



Einstein-æther scalar–tensor cosmology

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Abstract We propose an Einstein-æther scalar–tensor cosmological model. In particular, in the scalar–tensor Action Integral, we introduce the æther field with æther coefficients to be functions of the scalar field. This cosmological model extends previous studies on Lorentz-violating theories. For a spatially flat Friedmann–Lemaître–Robertson–Walker background space, we write the field equations which are of second order with dynamical variables the scale factor and the scalar field. The physical evolution of the field equations depends upon three unknown functions which are related to the scalar–tensor coupling function, the scalar field potential, and the æther coefficient functions. We investigate the existence of analytic solutions for the field equations and the integrability properties according to the existence of linear in the momentum conservation laws. We define a new set of variables in which the dynamical evolution depends only upon the scalar field potential. Furthermore, the asymptotic behavior and the cosmological history are investigated where we find that the theory provides inflationary eras similar to that of scalar–tensor theory but with Lorentz-violating terms provided by the æther field. Finally, in the new variables, we found that the field equations are integrable due to the existence of nonlocal conservation laws for arbitrary functional forms of the three free functions.

1 Introduction

A common property of various subjects of quantum gravity is the Lorentz violation [1]. Lorentz’s violation may not have been observed yet. However, gravitational models which violate Lorentz symmetry have been of special interest in the literature these last years [2–11]. A Lorentz violating gravitational theory has been widely studied is the Einstein-æther theory [12–16]. In the Einstein–Hilbert Action Integral, there are introduced the kinematic quantities of a unitary time-like vector field, the æther field. The Lorentz symmetry is violated by the definition of the preferred frame when the æther field is selected. The limit of Einstein’s General Relativity exists while Einstein-æther theory preserves locality and covariance formulation. Einstein-æther theory is a second-order theory and it has been used for the description of various gravitational systems [18–25]. An important characteristic of

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Einstein-æther theory is that it can describe the classical limit of Hořava–Lifshitz [26]. The inverse is not true, see the discussion in [27,28].

On the other hand, scalar fields play an important role in the description of the universe. The main mechanism for the description of the early acceleration era of the universe is attributed to a scalar field, the inflaton. Additionally, scalar fields have been used to describe the late-time acceleration as possible solutions to the dark energy problem [29–36]. Another important feature of the scalar fields is that they can attribute the degrees of freedom provided by higher-order derivatives in gravitational Action Integral as a result of quantum corrections or modifications of General Relativity [37–39]. Consequently, gravitational models where they describe æther Lorentz violating inflationary solutions have been introduced in the literature. A first attempt was proposed by Kanno and Soda in [40] while a more general consideration was investigated by Donnelly and Jacobson in [17]. A common characteristic of these studies is that the scalar field has been defined to be minimally coupled to gravity. In this work, we extend the previous considerations and specifically the model proposed in [40] for which we assume that the scalar field is nonminimally coupled to gravity, that is, we select as natural frame the Jordan frame. We shall call this model Einstein-æther scalar-tensor theory.

The commonest scalar-tensor theory is the Brans–Dicke theory. It was proposed in [41] and it is inspired by Mach’s Principle. In Mach’s Principle, gravity is described by the metric tensor and by a scalar field nonminimally coupled to gravity. The field plays an important role in the description of the early universe [42] and in general on the construction of the physical space. For a debate on which is the natural frame, the Jordan frame, or the Einstein frame we refer the reader in [43–45] and references therein. Another well-known scalar-tensor model is the dilaton theory, which is the fundamental Action Integral for string cosmology [46].

In this study, we investigate the effects of the introduction of the aether field in scalar-tensor theory in cosmology. For the description of the natural space, we assume an isotropic and homogeneous universe described by the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric. For this spacetime, and for the comoving time-like aether field, we derive the field equations which are of second order as in scalar-tensor theory. The field equations depend on three unknown variables which are constructed by the scalar field potential, the scalar field coupling function to gravity, and the coefficient functions which relate the aether and the scalar fields. We investigate the existence of integrable cosmological models by defining new variables and by using Noether’s symmetry approach [47]. The latter approach has been applied for the classification of various cosmological models and the determination of new analytic solutions [48–51].

Moreover, we study the cosmological asymptotic solutions provided by this model by determining the stationary points of the field equations [52–54]. Such analysis is essential to infer the viability of the model [55]. We found that the Einstein-aether scalar-tensor model can provide more than one inflationary eras, as also it provides a richer cosmological history is contrary to the scalar-tensor theory. The plan of the paper is as follows.

In Sect. 2, we define the model of our consideration and we derive the cosmological field equations. In Sect. 3, we define new variables in which we can reduce the cosmological field equations into a Newtonian integrable model. The asymptotic dynamics are investigated in Sect. 4. Finally, the symmetry classification scheme and the derivation of a new analytic solution are presented in Sect. 5. Finally, in Sect. 6 we draw our conclusions.

