code $\mathfrak{C}$ is $d_H(\mathfrak{C}) = \min\{d_H(\mathfrak{c}, \mathfrak{c}') \mid \mathfrak{c} \neq \mathfrak{c}'$, for all $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{C}\}$, where $d_H(\mathfrak{c}, \mathfrak{c}')$ is the Hamming distance between $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{C}$ and $d_H(\mathfrak{c}, \mathfrak{c}') = wt_H(\mathfrak{c} - \mathfrak{c}')$. Also, the Gray weight of an element $\mathfrak{r} \in \mathfrak{R}_\ell$ is defined as $wt_G(\mathfrak{r}) = wt_H(\Psi(\mathfrak{r}))$ and Gray weight for $\bar{r} = (r_0, r_1, \dots, r_{n-1}) \in \mathfrak{R}_\ell^n$ is $wt_G(\bar{r}) = \sum_{i=0}^{n-1} wt_G(r_i)$. Further, the Gray distance between the codewords $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{C}$ is defined as $d_G(\mathfrak{c}, \mathfrak{c}') = wt_G(\mathfrak{c} - \mathfrak{c}')$.

Let $\mathfrak{C}$ be a linear code of length n over $\mathfrak{R}$, and for each $J \subseteq \Lambda$, $\mathfrak{C}_J = \{\alpha_J \in \mathbb{F}_{q^2}^n \mid$ there exists $\beta_{J'} \in \mathbb{F}_{q^2}^n$ for some $J' \subseteq \Lambda$ distinct from J such that $\alpha_J \gamma_J + \sum_{J' \subseteq \Lambda} \beta_{J'} \gamma_{J'} \in \mathfrak{C}\}$. Then for every $J \subseteq \Lambda$, $\mathfrak{C}_J$ is a linear code of length n over $\mathbb{F}_{q^2}$. We observe that if $\mathfrak{C}$ is a linear code of length n over the ring $\mathfrak{R}_\ell$, then we can uniquely write $\mathfrak{C} = \bigoplus_{J \subseteq \Lambda} \mathfrak{C}_J$ and $|\mathfrak{C}| = \prod_{J \subseteq \Lambda} |\mathfrak{C}_J|$, where $\mathfrak{C}_J$ is a linear code of length n over $\mathbb{F}_{q^2}$ for all $J \subseteq \Lambda$. Further, if M_J is the generator matrix of $\mathfrak{C}_J$ over $\mathbb{F}_{q^2}$ for $J \subseteq \Lambda$, then generator matrix G of $\mathfrak{C}$ over $\mathfrak{R}_\ell$ can be given as $G = \left(\gamma_J M_J\right)_{J \subseteq \Lambda}$, and thus, $\Psi(C)$ has generator matrix $G' = \left(\Psi(\gamma_J M_J)\right)_{J \subseteq \Lambda}$. In particular, the following results hold.

Theorem 2 *Let $\mathfrak{C}$ be an $[n, K, d_L]$ linear code over $\mathfrak{R}_\ell$. Then, $\Psi(\mathfrak{C})$ is a $[2^\ell n, K, d_H]$ linear code over $\mathbb{F}_{q^2}$ where $d_L = d_H$.*

Proof As Ψ is linear and distance preserving bijection from $\mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}^{2^\ell}$, it follows easily. □

Theorem 3 *Let $\mathfrak{C}$ be a Hermitian self-orthogonal linear code of length n over $\mathfrak{R}_\ell$ and $M \in GL_{2^\ell}(\mathbb{F}_{q^2})$ such that $MM^T = \kappa I_{2^\ell}$. Then, $\Psi(\mathfrak{C})$ is a Hermitian self-orthogonal linear code of length $2^\ell n$ over $\mathbb{F}_{q^2}$. Moreover, $\mathfrak{C}$ is a Hermitian self-dual code if and only if $\Psi(\mathfrak{C})$ is a Hermitian self-dual code.*

Proof Let $a = (a_0, a_1, \ldots, a_{2^\ell-1})$ and $b = (b_0, b_1, \ldots, b_{2^\ell-1})$ be any two arbitrary elements of $\Psi(\mathfrak{C})$. Then, there exist $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ in $\mathfrak{C}$ such that $a = \Psi(x)$ and $b = \Psi(y)$. Now, $MM^T = \kappa I_{2^\ell}$ and $\mathfrak{C}$ is Hermitian self-orthogonal, and we have

$$< a, b >_H = < \Psi(x), \Psi(y) >_H = \sum_{i=0}^{n-1} x_i M (M^T y_i)^q = 0.$$

Therefore, $\Psi(\mathfrak{C})$ is a Hermitian self-orthogonal linear code of length $2^\ell n$ over $\mathbb{F}_{q^2}$. Further, if $\mathfrak{C}$ is Hermitian self-dual, then $\mathfrak{C}^{\perp_H} = \mathfrak{C}$. Again, since Ψ is a linear bijection, $\Psi(\mathfrak{C})^{\perp_H} = \Psi(\mathfrak{C})$. □

Theorem 4 *If* $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \mathfrak{C}_J$ *is a linear code of length n over* $\mathfrak{R}_\ell$*, then following holds*

1. $\mathfrak{C}^{\perp_H} = \bigoplus_{J\subseteq\Lambda} \mathfrak{C}_J^{\perp_H}$.
2. $\mathfrak{C}$ *is Hermitian self-dual if and only if for all* $J \subseteq \Lambda$, $\mathfrak{C}_J$ *is Hermitian self-dual.*

Proof It follows by using Hermitian inner product along with the same line of arguments as given in [[8], Theorem 3.5]. □

In the next result, we classify the units of $\mathfrak{R}_\ell$. Here, $\mathfrak{R}_\ell^*$ represents the set of all units in $\mathfrak{R}_\ell$.

Lemma 1 [[25], Lemma 3.2] *Let* $\lambda = \sum_{J\subseteq\Lambda} \lambda_J \upsilon_J = \sum_{J\subseteq\Lambda} \delta_J \gamma_J \in \mathfrak{R}_\ell$. *Then,* λ *is a unit in* $\mathfrak{R}_\ell$ *if and only if* δ_J *is a unit in* $\mathbb{F}_{q^2}$, *for all* $J \subseteq \Lambda$.

3 Skew constacyclic codes over $\mathfrak{R}_\ell$

In this section, the structure of skew constacyclic codes over $\mathfrak{R}_\ell$ and their Hermitian duals are discussed. We begin with the following definition.

Definition 1 Let $\lambda \in \mathfrak{R}_\ell^*$ and μ be the automorphism on $\mathfrak{R}_\ell$. A linear code $\mathfrak{C}$ of length n over $\mathfrak{R}_\ell$ is said to be a skew λ-constacyclic code with respect to μ if $\mathfrak{C}$ is closed under the skew λ-constacyclic shift $\tau_{(\mu,\lambda)} : \mathfrak{R}_\ell^n \longrightarrow \mathfrak{R}_\ell^n$ defined by $\tau_{(\mu,\lambda)}(\mathfrak{c}) = (\lambda\mu(c_{n-1}), \mu(c_0), \ldots, \mu(c_{n-2})) \in \mathfrak{C}$ for $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathfrak{C}$. In particular, if $\lambda = 1$ and $\lambda = -1$, then $\mathfrak{C}$ is called skew cyclic and skew negacyclic code, respectively. Moreover, if μ is the identity automorphism, then $\mathfrak{C}$ is a λ-constacyclic code over $\mathfrak{R}_\ell$.

Let $\mathfrak{C}$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. Then similar to polynomial representation of constacyclic codes, we can also identify each codeword $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathfrak{C}$ by a polynomial $\mathfrak{c}(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in \mathfrak{R}_\ell[x; \mu]/\langle x^n - \lambda\rangle$ under the correspondence $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \mapsto \mathfrak{c}(x) = (c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) \bmod \langle x^n - \lambda\rangle$. Note that the problem to find all λ-constacyclic codes of length n over the ring $\mathfrak{R}_\ell$ is equivalent to find all the ideals of the quotient ring $\frac{\mathfrak{R}_\ell[x]}{\langle x^n-\lambda\rangle}$. Since skew polynomial ring $\mathfrak{R}_\ell[x; \mu]$ is non-commutative, therefore, the quotient $\frac{\mathfrak{R}_\ell[x;\mu]}{\langle x^n-\lambda\rangle}$ need not be a ring, but module structure is always possible for $\mathfrak{R}_\ell[x; \mu]$, where the scalar multiplication is defined by

$$a(x)(b(x) - \langle x^n - \lambda\rangle) = a(x)b(x) + \langle x^n - \lambda\rangle.$$

Thus, to construct all skew λ-constacyclic codes of length n over $\mathfrak{R}_\ell$, it is enough to find all $\mathfrak{R}_\ell[x; \mu]$-submodule of $\frac{\mathfrak{R}_\ell[x;\mu]}{\langle x^n-\lambda\rangle}$. From Theorem 1, we observed that if n is even and $\lambda \in \mathfrak{R}_\ell^*$ such that $\mu(\lambda) = \lambda$, then $\langle x^n - \lambda\rangle$ is a two-sided ideal in $\mathfrak{R}_\ell[x; \mu]$; consequently, $\frac{\mathfrak{R}_\ell[x;\mu]}{\langle x^n-\lambda\rangle}$ forms a ring structure. However, in both cases skew λ-constacyclic code $\mathfrak{C}$ of length n over $\mathfrak{R}_\ell$ is generated by a monic right divisor of $x^n - \lambda$.

Remark 1 Throughout this paper, we assume $\lambda \in \mathfrak{R}_\ell^*$ is a unit fixed by the automorphism μ and the length of a skew λ-constacyclic code is divisible by the order of the automorphism μ.

Theorem 5 *A linear code* $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J\mathfrak{C}_J$ *of length n over* $\mathfrak{R}_\ell$ *is a skew* λ*-constacyclic code if and only if each* $\mathfrak{C}_J$ *is a skew* δ_J*-constacyclic code of length n over* $\mathbb{F}_{q^2}$*, for all* $J \subseteq \Lambda$.

Proof Let $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J\mathfrak{C}_J$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. For $J \subseteq \Lambda$, let $a^J = (a_0^J, a_1^J, \ldots, a_{n-1}^J) \in \mathfrak{C}_J$. Suppose $r_i = \sum_{J\subseteq\Lambda} \gamma_J a_i^J$ for $0 \le i \le n-1$, then $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathfrak{C}$. Therefore, $\tau_{(\mu,\lambda)}(r) = \sum_{J\subseteq\Lambda} \gamma_J \tau_{(\sigma,\delta_J)}(a^J) \in \mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J\mathfrak{C}_J$. Hence, $\tau_{(\sigma,\delta_J)}(a^J) \in \mathfrak{C}_J$, for $J \subseteq \Lambda$. Thus, $\mathfrak{C}_J$ is a skew δ_J-constacyclic code of length n over $\mathbb{F}_{q^2}$, for $J \subseteq \Lambda$.

Conversely, for $J \subseteq \Lambda$, let $\mathfrak{C}_J$ be a skew δ_J-constacyclic code of length n over $\mathbb{F}_{q^2}$ with respect to σ. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathfrak{C}$ where $r_i = \sum_{J\subseteq\Lambda} \gamma_J a_i^J$, for some $a_i^J \in \mathbb{F}_{q^2}$, $0 \le i \le n-1$. Now, $a^J = (a_0^J, a_1^J, \ldots, a_{n-1}^J) \in \mathfrak{C}_J$ for $J \subseteq \Lambda$. Therefore, $\tau_{(\sigma,\delta_J)}(a^J) \in \mathfrak{C}_J$. Hence, $\tau_{(\mu,\lambda)}(r) = \sum_{J\subseteq\Lambda} \gamma_J \tau_{(\sigma,\delta_J)}(a^J) \in \bigoplus_{J\subseteq\Lambda} \gamma_J\mathfrak{C}_J = \mathfrak{C}$. Thus, $\mathfrak{C}$ is a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. □

For a unit element $\delta \in \mathbb{F}_{q^2}$, Boucher et al. [4] obtained the generator polynomials of skew δ-constacyclic codes of length n over $\mathbb{F}_{q^2}$. We use this result to get the generator of skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$.

Theorem 6 [4] *Let $\delta \in \mathbb{F}^*_{q^2}$ and $\sigma \in Aut(\mathbb{F}_{q^2})$. Let $\mathfrak{C}$ be a linear code of length n over $\mathbb{F}_{q^2}$ such that $o(\sigma)$ (the order of σ) divides n and $\sigma(\delta) = \delta$. Then, $\mathfrak{C}$ is skew δ-constacyclic over $\mathbb{F}_{q^2}$ if and only if there exists a monic polynomial $g(x) \in \mathbb{F}_{q^2}[x; \sigma]$ such that $\mathfrak{C} = \langle g(x)\rangle$ and $x^n - \delta$ is right divisible by $g(x)$ in $\mathbb{F}_{q^2}[x; \sigma]$.*

Theorem 7 *If $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ is a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$, then there exists a monic polynomial $f(x)$ in $\mathfrak{R}[x; \mu]$ such that $\mathfrak{C} = \langle f(x)\rangle$ and $x^n - \lambda$ is right divisible by $f(x)$ in $\mathfrak{R}[x; \mu]$. Moreover, if $f_J(x)$ is the generator polynomial of skew δ_J-constacyclic codes over $\mathbb{F}_{q^2}$ for $J \subseteq \Lambda$, then $f(x) = \sum_{J\subseteq\Lambda} \gamma_J f_J(x)$.*

Proof Let $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. Then by Theorem 5, for each $J \subseteq \Lambda$, $\mathfrak{C}_J$ is a skew δ_J-constacyclic code of length n over $\mathbb{F}_{q^2}$. Also, by Theorem 6, there exists a monic polynomial $f_J(x) \in \mathbb{F}_{q^2}[x; \sigma]$ which is a right divisor of $x^n - \delta_J$ and $\mathfrak{C}_J = \langle f_J(x)\rangle$. Therefore, $\gamma_J f_J(x)$ is the generator of $\mathfrak{C}$ for all $J \subseteq \Lambda$. Let $f(x) = \sum_{J\subseteq\Lambda} \gamma_J f_J(x)$. Then, $\langle f(x)\rangle \subseteq \mathfrak{C}$. On the other hand, $\gamma_J f_J(x) = \gamma_J f(x) \in \langle f(x)\rangle$ for all $J \subseteq \Lambda$. Consequently, $\mathfrak{C} \subseteq \langle f(x)\rangle$ and hence, $\mathfrak{C} = \langle f(x)\rangle$.

Since for every $J \subseteq \Lambda$, $f_J(x)$ is a right divisor of $x^n - \delta_J$ in $\mathbb{F}_{q^2}[x; \sigma]$, so there exists skew polynomial $h_J(x)$ such that $x^n - \delta_J = h_J(x) f_J(x)$. Now, $[\sum_{J\subseteq\Lambda} \gamma_J h_J(x)] f(x) = \sum_{J\subseteq\Lambda} \gamma_J h_J(x) f_J(x) = \sum_{J\subseteq\Lambda} \gamma_J (x^n - \delta_J) = x^n - \lambda$. Hence, $f(x)$ is a right divisor of $x^n - \lambda$ in $\mathfrak{R}_\ell[x; \mu]$. □

For a polynomial $f(x) = \sum_{i=0}^{k} a_i x^i \in \mathbb{F}_{q^2}[x; \sigma]$ with $a_0 \neq 0$, the left monic skew reciprocal polynomial of $f(x)$ is defined as $f(x)^* = \frac{1}{\sigma^k(a_0)}(\sum_{i=0}^{k} \sigma^i(a_{k-i})x^i)$, and the conjugate polynomial of $f(x)$ is $\overline{f(x)} = \sum_{i=0}^{k} \sigma(a_i)x^i$. Now, it is easy to see that $\overline{f(x)^*} = \overline{f(x)}^* = f^\dagger(x)$. The polynomial $f^\dagger(x)$ is known as the skew Hermitian reciprocal polynomial of $f(x)$. From [[45], Theorem 1], if δ is a unit fixed under the automorphism σ such that $\delta^2 = 1$ and $\mathfrak{C} = \langle f(x)\rangle$ is a skew δ-constacyclic code of length n over $\mathbb{F}_{q^2}$, then Hermitian dual $\mathfrak{C}^{\perp_H}$ of $\mathfrak{C}$ is a skew δ^{-1}-constacyclic code generated by $h^\dagger(x)$ where $f(x)h(x) = h(x)f(x) = x^n - \delta$. Hence, we have the following result.