2 Einstein-aether scalar–tensor cosmology

Einstein-aether cosmological models with the scalar field have been introduced before in the literature. In [17], it has been considered the scalar field potential for a quintessence field to be an arbitrary function of the kinematic invariants of the aether field. This is a generic model which has been used as a base model of study. The attention has been drawn by the proposed Lagrangian by Kanno and Soda [40]. They introduced a gravitational action integral in which the Einstein-aether coupling parameters are functions of the scalar field. Such a cosmological model admits two stages for the inflationary era, the slow-roll era when the scalar field dominates, and a Lorentz violating state when the aether field contributes to the cosmological fluid.

The model proposed in [40] affects the dynamics of the chaotic inflationary model. It has been used as a toy model for the definition of a Lorentz violating DGP model with no ghosts [56]. Lorentz violating inflationary models has been widely studied in the literature, see for instance [57–60]. In [61], the authors investigated the effects of Lorentz violation in cosmological history. Moreover, in [59] the Lorentz violation in cosmology was constrained on the cosmological observations. It was found that Einstein-aether cosmology can explain cosmological observations.

The dynamics of cosmological models with an aether field was the subject of study for many studies. In [62], the dynamics of the model proposed in [40] was investigated in detail. For a purely scalar field, it was found that two attractor solutions can describe inflationary epochs, which is in agreement with the two inflationary eras as described in [40]. Other studies on the dynamics of Einstein-aether scalar field models can be found in [63–69].

Exact and analytic solutions in Einstein-aether scalar field cosmology were found by using the symmetry analysis in [70]. Moreover, the first attempt to quantize in Einstein-aether scalar field cosmology was performed in [71]. Specifically, by using the minisuperspace description was proposed a canonical quantization approach where the Wheeler–DeWitt equation was defined and was solved by investigating the existence of quantum operators which keep invariant the Wheeler–DeWitt equation.

In this study, we generalize the gravitational model proposed in [40] and assume that the scalar field is defined in the Jordan frame, that is, the field $\phi(x^\nu)$ is coupled to gravity. We propose the Einstein-aether scalar–tensor gravitational model defined by the Action Integral

$$S = S_{ST} + S_{Aether}, \tag{1}$$

where S_{ST} is the Action Integral for the scalar–tensor theory [82]

$$S_{ST} = \int dx^4 \sqrt{-g} \left(F(\phi) R + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + V(\phi) \right), \tag{2}$$

function $F(\phi)$ is the coupling function of the scalar field $\phi(x^\nu)$ with gravity which we assume that it is nonconstant, and $V(\phi(x^\nu))$ is the scalar field potential. The second term, S_{Aether} , in expression (1) corresponds to the aether field u^μ as defined in [40]

$$S_{Aether} = - \int dx^4 \sqrt{-g} \left[\beta_1(\phi) u^{\nu;\mu} u_{\nu;\mu} + \beta_2(\phi) (g^{\mu\nu} u_{\mu;\nu})^2 + \beta_3(\phi) u^{\nu;\mu} u_{\mu;\nu} + \beta_4(\phi) u^\mu u^\nu u_{\mu} u_{\nu} - \lambda (u^\mu u_\mu + 1) \right]. \tag{3}$$

Functions $\beta_1(\phi)$, $\beta_2(\phi)$, $\beta_3(\phi)$ and $\beta_4(\phi)$ are the coefficient functions which define the coupling between the aether field and the scalar field. Moreover, λ is a Lagrange multiplier which ensure the unitarity of the aether field, i.e., $u^\mu u_\mu + 1 = 0$.

According to the Cosmological principle, in large scale the universe is assumed to be isotropic and homogeneous described by the spatially flat FLRW line element

$$ds^2 = -N^2(t) dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \tag{4}$$

in which $N(t)$ is the lapse function, $a(t)$ is the scale factor and describes the radius of the three-dimensional Euclidean space. For the comoving observer, the expansion rate θ is defined as $\theta = 3H^2$, where $H = \frac{\dot{a}}{Na}$ is the Hubble function and dot means total derivative with respect to the independent variable t .

For the line element (4), we calculate the Ricciscalar $R = 12H^2 + \frac{6}{N}\dot{H}$. For the scalar field and the Aether field, we assume that they inherits the symmetries of the background space, $\phi = \phi(t)$, while the unitarity condition for the Aether field provides $u^\mu = \frac{1}{N}\delta_t^\mu$. We replace in (1) and integrating by parts the gravitational action integral is written in the minisuperspace description

$$S = \int \frac{dt}{N} \left(6F(\phi) a\dot{a}^2 + 6F_{,\phi} a^2 \dot{a} \dot{\phi} + \frac{1}{2} a^3 \dot{\phi}^2 - N^2 a^3 V(\phi) + 3(\beta_1(\phi) + 3\beta_2(\phi) + \beta_3(\phi)) a\dot{a}^2 \right). \tag{5}$$