Corollary 1 *Let $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$ and $\mathfrak{C}_J = \langle f_J(x)\rangle$ such that $x^n - \delta_J = h_J(x) f_J(x)$ for $J \subseteq \Lambda$. Then, $\mathfrak{C}^{\perp_H} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J^{\perp_H}$ is a skew λ^{-1}-constacyclic code over $\mathfrak{R}_\ell$. Moreover, $\mathfrak{C}^{\perp_H} = \langle \sum_{J\subseteq\Lambda} \gamma_J h_J^\dagger(x)\rangle$, where $h_J^\dagger(x)$ is the skew Hermitian reciprocal polynomial of $h_J(x)$, for all $J \subseteq \Lambda$.*

4 Quantum codes from skew λ-constacyclic codes over $\mathfrak{R}_\ell$

In this section, we employ the skew λ-constacyclic codes over the ring $\mathfrak{R}_\ell$ to construct non-binary quantum codes over $\mathbb{F}_q$. Toward this, first we provide the necessary and sufficient conditions for skew λ-constacyclic codes over $\mathbb{F}_{q^2}$ to contain their Hermitian duals. Then, we establish the Hermitian dual-containing condition for skew λ-constacyclic codes over the ring $\mathfrak{R}_\ell$.

Theorem 8 [46] [Hermitian construction] *Let $\mathcal{C}$ be a linear code over $\mathbb{F}_{q^2}$ with parameters $[n, k, d_H]$ satisfying $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$. Then, there exists a quantum code over $\mathbb{F}_q$ with parameters $[[n, 2k - n, \geq d_H]]_q$.*

Lemma 2 [[40], Lemma 6] *Let $\delta \in \mathbb{F}^*_{q^2}$ such that $\delta^2 = 1$ and $\mathfrak{C} = \langle g(x)\rangle$ be a skew δ-constacyclic code of even length n over $\mathbb{F}_{q^2}$. Then, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ if and only if $h^\dagger(x)h(x)$ is right divisible by $x^n - \delta$ where $x^n - \delta = h(x)g(x)$ and $h^\dagger(x)$ is the skew Hermitian reciprocal polynomial of $h(x)$.*

Theorem 9 *Let $\lambda \in \mathfrak{R}_\ell^*$ and $\lambda^2 = 1$. Let $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. Then, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ if and only if for all subsets $J \subseteq \Lambda$, the polynomial $h_J^\dagger(x)h_J(x)$ is right divisible by $x^n - \delta_J$, where $x^n - \delta_J = h_J(x) f_J(x)$ and $h_J^\dagger(x)$ is the skew Hermitian reciprocal polynomial of $h_J(x)$.*

Proof First assume that $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ is a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$ and $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Therefore, by Corollary 1,

$$\bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J^{\perp_H} \subseteq \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J. \quad (1)$$

Again, since n is even and $\lambda^2 = 1$, thus $\delta_J^2 = 1$. Now, for each $J \subseteq \Lambda$ taking modulo γ_J in equation (1), we have $\mathfrak{C}_J^{\perp_H} \subseteq \mathfrak{C}_J$. Therefore, by Lemma 2, the polynomial $h_J^\dagger(x)h_J(x)$ is right divisible by $x^n - \delta_J$ for all $J \subseteq \Lambda$ where $x^n - \delta_J = h_J(x)g_J(x)$ and $h_J^\dagger(x)$ is the Hermitian skew reciprocal polynomial of $h_J(x)$.

Conversely, suppose for each $J \subseteq \Lambda$, $h_J^\dagger(x)h_J(x)$ is right divisible by $x^n - \delta_J$, then again by Lemma 2, for all $J \subseteq \Lambda$, we have $\mathfrak{C}_J^{\perp_H} \subseteq \mathfrak{C}_J$. Therefore, $\bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J^{\perp_H} \subseteq \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$, or in other words, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. □

Corollary 2 *Let $\lambda \in \mathfrak{R}_\ell^*$ and $\lambda^2 = 1$. Let $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$. Then, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ if and only if for all subsets $J \subseteq \Lambda$, $\mathfrak{C}_J^{\perp_H} \subseteq \mathfrak{C}_J$.*

Theorem 10 *Let $\lambda \in \mathfrak{R}_\ell^*$ and $\lambda^2 = 1$. If $\mathfrak{C} = \bigoplus_{J\subseteq\Lambda} \gamma_J \mathfrak{C}_J$ is a skew λ-constacyclic code of length n over $\mathfrak{R}_\ell$ with Gray image $\Psi(\mathfrak{C})$ such that $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$, then there exists a quantum code with parameters $[[2^\ell n, 2k - 2^\ell n, \geq d_H]]$ over $\mathbb{F}_q$.*

Proof By using Theorem 9 and the Hermitian construction given in Theorem 8, we get the desired result. □

5 $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic codes

Let $\mathbb{F}_{q^2}\mathfrak{R}_\ell = \{(a, b) : a \in \mathbb{F}_{q^2}, b \in \mathfrak{R}_\ell\}$ and $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \mathbb{F}_{q^2}^n, \boldsymbol{b} \in \mathfrak{R}_\ell^m\}$. We define a projection map $\tau : \mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}$ by $\tau(f(v_1, v_2, \ldots, v_\ell)) = f(0)$ for all $f(v_1, v_2, \ldots, u_\ell) \in \mathfrak{R}_\ell$. With the help of the map τ, we define a multiplication $* : \mathfrak{R}_\ell \times (\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m) \longrightarrow \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$ by $c * (\boldsymbol{a}, \boldsymbol{b}) = (\tau(c)\boldsymbol{a}, c\boldsymbol{b})$. Now, it is checked that $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$ forms an $\mathfrak{R}_\ell$-module. In this case, any non-empty subset $\mathfrak{C}$ of $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$ is said to be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m) if it is an $\mathfrak{R}_\ell$-submodule of $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$. Denote $R_{(n,m)} = \frac{\mathbb{F}_{q^2}[x]}{\langle x^n-1\rangle} \times \frac{\mathcal{R}}{\langle x^m-\lambda\rangle}$, where $\mathcal{R} = \mathfrak{R}_\ell[x; \mu]$ is the skew polynomial ring over $\mathfrak{R}_\ell$ with automorphism μ on $\mathfrak{R}_\ell$ and λ is a unit in $\mathfrak{R}_\ell$. We identify each codeword $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$ by a polynomial $(a(x), b(x)) \in R_{(n,m)}$ under the correspondence $(\boldsymbol{a}, \boldsymbol{b}) \longmapsto (a(x), b(x))$, where $a(x) = a_0 + a_1x + \cdots + a_{n-1}x^{m-1} \in \frac{\mathbb{F}_{q^2}[x]}{\langle x^n-1\rangle}$, $b(x) = b_0 + b_1x + \cdots + b_{m-1}x^{m-1} \in \frac{\mathcal{R}}{\langle x^m-\lambda\rangle}$ for $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_{q^2}^n$, $\boldsymbol{b} = (b_0, b_1, \ldots, b_{m-1}) \in \mathfrak{R}_\ell^m$. Again, $R_{(n,m)}$ is a left $\mathcal{R}$-module where the left multiplication is defined by $c(x) * (a(x), b(x)) = (\tau(c(x))a(x), c(x)b(x))$ for $a(x) \in \frac{\mathbb{F}_{q^2}[x]}{\langle x^n-1\rangle}$, $b(x), c(x) \in \frac{\mathcal{R}}{\langle x^m-\lambda\rangle}$.

Definition 2 Let $\mathfrak{C}$ be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m). Then, it is called an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code if for any

$$(\boldsymbol{a}, \boldsymbol{b}) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_2, \ldots, b_{m-1}) \in \mathcal{C}$$

implies

$$T(\boldsymbol{a}, \boldsymbol{b}) := (a_{n-1}, a_0, \ldots, a_{n-2}, \lambda\mu(b_{m-1}), \mu(b_0), \ldots, \mu(b_{m-2})) \in \mathcal{C}.$$

Theorem 11 *Let $\mathfrak{C}$ be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m). Then, it is an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code if and only if $\mathfrak{C}$ is a left $\mathfrak{R}_\ell[x; \mu]$-submodule of $R_{(n,m)}$.*

Proof Let $\mathfrak{C}$ be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code of length (n, m). If $(a(x), b(x)) \in \mathfrak{C}$, where $a(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, $b(x) = b_0 + b_1x + \cdots + b_{m-1}x^{m-1}$, then in $R_{(n,m)}$, we have

$$\begin{aligned} x * (a(x), b(x)) = (&a_{n-1} + a_0x + a_1x^2 + \cdots + a_{n-2}x^{n-1}, \\ &\lambda\mu(b_{m-1}) + \mu(b_0)x + \mu(b_1)x^2 + \cdots + \mu(b_{m-2})x^{m-1}) \in \mathfrak{C}. \end{aligned}$$

In this way, for any $j \geq 1$, $x^j(a(x), b(x)) \in \mathfrak{C}$. Therefore, for any $c(x) \in \mathfrak{R}_\ell[x; \mu]$ we have $c(x) * (a(x), b(x)) \in \mathfrak{C}$. Hence, $\mathfrak{C}$ is a left $\mathfrak{R}_\ell[x; \mu]$-submodule of $R_{(n,m)}$. Conversely, let $\mathfrak{C}$ be a left $\mathfrak{R}_\ell[x; \mu]$-submodule of $R_{(n,m)}$. Then for any $(a(x), b(x)) \in \mathfrak{C}$, we have $x * (a(x), b(x)) \in \mathfrak{C}$. Since $T(\boldsymbol{a}, \boldsymbol{b}) = x * (a(x), b(x))$, thus $\mathfrak{C}$ is an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$ additive skew λ-constacyclic code. □

For any two codewords $\boldsymbol{v} = (\boldsymbol{a}, \boldsymbol{b}) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_2, \ldots, b_{m-1})$, $\boldsymbol{v'} = (\boldsymbol{a'}, \boldsymbol{b'}) = (a'_0, a'_1, \ldots, a'_{n-1}, b'_0, b'_2, \ldots, b'_{m-1})$ in $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m$, the Hermitian inner product is defined by

$$\langle \boldsymbol{v}, \boldsymbol{v'}\rangle_H = \sum_{i=1}^{\ell} v_i \sum_{i=0}^{n-1} a_i\mu(a_i) + \sum_{i=0}^{m-1} b_i\mu(b_i).$$

Further, the Hermitian dual is defined as $\mathcal{C}^{\perp_H} = \{\boldsymbol{v} \in \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m : \langle \boldsymbol{v}, \boldsymbol{v'}\rangle_H = 0 \text{ for all } \boldsymbol{v'} \in \mathcal{C}\}$. It is easy to verify that for an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code, its dual $\mathcal{C}^{\perp_H}$ is also an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code. Now, we define a Gray map $\Phi : \mathbb{F}_{q^2}\mathfrak{R}_\ell \longrightarrow \mathbb{F}_{q^2}^{2^\ell+1}$ by

$$\Phi(a, r) = (a, \Psi(r)) = (a, (r_0, r_1, \ldots, r_{2^\ell})M),$$

where $a \in \mathbb{F}_{q^2}$, $r = \sum_{i=1}^{2^\ell} \gamma_i r_i \in \mathfrak{R}_\ell$. It is a linear bijection and can be extended over $\mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m \longrightarrow \mathbb{F}_{q^2}^{(n+2^\ell m)}$ componentwise. Based on above discussion, we have the following result.

Lemma 3 *The map $\Phi : \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m \longrightarrow \mathbb{F}_{q^2}^{(n+2^\ell m)}$ is an $\mathbb{F}_{q^2}$-linear distance preserving map. Further, if $\mathfrak{C}$ is an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code with parameters $(n+m, M, d)$, then $\Phi(\mathfrak{C})$ is an $(n + 2^\ell m, \log_{q^2} M, d)$ where M represents the size of $\mathfrak{C}$.*

Proof Same as the proof of [[40], Proposition 2]. □

Lemma 4 *Let $\mathfrak{C}$ be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m). Then, $\Phi(\mathfrak{C})^{\perp_H} = \Phi(\mathfrak{C}^{\perp_H})$. In particular, $\mathfrak{C}$ is Hermitian self-dual if and only if $\Phi(\mathfrak{C})$ is so.*

Proof Same as the proof of [[24], Lemma 6]. □

Let $\pi_n : \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m \longrightarrow \mathbb{F}_{q^2}^n$ defined by $\pi_n(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a}$ and $\pi_m : \mathbb{F}_{q^2}^n\mathfrak{R}_\ell^m \longrightarrow \mathfrak{R}_\ell^m$ defined by $\pi_m(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{b}$ are projection maps. Clearly, these maps are also $\mathfrak{R}_\ell$-module homomorphisms. Therefore, for an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code $\mathfrak{C}$, $\pi_n(\mathfrak{C}) = \mathfrak{C}_n$ and $\pi_m(\mathfrak{C}) = \mathfrak{C}_m$ are linear codes over $\mathbb{F}_{q^2}$ and $\mathfrak{R}_\ell$, respectively. In particular, if $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$, then $\mathfrak{C}$ is called separable. In that case, $\mathfrak{C}^{\perp_H} = \mathfrak{C}_n^{\perp_H} \times \mathfrak{C}_m^{\perp_H}$.

Theorem 12 *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a separable $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m). Then, $\mathfrak{C}$ is an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code if and only if $\mathfrak{C}_n$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ and $\mathfrak{C}_m$ is a skew λ-constacyclic code of length m over $\mathfrak{R}_\ell$.*

Proof Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code of length (n, m). Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathfrak{C}_n$, $\boldsymbol{b} = (b_0, b_1, \ldots, b_{m-1}) \in \mathfrak{C}_m$. Then, $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{C}$, and hence

$$T(\boldsymbol{a}, \boldsymbol{b}) = (a_{n-1}, a_0, a_1, \ldots, a_{n-2}, \lambda\mu(b_{m-1}), \mu(b_0), \ldots, \mu(b_{m-2})) \in \mathfrak{C}_n \times \mathfrak{C}_m = \mathfrak{C}.$$

Then, $(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in \mathfrak{C}_n$, $(\lambda\mu(b_{m-1}), \mu(b_0), \ldots, \mu(b_{m-2})) \in \mathfrak{C}_m$. Therefore, $\mathfrak{C}_n$ is a cyclic code of length n over $\mathbb{F}_{q^2}$ and $\mathfrak{C}_m$ is a skew λ-constacyclic code of length m over $\mathfrak{R}_\ell$.

Conversely, let $\mathfrak{C}_n$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ and $\mathfrak{C}_m$ be a skew λ-constacyclic code of length m over $\mathfrak{R}_\ell$. Let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{C}$, where $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathfrak{C}_n$, $\boldsymbol{b} = (b_0, b_1, \ldots, b_{m-1}) \in \mathfrak{C}_m$. Then,

$$(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in \mathfrak{C}_n, (\lambda\mu(b_{m-1}), \mu(b_0), \ldots, \mu(b_{m-2})) \in \mathfrak{C}_m,$$

and $T(\boldsymbol{a}, \boldsymbol{b}) \in \mathfrak{C}_n \times \mathfrak{C}_m = \mathfrak{C}$. Therefore, $\mathfrak{C}$ is an $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code. □

Theorem 13 *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a separable $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive code of length (n, m). Then, $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$ if and only if $\mathfrak{C}_n^{\perp_H} \subseteq \mathfrak{C}_n$ and $\mathfrak{C}_m^{\perp_H} \subseteq \mathfrak{C}_m$.*

Proof Since $\mathfrak{C}^{\perp_H} = \mathfrak{C}_n^{\perp_H} \times \mathfrak{C}_m^{\perp_H}$, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ if and only if $\mathfrak{C}_n^{\perp_H} \subseteq \mathfrak{C}_n$ and $\mathfrak{C}_m^{\perp_H} \subseteq \mathfrak{C}_m$. □