Hence, the point-like Lagrangian which describes the field equations is

$$L(N, a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{N} \left(6A(\phi) a\dot{a}^2 + 6B(\phi) a^2 \dot{a} \dot{\phi} + \frac{1}{2} a^3 \dot{\phi}^2 \right) - Na^3 V(\phi), \tag{6}$$

where the new functions $A(\phi)$ and $B(\phi)$ are defined as $A(\phi) = F(\phi) + \frac{1}{2}(\beta_1(\phi) + 3\beta_2(\phi) + \beta_3(\phi))$ and $B(\phi) = F_{,\phi}$. The minisuperspace description is an important feature of gravitational models. The field equations describe the evolution of point-like particles while methods from analytic mechanics can be applied. Moreover, the existence of the minisuperspace is essential for the canonical quantization of the theory which leads to the Wheeler–DeWitt equation of quantum cosmology. In this work, we are interested on classical solutions for the field equations described by the singular point-like Lagrangian (6).

Variation concerning dependent variables N , a and ϕ of the action integral (5) provides the cosmological field equations

$$\frac{1}{N^2} \left(6A(\phi) a\dot{a}^2 + 6B(\phi) a^2 \dot{a} \dot{\phi} + \frac{1}{2} a^3 \dot{\phi}^2 \right) + a^3 V(\phi) = 0, \tag{7}$$

$$2a\ddot{a} + \dot{a}^2 - 2a\dot{a} \frac{\dot{N}}{N} + 2a\dot{a}\dot{\phi} - \frac{a^2 \left(\frac{1}{2} \dot{\phi}^2 - N^2 V(\phi) \right)}{2A} + \frac{a^2 B}{A} \left(\frac{B_{,\phi}}{B} \dot{\phi}^2 + \ddot{\phi} - \dot{\phi} \frac{\dot{N}}{N} \right) = 0, \tag{8}$$

$$\ddot{\phi} - \frac{\dot{N}}{N} \dot{\phi} + \frac{3}{a} \dot{a} \dot{\phi} + \frac{6}{a^2} B(\phi) (a\ddot{a} + 2\dot{a}^2 - a\dot{a}\dot{N}) - 6A_{,\phi}(\phi) \left(\frac{\dot{a}}{a} \right)^2 + N^2 V_{,\phi} = 0, \tag{9}$$

Equivalently,

$$6A(\phi) H^2 + 6B(\phi) H \frac{\dot{\phi}}{N} + \frac{\dot{\phi}^2}{2N^2} + V(\phi) = 0, \tag{10}$$

$$\left(\frac{2}{N} \dot{H} + 3H^2 \right) + 2 \frac{A_{,\phi}}{A} H \frac{\dot{\phi}}{N} - \frac{\left(\frac{1}{2N^2} \dot{\phi}^2 - V(\phi) \right)}{2} + \frac{B}{A} \left(\frac{B_{,\phi}}{B} \dot{\phi}^2 + \ddot{\phi} - \dot{\phi} \frac{\dot{N}}{N} \right) = 0, \tag{11}$$

$$\ddot{\phi} - \frac{\dot{N}}{N} \dot{\phi} + 3H\dot{\phi} + 6B(\phi) (\dot{H} + 3NH^2) - 6A_{,\phi}(\phi) H^2 + N^2 V_{,\phi} = 0. \tag{12}$$

Therefore, the field equations (10) and (11) can be written as

$$3H^2 = G_{\text{eff}} \rho_{\text{eff}}, \tag{13}$$

$$\frac{2}{N} \dot{H} + 3H^2 = G_{\text{eff}} p_{\text{eff}}, \tag{14}$$

in which ρ_{eff} and p_{eff} are the energy density and pressure for the effective fluid defined as

$$\rho_{\text{eff}} = \frac{1}{2} \left(6B(\phi) H \frac{\dot{\phi}}{N} + \frac{\dot{\phi}^2}{2N^2} + V(\phi) \right), \tag{15}$$

$$p_{\text{eff}} = - \left(2A(\phi)_{,\phi} H \frac{\dot{\phi}}{N} - \frac{1}{2} \left(\frac{1}{2N^2} \dot{\phi}^2 - V(\phi) \right) + B(\phi)_{,\phi} \frac{\dot{\phi}^2}{N^2} + B(\phi) \left(\ddot{\phi} - \dot{\phi} \frac{\dot{N}}{N} \right) \right), \tag{16}$$

and $G_{\text{eff}} = \frac{1}{(-A(\phi))}$ is the time-dependent gravitational constant. We remark that in contrary to the scalar–tensor theory in which the effective gravitational constant depends on the coupling function $F(\phi)$, in the Einstein-aether scalar tensor model it depends also on the aether coefficients functions $\beta_1 - \beta_4$. In addition, we observe that if the aether coefficient functions are constant, the usual scalar–tensor theory is recovered.

2.1 Conformal transformation

It is well known that there exists a unique relation of the minimally coupled scalar field with the scalar–tensor theories through conformal transformations [72]. The conformal transformation is nothing else than a geometric map that relates the gravitational action integrals of two theories defined in the Einstein and the Jordan frames. However, the question of which frame is physical is still without an answer. There are various studies in the literature that discuss this issue [73–76]. However, while the question has not been answered yet, we know that there are some features independent of the frame. In the following, we discuss the conformal transformation for the scalar–tensor aether model of our analysis and we investigate the evolution of the coupling functions between the two frames.

Consider the conformal transformation $g_{ij} = N^{-2} \bar{g}_{ij}$, with $N^{-1} = \sqrt{2F(\phi)}$, then the component S_{ST} of the gravitational action integral reads [77]

$$S_{\text{ST}} = \int dx^4 \sqrt{-g} \left[\frac{R}{2} - \frac{1}{2} \Psi_{;i} \Psi_{;j} g^{ij} + \bar{V}(\Psi) \right], \tag{17}$$

where $d\Psi = \sqrt{\left(\frac{3F_\phi^2 - F}{2F^2}\right)} d\phi$ and $\bar{V}(\Psi) = \frac{V(\Psi)}{4F(\Psi)^2}$. The later action integral is nothing else than that of a minimally coupled scalar field.

Moreover, the component of the aether field in the gravitational action integral becomes

$$S_{\text{Aether}} = - \int dx^4 \sqrt{-g} \left[\bar{\beta}_1(\Psi) \bar{u}^{\nu|\mu} \bar{u}_{\nu|\mu} + \bar{\beta}_2(\Psi) (g^{\mu\nu} \bar{u}_{\mu\nu})^2 + \bar{\beta}_3(\Psi) \bar{u}^{\nu|\mu} u_{\mu|\nu} + \bar{\beta}_4(\Psi) \bar{u}^\mu \bar{u}^\nu \bar{u}_\mu \bar{u}_\nu - \lambda (\bar{u}^\mu \bar{u}_\nu + 1) \right], \tag{18}$$

where “|” means covariant derivative with respect to the metric tensor $\bar{g}_{\mu\nu}$, $\bar{u}^\mu = N^{-1} u^\mu$, such that to be unitary the aether field and $\bar{\beta}(\Psi) = N^{-2} \beta(\Psi)$. Hence, under a conformal transformation in the scalar–tensor aether theory the scalar field aether theory of [40] is recovered. In addition, because $\sqrt{\bar{\beta}}$ corresponds to the mass scale of the symmetry breaking [40], by definition parameters β should be positive defined.

Consequently, in the absence of the scalar field, that is $\Psi = \Psi_0$, the Einstein-aether theory is recovered, and coefficient functions $\bar{\beta}$ become constants. Hence, the constraint values of these parameters as provided by gravitational waves [78] observations and massive objects and others [79–81] can be applied.

We continue our analysis by investigating the existence of analytic solutions for the field equations (10)–(12).

3 Analytic solution

Before we proceed, we can define without loss of generality the new scalar field ψ with the relation $d\phi = \sqrt{A(\psi)}d\psi$. Hence, the point-like Lagrangian (6) reads

$$L(N, a, \dot{a}, \psi, \dot{\psi}) = \frac{1}{N} \left(6A(\psi)a\dot{a}^2 + 6B(\psi)a^2\dot{\psi} + \frac{A(\psi)}{2}a^3\dot{\psi}^2 \right) - Na^3V(\psi). \tag{19}$$

The field equations of this cosmological model depend on three arbitrary functions, $A(\psi)$, $B(\psi)$ and $V(\psi)$ which should be defined. For the lapse function, we consider $N = na^{-3}A(\psi)$, the point-like Lagrangian (19) is written as follows

$$L(n, a, \dot{a}, \psi, \dot{\psi}) = \frac{a^3}{n} \left(6a\dot{a}^2 + 6\beta(\psi)a^2\dot{\psi} + \frac{1}{2}a^3\dot{\psi}^2 \right) - nV(\psi)A(\psi), \tag{20}$$

in which $B(\psi) = \beta(\psi)A(\psi)$.

We continue by defining the new variable $X = \ln(a) + \frac{1}{2} \int \beta(\psi) d\psi$, that is $a = \exp(X - \frac{1}{2} \int \beta(\psi) d\psi)$, then Lagrangian function (20) becomes

$$L(n, X, \dot{X}, \psi, \dot{\psi}) = \frac{1}{2n} e^{-6X-3 \int \beta(\psi)d\psi} (12\dot{X}^2 + (1 - 3\beta(\psi)^2) \dot{\psi}^2) - nV(\psi)A(\psi). \tag{21}$$

We select again the new lapse function $n = e^{-6X-3 \int \beta(\psi)d\psi}$, in which the point-like Lagrangian reads

$$L(n, X, \dot{X}, \psi, \dot{\psi}) = \frac{1}{2n} e^{-6X} (12\dot{X}^2 + (1 - 3\beta(\psi)^2) \dot{\psi}^2) - e^{-6X} V(\psi)A(\psi) e^{-3 \int \beta(\psi)d\psi}. \tag{22}$$