Now, we review some basic concepts from [47] that are useful for further discussion. For an integer s with $0 \leq s \leq n-1$, a q^2-cyclotomic coset modulo n containing s is denoted by C_s and defined as $C_s = \{sq^{2j} \pmod n) : 0 \leq j \leq j_s - 1\}$, where j_s is the least positive integer such that $sq^{2j_s} \equiv s \pmod n$ and $| C_s |= j_s$. The smallest integer in C_s is called the coset leader of C_s, and we use $\Gamma_{(n,q^2)}$ for the set of all coset leaders. Note that $C_s \cap C_{s'} = \phi$, for any two coset leaders s and s' in $\Gamma_{(n,q^2)}$. Further, let $n \in \mathbb{N}$ such that $\gcd(n, q) = 1$ and $\mathrm{ord}_n(q^2) = l$ be the multiplicative order of q^2 modulo n. Let η be the generator of $\mathbb{F}_{q^{2l}}^*$. Put $\xi = \eta^{(q^{2l}-1)/n}$. Then, ξ is a primitive n-th roots of unity in $\mathbb{F}_{q^{2l}}$. The minimal polynomial $m_s(x)$ of ξ^s over $\mathbb{F}_{q^2}$ is the smallest degree monic polynomial over $\mathbb{F}_{q^2}$ with ξ^s as a root. Based on above discussion, one can easily check that

$$m_s(x) = \prod_{i \in C_s} (x - \xi^i) \in \mathbb{F}_{q^2}[x],$$

and

$$x^n - 1 = \prod_{s \in \Gamma_{(n,q^2)}} m_s(x).$$

Let $\mathfrak{C}$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ with generator polynomial $g(x)$. Then, the set

$$\mathcal{T} = \{\jmath \mid g(\xi^\jmath) = 0, 0 \leq \jmath \leq n-1\}$$

is called the defining set of $\mathfrak{C}$. Note that the defining set of $\mathfrak{C}$ is the union of some q^2-cyclotomic cosets modulo n and $\dim(\mathfrak{C}) = n - |\mathcal{T}|$. Also, one can see that the defining set of $\mathfrak{C}^{\perp_H}$ is

$$\mathcal{T}^{-q} = \{-q\jmath \pmod n) \mid \jmath \in \mathcal{T}\}.$$

Lemma 5 [[48], Lemma 8] *Let $\mathfrak{C}$ be a cyclic code over $\mathbb{F}_{q^2}$ of length n such that $\gcd(n, q) = 1$ with defining set $\mathcal{T}$. Then, $\mathfrak{C}$ contains its Hermitian dual if and only if $\mathcal{T} \cap \mathcal{T}^{-q} = \phi$ where $\mathcal{T}^{-q}$ is the defining set of $\mathfrak{C}^{\perp_H}$.*

Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a separable $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code of length (n, m) such that $\gcd(n, q) = 1$ and the order of the automorphism μ of $\mathfrak{R}_\ell$ divides m. Let $\lambda \in \mathfrak{R}_\ell^*$ such that $\mu(\lambda) = \lambda$ and $\lambda^2 = 1$. Let δ_J be the corresponding units in $\mathbb{F}_{q^2}$. Then, we have the following results.

Theorem 14 *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a separable $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code of length (n, m). Also, let $\mathfrak{C}_n = \langle g(x)\rangle$ with defining set $\mathcal{T}$ and $\mathfrak{C}_m = \langle \sum_{J\subseteq\Lambda} \gamma_J f_J(x)\rangle$ where $x^m - \delta_J = h_J(x) f_J(x)$ for every subset $J \subseteq \Lambda$. Then, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ if and only if $\mathcal{T} \cap \mathcal{T}^{-q} = \phi$ and for every subset $J \subseteq \Lambda$, $x^m - \delta_J$ is a right divisor of $h_J^{\dagger}(x)h_J(x)$, where $\mathcal{T}^{-q}$ is the defining set of $\mathfrak{C}_n^{\perp_H}$ and $h_J^{\dagger}(x)$ is the skew Hermitian reciprocal polynomial of $h_J(x)$.*

Proof It follows from Theorem 9, Theorem 13 and Lemma 5. □

Now, we can construct quantum codes by using the Hermitian construction given in Lemma 9 and Theorem 14 as follows.

Theorem 15 *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a separable $\mathbb{F}_{q^2}\mathfrak{R}_\ell$-additive skew λ-constacyclic code of length (n, m). Also, let $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ and $\Phi(\mathfrak{C})$ has parameters $[n + 2^\ell m, k, d_H]_{q^2}$. Then, there exists a quantum code $[[n + 2^\ell m, 2k - n - 2^\ell m, \geq d_H]]_q$.*

5.1 Computational results

In this subsection, we provide some examples in support of our study.

Example 1 Consider the ring $\mathfrak{R}_1 := \mathbb{F}_{7^2}[v_1]/\langle v_1^2 - 1\rangle$, where $\mathbb{F}_{7^2} = \mathbb{F}_7(t)$ and t satisfies $t^2 = t + 4$. Then, $J = \phi$, $J' = \{1\}$ and $\gamma_J = \frac{1}{2}(1 + v_1)$, $\gamma_{J'} = \frac{1}{2}(1 - v_1)$. Let μ be the automorphism over $\mathfrak{R}_1$ defined by $\mu(r_0 + r_1 v_1) = r_0^7 + r_1^7 v_1$ and $\mathfrak{C}$ be a skew λ-constacyclic code of length 56 over $\mathfrak{R}_1$ under μ, where $\lambda = -v_1$. Let $g(x) = \gamma_J g_J(x) + \gamma_{J'} g_{J'}(x)$ be the generator polynomial of $\mathfrak{C}$ where $g_J(x) = x^2 + x + t^{30}$ and $g_{J'}(x) = x^3 + 6x^2 + t^{14}x + t^3$ are generator polynomials of skew negacyclic code $\mathfrak{C}_J$ and skew cyclic code $\mathfrak{C}_{J'}$ over $\mathbb{F}_{7^2}$, respectively. Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix} \in GL_2(\mathbb{F}_{7^2}),$$

satisfying $MM^t = 2I_2$. Then, the Gray image $\Psi(C)$ has the parameters [112, 107, 4]. Also,

$$\begin{aligned}
h_J(x) = {} & x^{54} + 6x^{53} + t^{47}x^{52} + 3x^{51} + t^{11}x^{50} + 6x^{49} + t^{18}x^{48} + 6x^{46} + x^{45} + t^{23}x^{44} + 4x^{43} + t^{35}x^{42} \\
& + x^{41} + t^{42}x^{40} + x^{38} + 6x^{37} + t^{47}x^{36} + 3x^{35} + t^{11}x^{34} + 6x^{33} + t^{18}x^{32} + 6x^{30} + x^{29} + t^{23}x^{28} \\
& + 4x^{27} + t^{35}x^{26} + x^{25} + t^{42}x^{24} + x^{22} + 6x^{21} + t^{47}x^{20} + 3x^{19} + t^{11}x^{18} + 6x^{17} + t^{18}x^{16} + 6x^{14} \\
& + x^{13} + t^{23}x^{12} + 4x^{11} + t^{35}x^{10} + x^9 + t^{42}x^8 + x^6 + 6x^5 + t^{47}x^4 + 3x^3 + t^{11}x^2 + 6x + t^{18}, \\
h_{J'}(x) = {} & x^{53} + x^{52} + t^{12}x^{51} + t^3x^{50} + t^{38}x^{49} + t^{21}x^{48} + x^{45} + x^{44} + t^{12}x^{43} + t^3x^{42} + t^{38}x^{41} + t^{21}x^{40} \\
& + x^{37} + x^{36} + t^{12}x^{35} + t^3x^{34} + t^{38}x^{33} + t^{21}x^{32} + x^{29} + x^{28} + t^{12}x^{27} + t^3x^{26} + t^{38}x^{25} + t^{21}x^{24} \\
& + x^{21} + x^{20} + t^{12}x^{19} + t^3x^{18} + t^{38}x^{17} + t^{21}x^{16} + x^{13} + x^{12} + t^{12}x^{11} + t^3x^{10} + t^{38}x^9 + t^{21}x^8 \\
& + x^5 + x^4 + t^{12}x^3 + t^3x^2 + t^{38}x + t^{21}, \\
h_J^{\dagger}(x) = {} & x^{54} + t^{42}x^{53} + t^{47}x^{52} + t^{26}x^{51} + t^{11}x^{50} + t^{42}x^{49} + t^{18}x^{48} + 6x^{46} + t^{18}x^{45} + t^{23}x^{44} + t^2x^{43} \\
& + t^{35}x^{42} + t^{18}x^{41} + t^{42}x^{40} + x^{38} + t^{42}x^{37} + t^{47}x^{36} + t^{26}x^{35} + t^{11}x^{34} + t^{42}x^{33} + t^{18}x^{32} + 6x^{30} \\
& + t^{18}x^{29} + t^{23}x^{28} + t^2x^{27} + t^{35}x^{26} + t^{18}x^{25} + t^{42}x^{24} + x^{22} + t^{42}x^{21} + t^{47}x^{20} + t^{26}x^{19} + t^{11}x^{18} \\
& + t^{42}x^{17} + t^{18}x^{16} + 6x^{14} + t^{18}x^{13} + t^{23}x^{12} + t^2x^{11} + t^{35}x^{10} + t^{18}x^9 + t^{42}x^8 + x^6 \\
& + t^{42}x^5 + t^{47}x^4 + t^{26}x^3 + t^{11}x^2 + t^{42}x + t^{18}, \\
h_{J'}^{\dagger}(x) = {} & x^{53} + t^5x^{52} + t^{30}x^{51} + t^{15}x^{50} + t^{27}x^{49} + t^{27}x^{48} + x^{45} + t^5x^{44} + t^{30}x^{43} + t^{15}x^{42} + t^{27}x^{41} \\
& + t^{27}x^{40} + x^{37} + t^5x^{36} + t^{30}x^{35} + t^{15}x^{34} + t^{27}x^{33} + t^{27}x^{32} + x^{29} + t^5x^{28} + t^{30}x^{27} + t^{15}x^{26} \\
& + t^{27}x^{25} + t^{27}x^{24} + x^{21} + t^5x^{20} + t^{30}x^{19} + t^{15}x^{18} + t^{27}x^{17} + t^{27}x^{16} + x^{13} + t^5x^{12} + t^{30}x^{11} \\
& + t^{15}x^{10} + t^{27}x^9 + t^{27}x^8 + x^5 + t^5x^4 + t^{30}x^3 + t^{15}x^2 + t^{27}x + t^{27},
\end{aligned}$$