Now for the scalar field potential $V(\psi) = V_0(A(\psi))^{-1} e^{3 \int \beta(\psi)d\psi}$, the field equations are

$$(12\dot{X}^2 + (1 - 3\beta(\psi)^2) \dot{\psi}^2) + 2V_0e^{6X} = 0, \tag{23}$$

$$\ddot{X} + V_0e^{6X} = 0, \tag{24}$$

$$\ddot{\psi} - 3\beta(\psi)(\beta_{,\psi}\dot{\psi}^2 + \beta(\psi)\ddot{\psi}) = 0, \tag{25}$$

while the element for the background space is

$$ds^2 = -\exp(-12X)A(\psi)dt^2 + \exp\left(2X - \int \beta(\psi) d\psi\right)(dx^2 + dy^2 + dz^2). \tag{26}$$

The system of differential equations (23)–(25) describes the motion of particle in the two-dimensional minisuperspace with line element

$$ds_{(ms)}^2 = (12dX^2 + (1 - 3\beta(\psi)^2) d\psi^2), \tag{27}$$

and effective potential $V_{\text{eff}} = V_0e^{6X}$.

The analytic solution of the field equations (23), (24), (25) is

$$X(t) = \frac{1}{6} \ln \left(\cosh \left(\frac{\sqrt{3}t}{X_0} + X_2 \right) \right) - \frac{1}{3} \ln(X_0), \tag{28}$$

$$\Psi(t) = \Psi_1 t + \Psi_0, \tag{29}$$

with $d\Psi = \sqrt{1 - 3\beta(\psi)}d\psi$. Consequently, for the integrability of the field equations the following Proposition 1 follows.

Proposition 1 *The cosmological field equations for the Einstein-aether scalar–tensor described by the point-like Lagrangian are superintegrable for arbitrary functions $A(\psi)$ and $B(\psi)$ when $V(\psi) = V_0(A(\psi))^{-1} e^{3 \int \frac{B(\psi)}{A(\psi)} d\psi}$. The field equations describe the motion of a particle in the two-dimensional flat space in Cartesian coordinates for the lapse function $N = \exp(-12X)A(\psi)$ with $X = \ln(a) + \frac{1}{2} \int \frac{B(\psi)}{A(\psi)} d\psi$.*

In the special case in which $\frac{B(\psi)}{A(\psi)} = \text{const}$, then for the scalar field potential and the coupling function $A(\psi)$ follows $V(\psi) = \bar{V}_0 A(\psi)^{-1}$.

The determination of integrability for a given cosmological model, and in general for a physical system is an important property. Nowadays, it is popular to solve a dynamical system by using numerical techniques. However, there are two important issues in that approach. When we solve a dynamical system numerically, we do not know if the evolution is sensitive to the initial conditions, especially when chaos exists. Moreover, it is unknown if the numerical solution corresponds to an actual solution to the problem. These issues are solved when we determine the integrability of the given dynamical system.

In cosmology, the knowledge that a dynamical system is integrable is of special interest. Integrable cosmological models have been found that they can describe various areas in the evolution of cosmological history. Moreover, the initial value problem for inflation [83] can be solved easily for integrable models, while we know that the provided cosmologically history is not sensitive to the small changes of the initial conditions. Although an integrable model may not describe the complete cosmological history, it can be used always as a reference model. For instance, the multi-body gravitational system is a chaotic dynamical system, the integrable two-body system describes well the Sun–Earth orbits.

To understand the general evolution of the given cosmological model in the following section, we investigate the asymptotic behavior of the field equations.

4 Asymptotic dynamics

Let us perform a detailed study on the evolution of the field equations in the Einstein-aether scalar–tensor model. We prefer to work with the scalar field ψ , where the field equations are described by the Lagrangian function (19). We select to work in the H -normalization [52] and to define the new dimensionless variables

$$x = \frac{\dot{\psi}}{2\sqrt{3}H}, \quad y = \frac{V(\psi)}{A(\psi)H^2}, \quad \lambda = \frac{V_{,\psi}(\psi)}{V(\psi)}. \tag{30}$$

We remark that in general someone should consider a more general consideration which will allow for the Hubble function to take the value zero. Here, we work on the branch $H > 0$ and we focus on the existence of asymptotic solutions which describe de Sitter universes or scaling solutions.

For the arbitrary functions of the dynamical system, we assume $B(\psi) = \frac{\beta_0}{\sqrt{3}}A(\psi)$, $\beta_0^2 \neq 1$, with $A(\psi) = A_0e^{-\nu\psi}$. Therefore, in the new variables the field equations read

$$y = -6(1 + 2\beta_0x + x^2), \tag{31}$$

and

$$\frac{dx}{d \ln a} = \frac{1}{1 - \beta_0^2} \left(\sqrt{3}(\lambda + \nu)(1 + \beta_0x) - 3(\beta_0 + x) \right) (1 + 2\beta_0x + x^2), \tag{32}$$

$$\frac{d\lambda}{d \ln a} = 2\sqrt{3}\lambda^2x(\Gamma(\lambda) - 1), \quad \Gamma(\lambda) = \frac{V_{,\psi}\psi V}{(V_\psi)^2}. \tag{33}$$

Every stationary point for the dynamical system (32)–(33) describes an exact solution for the scale factor for the background space. Indeed, the effective equation of state $w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}}$ in the new variables is

$$w_{\text{eff}}(x, \lambda) = -1 + \frac{2\beta_0(3\beta_0 - \sqrt{3}\lambda) - 2(\sqrt{3}\beta_0\lambda - 3)(2\beta_0 + x)x - 2\sqrt{3}\nu(\beta_0 + x)(2 + \beta_0x)}{3(1 - \beta_0^2)}. \tag{34}$$

Hence, at the stationary point P with coordinates $P = (x(P), \lambda(P))$, $w_{\text{eff}}(P) = \text{const}$, that is, by definition $a(t) = a_0t^{\frac{2}{3(1+w_{\text{eff}}(P))}}$ for $w_{\text{eff}}(P) \neq -1$ and $a(t) = a_0e^{H_0t}$ for $w_{\text{eff}} = -1$. Consequently, when we determine the stationary points of the system (32)–(33) and investigate their stability we are able to infer about the asymptotic behavior for the Einstein-aether scalar-tensor model.

For the scalar field potential, we consider the following potential function $V(\psi) = V_0e^{\lambda\psi}$.

4.1 Exponential potential

For the exponential potential, $V(\psi) = V_0e^{\lambda\psi}$, parameter λ is always a constant, thus the dynamical system is reduced to the one-dimension equation (32). The right-hand side of equation (32) vanishes when

$$x_1 = -\frac{3\beta_0 - \sqrt{3}(\lambda + \nu)}{3 - \sqrt{3}\beta_0(\lambda + \nu)},$$

$$x_2 = -\beta_0 + \sqrt{\beta_0^2 - 1},$$

$$x_3 = -\beta_0 - \sqrt{\beta_0^2 - 1}.$$

Consequently, we determine three stationary points, namely A_1 , A_2 and A_3 with coordinates x_1 , x_2 and x_3 , respectively.

For point A_1 , we derive $y(A_1) = \frac{18(1-\beta_0^2)(2\sqrt{3}\beta_0(\lambda+\nu)-3-(\lambda+\nu)^2)}{(3-\sqrt{3}\beta_0(\lambda+\nu))^2}$, and $w_{\text{eff}}(A_1) = \frac{-3+2(\lambda^2-\nu^2)+\sqrt{3}\beta_0(\nu-3\lambda)}{\sqrt{3}\beta_0(\lambda+\nu)-3}$. Therefore, the asymptotic solution at A_1 describes a universe where the kinetic part, the potential part of the scalar field as also the aether field contributes in the cosmological fluid. The scalar factor is scaling, except from the case in which $\nu = \sqrt{3}\beta_0 - \lambda$ or $\nu = \lambda$ where the asymptotic solution describes a de Sitter universe and $w_{\text{eff}}(A_1) = -1$. The linearized system around the stationary point A_1 becomes $\frac{d\delta x}{d \ln a} = e_1(\lambda, \nu, \beta_0)\delta x$ in which

$e_1(\lambda, \nu, \beta_0) = \frac{3(3+(\lambda+\nu)(\lambda+\nu-2\sqrt{3}\beta_0))}{\sqrt{3}\beta_0(\lambda+\nu)-3}$. When $e_1 < 0$, the stationary point is an attractor and the asymptotic solution is stable. Otherwise, the point A_1 is a source. Let us focus in the special

case that we studied before for the integrable model in which $\nu = -\lambda$. In this case, we calculate $y(A_1) = -6(1 - \beta_0^2)$, $w_{\text{eff}}(A_1) = -1 + \frac{4}{3}\beta_0\lambda$ and $e_1(\lambda, -\lambda, \beta_0) = -3$. We conclude that the stationary point is always an attractor and $w_{\text{eff}}(A_1) < -\frac{1}{3}$ when $\beta_0\lambda < \frac{1}{2\sqrt{3}}$. In addition, $y > 0$ when $\beta_0^2 > 1$. On the other hand, for the two de Sitter $\nu = \lambda$ we derive $y(A_1) = \frac{18(1-\beta_0^2)(4\sqrt{3}\beta_0\lambda-3-4\lambda^2)}{(3-2\sqrt{3}\beta_0\lambda)^2}$, $w_{\text{eff}}(A_1) = -1$ and $e_1(\lambda, \lambda, \beta_0) = -6 + \frac{3(3-4\lambda^2)}{3-2\sqrt{3}\beta_0\lambda}$.