and

$$\begin{aligned} h_J^{\dagger}(x)h_J(x) &= (x^{52}+t^{45}x^{51}+t^{12}x^{50}+t^{13}x^{49}+3x^{48}+t^{37}x^{47}+t^{34}x^{46}+t^{37}x^{45}+t^{9}x^{44}+t^{47}x^{42}\\ &\quad +t^{29}x^{41}+t^{2}x^{40}+t^{29}x^{39}+t^{26}x^{38}+t^{29}x^{37}+t^{31}x^{36}+t^{21}x^{35}+t^{33}x^{34}+t^{29}x^{33}\\ &\quad +t^{30}x^{32}+t^{45}x^{31}+t^{42}x^{30}+t^{45}x^{29}+t^{21}x^{28}+t^{13}x^{27}+t^{34}x^{26}+t^{13}x^{25}+t^{39}x^{24}\\ &\quad +t^{45}x^{23}+t^{42}x^{22}+t^{45}x^{21}+t^{6}x^{20}+t^{29}x^{19}+t^{27}x^{18}+t^{21}x^{17}+t^{13}x^{16}+t^{29}x^{15}\\ &\quad +t^{26}x^{14}+t^{29}x^{13}+t^{2}x^{12}+t^{29}x^{11}+t^{29}x^{10}+t^{3}x^{8}+t^{37}x^{7}+t^{34}x^{6}\\ &\quad +t^{37}x^{5}+t^{44}x^{4}+t^{13}x^{3}+6x^{2}+t^{45}x+t^{36})(x^{56}+1),\\ h_{J'}^{\dagger}(x)h_{J'}(x) &= (x^{50}+t^{38}x^{49}+t^{7}x^{47}+6x^{46}+t^{35}x^{45}+6x^{44}+t^{31}x^{43}+2x^{42}+t^{17}x^{41}+x^{40}+t^{23}x^{39}\\ &\quad +5x^{38}+t^{3}x^{37}+5x^{36}+t^{47}x^{35}+3x^{34}+t^{10}x^{33}+2x^{32}+t^{15}x^{31}+4x^{30}+t^{43}x^{29}+4x^{28}\\ &\quad +t^{39}x^{27}+4x^{26}+t^{15}x^{25}+3x^{24}+t^{39}x^{23}+3x^{22}+t^{19}x^{21}+3x^{20}+t^{15}x^{19}+5x^{18}\\ &\quad +t^{28}x^{17}+4x^{16}+t^{47}x^{15}+2x^{14}+t^{27}x^{13}+2x^{12}+t^{23}x^{11}+6x^{10}+t^{29}x^{9}\\ &\quad +5x^{8}+t^{31}x^{7}+x^{6}+t^{11}x^{5}+x^{4}+t^{7}x^{3}+4x+6)(x^{56}-1). \end{aligned}$$

Since $h_J^{\dagger}(x)h_J(x)$ and $h_{J'}^{\dagger}(x)h_{J'}(x)$ are right divisible by $x^{56}+1$ and $x^{56}-1$, respectively, by Theorem 9, we have $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Also, by Theorem 10, there exists a quantum code $[[112, 102, \geq 4]]_7$. It is noted that the constructed quantum code has better code rate than best-known quantum code $[[112, 92, 4]]_7$ in [49].

Example 2 Consider the ring $\mathfrak{R}_1 := \mathbb{F}_{5^2}[v_1]/\langle v_1^2-1\rangle$, where $\mathbb{F}_{5^2}=\mathbb{F}_5(t)$ and t satisfies $t^2=t+3$. Then, $J=\phi$, $J^{'}=\{1\}$ and $\gamma_J=\frac{1}{2}(1+v_1)$, $\gamma_{J'}=\frac{1}{2}(1-v_1)$. Let μ be the automorphism over $\mathfrak{R}_1$ defined by $\mu(r_0+r_1v_1)=r_0^3+r_1^3v_1$ and $\mathfrak{C}$ be a skew λ-constacyclic code of length 8 over $\mathfrak{R}_1$ with respect to μ where $\lambda=v_1$. Let $g(x)=\gamma_J g_J(x)+\gamma_{J'}g_{J'}(x)$ be the generator polynomial of $\mathfrak{C}$, where $g_J(x)=x^3+x^2+t^2x+t^{14}$ and $g_{J'}(x)=x^4+t^4x^3+t^{20}x^2+t^{16}x+1$ are generator polynomials of skew cyclic code $\mathfrak{C}_J$ and skew negacyclic code $\mathfrak{C}_{J'}$ over $\mathbb{F}_{5^2}$, respectively. Let

$$M=\begin{bmatrix}1 & 1\\ 1 & 4\end{bmatrix}\in GL_2(\mathbb{F}_{5^2}),$$

satisfying $MM^t=2I_2$. Then, the Gray image $\Psi(\mathfrak{C})$ has the parameters $[16, 9, 7]$. Also,

$$\begin{aligned} h_J(x) &= x^5+4x^4+t^{15}x^3+t^{13}x^2+t^{14}x+t^{22},\\ h_{J'}(x) &= x^4+t^{16}x^3+t^4x^2+t^4x+1,\\ h_J^{\dagger}(x) &= x^5+x^4+t^{15}x^3+t^5x^2+t^{14}x+t^2,\\ h_{J'}^{\dagger}(x) &= x^4+t^4x^3+t^{20}x^2+t^{16}x+1, \end{aligned}$$

and

$$\begin{aligned} h_J^{\dagger}(x)h_J(x) &= (x^2+4)(x^8-1),\\ h_{J'}^{\dagger}(x)h_{J'}(x) &= (1)(x^8+1). \end{aligned}$$

Since $h_J^{\dagger}h_J$ and $h_{J'}^{\dagger}h_{J'}$ are right divisible by x^8-1 and x^8+1, respectively, by Theorem 9, $\mathfrak{C}^{\perp_H}\subseteq\mathfrak{C}$. Again, by Theorem 10, there exists a quantum code $[[16, 2, \geq 7]]_5$ which has better parameters than $[[16, 1, 6]]_5$ given in [50].