Therefore, the de Sitter point is an attractor when $\left\{ \beta_0\lambda < \frac{\sqrt{3}}{2}, \beta_0\lambda > \frac{4\lambda^2+3}{4\sqrt{3}}, |\lambda| > \frac{\sqrt{3}}{2} \right\}$ or $\left\{ \beta_0\lambda > \frac{\sqrt{3}}{2}, \beta_0\lambda < \frac{4\lambda^2+3}{4\sqrt{3}}, |\lambda| < \frac{\sqrt{3}}{2} \right\}$, and $\left\{ |\lambda| = \frac{\sqrt{3}}{2}, \beta_0\lambda \neq \frac{\sqrt{3}}{2} \right\}$. For the second de Sitter solution $\nu = \sqrt{3}\beta_0 - \lambda$, we derive $y(A_1) = -6$, $w_{\text{eff}}(A_1) = -1$ and $e_1(\lambda, \sqrt{3}\beta_0 - \lambda, \beta_0) = -3$, from where we infer that the stationary point is always an attractor and $y(A_1) < 0$.

The stationary points A_2 and A_3 describe universes dominated by the kinetic part of the scalar field, that is, $y(A_2) = 0$ and $y(A_3) = 0$. The points are real for $|\beta_0| < 1$. The equation of state parameter is found at each point $w_{\text{eff}}(A_2) = 1 - \frac{4}{3}\nu \left(\beta_0 + \sqrt{\beta_0^2 - 1} \right)$ and $w_{\text{eff}}(A_2) = 1 - \frac{4}{3}\nu \left(-\beta_0 + \sqrt{\beta_0^2 - 1} \right)$. In a similar way as before for the linearized system near the stationary points, we find $e_1(A_2) = 2 \left(3 - \sqrt{3} \left(\beta_0 + \sqrt{\beta_0^2 - 1} \right) (\lambda + \nu) \right)$ and $e_1(A_3) = 2 \left(3 + \sqrt{3} \left(-\beta_0 + \sqrt{\beta_0^2 - 1} \right) (\lambda + \nu) \right)$. In the case with $\nu = -\lambda$, we find that $e_1(A_2) = e_2(A_3) = 6$, that is, the solutions are always unstable. Moreover, when $\nu = \frac{\sqrt{3}}{2} \left(\beta_0 - \sqrt{\beta_0^2 - 1} \right)$, point A_2 describes a de Sitter universe which is stable when $\left\{ \lambda < -\frac{\sqrt{3}}{2}, \frac{3+4\lambda^2}{4\sqrt{3}\lambda} < \beta_0 < -1 \right\}$, $\left\{ 0 < \lambda < \frac{\sqrt{3}}{2}, \beta_0 < \frac{3+4\lambda^2}{4\sqrt{3}\lambda} \right\}$ and $\left\{ \lambda > \frac{\sqrt{3}}{2}, \beta_0 > 1 \right\}$. Similarly, when $\nu = \frac{\sqrt{3}}{2} \left(\beta_0 + \sqrt{\beta_0^2 - 1} \right)$ point A_3 describes a de Sitter solution which is stable when $\left\{ \lambda < \frac{\sqrt{3}}{2}, \beta_0 < -1 \right\}$, $\left\{ 0 < \lambda < \frac{\sqrt{3}}{2}, \beta_0 < \frac{3+4\lambda^2}{4\sqrt{3}\lambda} \right\}$ and $\left\{ \lambda > \frac{\sqrt{3}}{2}, 1 < \beta_0 < \frac{3+4\lambda^2}{4\sqrt{3}\lambda} \right\}$.

From these results, it is clear that inflationary solutions are provided by this cosmological model. From (30), it follows $V(\psi) = yA(\psi)H^2$. Thus, when $V(\psi) \neq 0$ then we shall say that the contribution of the aether field, through $A(\psi)$, is nonzero. That is true in the solution described by the stationary point A_1 , which can be seen as Lorentz violated inflationary solution in the Jordan frame. Moreover, the asymptotic solutions described by points A_1 and A_2 are similar to that of the usual scalar-tensor theory [84].

4.2 Arbitrary potential

For an arbitrary potential function, that is, for arbitrary function $\Gamma(\lambda)$, then, the stationary points of the two-dimensional dynamical system (32)–(33) are

$$B_1(\lambda_0) = \left(-\frac{3\beta_0 - \sqrt{3}\nu}{3 - \sqrt{3}\beta_0\nu}, \lambda_0 \right),$$

$$\begin{aligned}
 B_2(\lambda_0) &= \left(-\beta_0 + \sqrt{\beta_0^2 - 1}, \lambda_0\right), \\
 B_3(\lambda_0) &= \left(-\beta_0 - \sqrt{\beta_0^2 - 1}, \lambda_0\right) \text{ and} \\
 B_4 &= \left(0, \sqrt{3}\beta_0 - \nu\right),
 \end{aligned}$$

in which λ_0 are solutions of the equation $\lambda^2 (\Gamma(\lambda) - 1) = 0$.

The family of stationary points B_1, B_2 and B_3 have the same physical properties with that of points A_1, A_2 and A_3 , respectively, with $\lambda = 0$. The new point B_4 is nothing else than the de Sitter solution described by A_1 and $\nu = \sqrt{3}\beta_0 - \lambda$. What is different for an arbitrary potential is the stability of the points which depends on the nature of function $\Gamma(\lambda)$. We summarize the results in the following Proposition 2.