Example 3 Suppose $\mathbb{F}_{3^2}=\mathbb{F}_3(t)$, t satisfies $t^2=t+1$ and $\mathfrak{C}_n$ is a cyclic code of length $n=40$ over $\mathbb{F}_{3^2}$ with defining set $\mathcal{T}=\{5, 12, 28\}$. Then, the generator polynomial of $\mathfrak{C}_n$ is $f_n(x)=x^3+tx^2+t^7x+t^5$. Also, as $\mathcal{T}^{-3}=\{4, 25, 36\}$ and $\mathcal{T}\cap\mathcal{T}^{-3}=\phi$, by Lemma 5 $\mathfrak{C}_n$ is a Hermitian dual-containing cyclic code over $\mathbb{F}_{3^2}$. Let $\mathfrak{R}_1 := \mathbb{F}_{3^2}[v_1]/\langle v_1^2-1\rangle$. Then, $J=\phi$, $J^{'}=\{1\}$ and $\gamma_J=\frac{1}{2}(1+v_1)$, $\gamma_{J'}=\frac{1}{2}(1-v_1)$. Let μ be the automorphism over $\mathfrak{R}_1$ defined by $\mu(r_0+r_1v_1)=r_0^3+r_1^3v_1$ and $\mathfrak{C}_m$ be a skew cyclic code of length $m=48$ over $\mathfrak{R}_1$ with respect to the automorphism μ. Let $g(x)=\gamma_J g_J(x)+\gamma_{J'}g_{J'}(x)$ be the generator polynomial of $\mathfrak{C}$ where $g_J(x)=x^2+t^2x+t^2$ and $g_{J'}(x)=x^3+t^3x^2+x+t$ are generator polynomials of skew cyclic codes $\mathfrak{C}_J$ and $\mathfrak{C}_{J'}$ over $\mathbb{F}_{3^2}$, respectively. Here,

$$\begin{aligned} h_J(x) &= x^{46} + t^6x^{45} + t^5x^{44} + t^6x^{43} + t^6x^{42} + 2x^{40} + t^2x^{39} + tx^{38} + t^2x^{37} + t^2x^{36} + x^{34} + t^6x^{33} \\ &\quad + t^5x^{32} + t^6x^{31} + t^6x^{30} + 2x^{28} + t^2x^{27} + tx^{26} + t^2x^{25} + t^2x^{24} + x^{22} + t^6x^{21} + t^5x^{20} \\ &\quad + t^6x^{19} + t^6x^{18} + 2x^{16} + t^2x^{15} + tx^{14} + t^2x^{13} + t^2x^{12} + x^{10} + t^6x^9 + t^5x^8 + t^6x^7 \\ &\quad + t^6x^6 + 2x^4 + t^2x^3 + tx^2 + t^2x + t^2, \\ h_{J'}(x) &= x^{45} + t^5x^{44} + x^{43} + t^7x^{42} + 2x^{41} + tx^{40} + x^{39} + x^{38} + t^2x^{36} + t^6x^{34} + 2x^{33} + t^3x^{32} + x^{29} \\ &\quad + t^5x^{28} + x^{27} + t^7x^{26} + 2x^{25} + tx^{24} + x^{23} + x^{22} + t^2x^{20} + t^6x^{18} + 2x^{17} + t^3x^{16} + x^{13} \\ &\quad + t^5x^{12} + x^{11} + t^7x^{10} + 2x^9 + tx^8 + x^7 + x^6 + t^2x^4 + t^6x^2 + 2x + t^3, \\ h^{\dagger}_J(x) &= x^{46} + 2x^{45} + t^5x^{44} + 2x^{43} + t^6x^{42} + 2x^{40} + x^{39} + tx^{38} + x^{37} + t^2x^{36} + x^{34} + 2x^{33} + t^5x^{32} \\ &\quad + 2x^{31} + t^6x^{30} + 2x^{28} + x^{27} + tx^{26} + x^{25} + t^2x^{24} + x^{22} + 2x^{21} + t^5x^{20} + 2x^{19} + t^6x^{18} \\ &\quad + 2x^{16} + x^{15} + tx^{14} + x^{13} + t^2x^{12} + x^{10} + 2x^9 + t^5x^8 + 2x^7 + t^6x^6 + 2x^4 + x^3 + tx^2 + x + t^2, \\ h^{\dagger}_{J'}(x) &= x^{45} + tx^{44} + t^3x^{43} + t^7x^{41} + t^5x^{39} + t^5x^{38} + t^6x^{37} + tx^{36} + 2x^{35} + t^5x^{34} + t^2x^{33} + t^5x^{32} \\ &\quad + x^{29} + tx^{28} + t^3x^{27} + t^7x^{25} + t^5x^{23} + t^5x^{22} + t^6x^{21} + tx^{20} + 2x^{19} + t^5x^{18} + t^2x^{17} + t^5x^{16} \\ &\quad + x^{13} + tx^{12} + t^3x^{11} + t^7x^9 + t^5x^7 + t^5x^6 + t^6x^5 + tx^4 + 2x^3 + t^5x^2 + t^2x + t^5, \end{aligned}$$

and

$$\begin{aligned} h^{\dagger}_J(x)h_J(x) &= (x^{44} + t^3x^{43} + 2x^{42} + 2x^{38} + t^7x^{37} + x^{36} + x^{32} + t^3x^{31} + 2x^{30} + 2x^{26} + t^7x^{25} + x^{24} \\ &\quad + x^{20} + t^3x^{19} + 2x^{18} + 2x^{14} + t^7x^{13} + x^{12} + x^8 + t^3x^7 + 2x^6 + 2x^2 + t^7x + 1)(x^{48} - 1), \\ h^{\dagger}_{J'}(x)h_{J'}(x) &= (x^{42} + t^6x^{41} + t^2x^{40} + t^2x^{39} + x^{37} + t^5x^{36} + t^3x^{35} + t^7x^{34} + t^6x^{33} + t^6x^{32} + t^5x^{31} \\ &\quad + t^5x^{30} + t^5x^{29} + t^5x^{28} + t^5x^{27} + t^3x^{26} + t^7x^{25} + x^{24} + x^{23} + t^5x^{22} + t^3x^{21} + tx^{20} \\ &\quad + t^2x^{19} + 2x^{18} + t^7x^{17} + t^7x^{16} + tx^{15} + tx^{14} + tx^{13} + tx^{12} + tx^{11} + t^2x^{10} + 2x^9 \\ &\quad + t^3x^8 + t^3x^7 + tx^6 + t^2x^5 + x^3 + t^6x^2 + 2x + 2)(x^{48} - 1). \end{aligned}$$

Since $h^{\dagger}_J(x)h_J(x)$ and $h^{\dagger}_{J'}(x)h_{J'}(x)$ both are right divisible by $x^{48} - 1$, by Theorem 9, we have $\mathfrak{C}_m^{\perp_H} \subseteq \mathfrak{C}_m$. Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a skew cyclic code of length (40, 48) over $\mathbb{F}_{3^2}\mathfrak{R}_1$. Then by Lemma 13, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Moreover, if

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in GL_2(\mathbb{F}_{3^2}),$$

satisfying $MM^t = 2I_2$, then the Gray image $\Phi(\mathfrak{C})$ has the parameters [136, 128, 3]. Thus, by Theorem 15, there exists quantum code with parameters $[[136, 120, \geq 3]]_3$ which has better code rate than the best-known quantum code $[[136, 118, 3]]_3$ available in [49].

Example 4 Let $\mathbb{F}_{7^2} = \mathbb{F}_7(t)$, t satisfies $t^2 = t+4$ and $\mathfrak{C}_n$ be a cyclic code of length $n = 80$ over $\mathbb{F}_{7^2}$ with defining set $\mathcal{T} = \{32, 45, 48\}$. Then, the generator polynomial of $\mathfrak{C}_n$ is $f_n(x) = x^3 + t^3x + t^3$. Also, as $\mathcal{T}^{-7} = \{5, 16, 64\}$ and $\mathcal{T} \cap \mathcal{T}^{-7} = \phi$. Therefore, by Lemma 5, $\mathfrak{C}_n$ is a Hermitian dual-containing cyclic code over $\mathbb{F}_{7^2}$. Let $\mathfrak{R}_1 := \mathbb{F}_{7^2}[v_1]/\langle v_1^2 - 1\rangle$. Then, $J = \phi$, $J' = \{1\}$ and $\gamma_J = \frac{1}{2}(1 + v_1)$, $\gamma_{J'} = \frac{1}{2}(1 - v_1)$. Let μ be the automorphism over $\mathfrak{R}_1$ defined by $\mu(r_0 + r_1v_1) = r_0^7 + r_1^7v_1$ and $\mathfrak{C}_m$ be a skew λ-constacyclic code of length $m = 8$ over $\mathfrak{R}_1$ with respect to the automorphism μ, where $\lambda = v_1$. Suppose $g(x) = \gamma_J g_J(x) + \gamma_{J'} g_{J'}(x)$ is the generator polynomial of $\mathfrak{C}_m$ where $g_J(x) = x + t^3$ and $g_{J'}(x) = x^2 + t^2x + 6$ are generator polynomials of skew cyclic codes $\mathfrak{C}_J$ and skew negacyclic code $\mathfrak{C}_{J'}$ over $\mathbb{F}_{7^2}$, respectively. Here,

$$\begin{aligned} h_J(x) &= x^7 + t^{45}x^6 + 6x^5 + t^{21}x^4 + x^3 + t^{45}x^2 + 6x + t^{21}, \\ h_{J'}(x) &= x^6 + t^{26}x^5 + 3x^4 + t^{10}x^3 + 4x^2 + t^{26}x + 6, \\ h^{\dagger}_J(x) &= x^7 + t^3x^6 + 6x^5 + t^{27}x^4 + x^3 + t^3x^2 + 6x + t^{27}, \\ h^{\dagger}_{J'}(x) &= x^6 + t^2x^5 + 3x^4 + t^{34}x^3 + 4x^2 + t^2x + 6, \end{aligned}$$

Table 1 New quantum codes from skew λ-constacyclic codes over $\mathfrak{R}_\ell$ for $\ell = 1$

n	λ	$(\delta_J, \delta_{J'})$	$f_J(x)$	$f_{J'}(x)$	$\Psi(\mathfrak{C})$	$[[n, k, d]]_q$	$[[n', k', d']]_q$
26	$-v_1$	$(-1, 1)$	$t1$	121	[52, 48, 3]	$[[52, 44, \geq 3]]_3$	$[[52, 43, 3]]_3$ [49]
12	-1	$(-1, -1)$	$t^{19}111$	$1t^{13}t^7t1$	[24, 17, 6]	$[[24, 10, \geq 6]]_5$	$[[24, 10, 5]]_5$ [40]
8	v_1	$(1, -1)$	$t^{14}t^211$	$1t^{16}t^{20}t^41$	[16, 9, 7]	$[[16, 2, \geq 7]]_5$	$[[16, 1, 6]]_5$ [50]
40	1	$(1, 1)$	$(t^4)1$	$t^{17}t^{10}t^41$	[80, 76, 3]	$[[80, 72, \geq 3]]_5$	$[[78, 70, 3]]_5$ [49]
14	v_1	$(1, -1)$	$1t^41$	t^31	[28, 25, 3]	$[[28, 22, \geq 3]]_7$	$[[28, 20, 3]]_7$ [49]
36	1	$(1, 1)$	t^5t1	t^52t^41	[72, 67, 3]	$[[72, 62, \geq 3]]_7$	$[[72, 40, 3]]_7$ [49]
38	-1	$(-1, -1)$	t^31	$1t^{33}t^31$	[76, 72, 3]	$[[76, 68, \geq 3]]_7$	$[[75, 69, 3]]_7$ [49]
42	$-v_1$	$(-1, 1)$	$t^{22}t^{30}1$	$62t1$	[84, 79, 4]	$[[84, 74, \geq 4]]_7$	$[[87, 73, 4]]_7$ [49]
56	$-v_1$	$(-1, 1)$	$1t^{41}1$	$6t^{31}1$	[112, 108, 3]	$[[112, 104, \geq 3]]_7$	$[[112, 101, 3]]_7$ [49]
56	$-v_1$	$(-1, 1)$	$t^{30}11$	$t^3t^{14}61$	[112, 107, 4]	$[[112, 102, \geq 4]]_7$	$[[112, 92, 4]]_7$ [49]
6	-1	$(-1, -1)$	$t^4t^{10}1$	$t^{128}11$	[12, 8, 5]	$[[12, 4, \geq 5]]_{13}$	$[[12, 2, 5]]_{13}$ [50]
24	v_1	$(1, -1)$	t^4t^31	$t^{44}11$	[48, 44, 4]	$[[48, 40, \geq 4]]_{17}$	$[[48, 38, 4]]_{17}$ [25]

and

$$\begin{aligned} h_J^\dagger(x)h_J(x) &= (x^6 + 6x^4 + x^2 + 6)(x^8 - 1), \\ h_{J'}^\dagger(x)h_{J'}(x) &= (x^4 + 4x^2 + 1)(x^8 + 1). \end{aligned}$$

Since $h_J^\dagger h_J$ and $h_{J'}^\dagger h_{J'}$ are right divisible by $x^8 - 1$ and $x^8 + 1$, respectively, by Theorem 9, we have $\mathfrak{C}_m^{\perp_H} \subseteq \mathfrak{C}_m$. Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a skew λ-constacyclic code of length (80, 8) over $\mathbb{F}_{7^2}\mathfrak{R}_1$. Then by Lemma 13, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Moreover, if

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix} \in GL_2(\mathbb{F}_{7^2}),$$

satisfying $MM^t = 2I_2$, then the Gray image $\Phi(\mathfrak{C})$ has the parameters [96, 90, 3]. Thus, by Theorem 15, there exists quantum code with parameters $[[96, 84, \geq 3]]_7$. Notice that the constructed quantum code has better code rate than the code $[[96, 80, 3]]_7$ obtained in [24].

Let $\mathfrak{C}$ be a skew λ-constacyclic code of length n over $\mathfrak{R}_1 := \mathbb{F}_{q^2}[v_1]/\langle v_1^2 - 1\rangle$. Then, $J = \phi$, $J' = \{1\}$ and $\mathfrak{C} = \gamma_J \mathfrak{C}_J \oplus \gamma_{J'}\mathfrak{C}_{J'}$, where $\mathfrak{C}_J$ and $\mathfrak{C}_{J'}$ are corresponding skew δ_J-constacyclic and skew $\delta_{J'}$-constacyclic codes over $\mathbb{F}_{q^2}$ of length n, respectively. Here, the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in GL_2(\mathbb{F}_{q^2})$$

is used to find the Gray image $\Psi(\mathfrak{C})$. By using the MAGMA computation software, we obtain (in Table 1) several better quantum codes than the best-known codes from dual-containing skew λ-constacyclic over $\mathfrak{R}_1$. First column of Table 1 denotes length of skew λ-constacyclic code $\mathfrak{C}$ over $\mathfrak{R}_1$, whereas second and third columns are used to write the units λ and the corresponding units δ_J, respectively. The generator polynomials $g_J(x)$ and $g_{J'}(x)$ of $\mathfrak{C}_J$ and $\mathfrak{C}_{J'}$ are written in columns fourth and fifth, respectively. Column sixth contains the Gray image $\Psi(\mathfrak{C})$, whereas column seventh is used to write the parameters of the obtained quantum codes. Last column of the table denotes the parameters of the existing quantum codes available in the literature to compare our obtained codes. Note that instead of writing the whole polynomial, we just write the coefficients of polynomial in ascending order of the powers of the variable. For example, the polynomial $t^{14}x^3 + t^2x^2 + x + 1$ is written as $t^{14}t^211$. In this way, we have shown that the skew constacyclic codes produced better quantum codes.

6 Conclusion

Here, we investigated the algebraic structure of skew constacyclic codes over a class of non-chain rings $\mathfrak{R}_\ell$. Then, we have extended our study to mixed alphabets $\mathbb{F}_{q^2}\mathfrak{R}_\ell$. Among others, we have established the conditions for these codes to contain their Hermitian duals, and consequently, under Hermitian construction, we obtained many new quantum codes. Recent literature shows that the constacyclic codes over non-chain rings are worthy to produce good quantum codes (see [20–25]). Thus, we have obtained many new quantum codes from skew constacyclic codes. To validate the novelty of the approach, we also compare our obtained codes to the existing codes that appeared in some recent articles. We believe that our study will inspire researchers a lot to study skew constacyclic codes over other non-chain rings and their application in the coming years.

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