Proposition 2 *The stationary points for the Einstein-aether scalar–tensor field equations (32)–(33) for arbitrary potential provide a de Sitter point as asymptotic solution, point B_4 , and three families of scaling solutions described by points B_1, B_2 and B_3 for every λ_0 which solves the equation $\lambda_0^2 (\Gamma(\lambda_0) - 1) = 0$.*

4.2.1 Potential $V(\psi) = V_0(e^{\sigma\psi} - \Lambda)$

We demonstrate the results of the latter Proposition 2 by considering the potential function $V(\psi) = V_0(e^{\sigma\psi} - \Lambda)$. For this potential, we calculate $\Gamma(\lambda) = \frac{\sigma}{\psi}$, such that Eq. (33) to become

$$\frac{d\lambda}{d \ln a} = -2\sqrt{3}\lambda(\lambda - \sigma)x. \tag{35}$$

Thus, the polynomial equation $\lambda(\lambda - \sigma) = 0$ has the following roots, $\lambda_1 = 0$ and $\lambda_2 = \sigma$. The stationary points for the field equations (32)–(33) are

$$\begin{aligned}
 B_1(0) &= \left(-\frac{3\beta_0 - \sqrt{3}\nu}{3 - \sqrt{3}\beta_0\nu}, 0\right), \\
 B_2(0) &= \left(-\beta_0 + \sqrt{\beta_0^2 - 1}, 0\right), \\
 B_3(0) &= \left(-\beta_0 - \sqrt{\beta_0^2 - 1}, 0\right), \\
 B_1(\sigma) &= \left(-\frac{3\beta_0 - \sqrt{3}(\lambda + \nu)}{3 - \sqrt{3}\beta_0(\lambda + \nu)}, \sigma\right), \\
 B_2(\sigma) &= \left(-\beta_0 + \sqrt{\beta_0^2 - 1}, \sigma\right), \\
 B_3(\sigma) &= \left(-\beta_0 - \sqrt{\beta_0^2 - 1}, \sigma\right).
 \end{aligned}$$

and

$$B_4 = \left(0, \sqrt{3}\beta_0 - \nu\right).$$

Let us focus on the stability conditions for point B_4 . The eigenvalues of the linearized system around the de Sitter point B_4 are

$$e^\pm(B_4) = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} \sqrt{\frac{27\beta_0^2 + 8\nu(\nu + \sigma) - 8\sqrt{3}(2\nu + \sigma)}{\beta_0^2 - 1}}. \tag{36}$$

In Fig. 1, we present region plots in the spaces $\{\sigma, \nu\}$, $\{\beta_0, \nu\}$ and $\{\sigma, \beta_0\}$ in which the real parts for the eigenvalues $e^\pm(B_4)$ are negative and the de Sitter universe described by B_4 is an attractor.

5 Symmetry classification

In Sect. 3, we determined a family of analytic solutions for arbitrary functions $A(\psi)$ and $B(\psi)$ with the empirical approach of reducing the field equations to a two-dimensional system in Cartesian coordinates and recognizing its integrable form by our experience in analytic mechanics. We now use a systematic method to determine the scalar field potential $V(\psi)$ such that the field equations form a Liouville integrable system. The Noether symmetry approach is applied to perform a classification of the scalar field potential similar to that of Ovsiannikov’s classification scheme. We briefly discuss the basic properties and definitions of Noether’s theory.

Consider the one-parameter point transformation in the space of dependent and independent variables $\{t, x^i(t)\}$:

$$t' = t + \varepsilon\xi(t, x^j), \quad x^i = x^i + \varepsilon\eta^i(t, x^j), \tag{37}$$

and generator $Z = \xi\partial_t + \eta^i\partial_i$. Then, according to Noether’s first theorem, for a dynamical system provided by the variational principle of the Lagrangian function $L(t, x^j, \dot{x}^k)$, the dynamical system remains invariant under the action of the one-parameter point transformation with generator X if and only if there exists a function f such that the following condition be true

$$Z^{[1]}L + L\dot{\xi} = \dot{f}, \tag{38}$$

where $Z^{[1]} = Z + (\eta^j - \dot{x}^j\xi)\partial_{\dot{x}^j}$ is the first extension of X in the jet space $\{t, x^i, \dot{x}^i\}$. Condition (38) is called the Noether symmetry condition and, when it is true, the vector field Z is a Noether symmetry for the dynamical system described by the Lagrangian function $L(t, x^j, \dot{x}^k)$.

The importance of the derivation of Noether symmetries for a dynamical system is twofold. Noether symmetries form a subalgebra of the Lie point symmetries of the dynamical system. Thus, the determination of Noether symmetries can be used to define invariant functions or under the action of similarity transformations to simplify the expression of the dynamical system. Moreover, from Noether’s second theorem there exists a formula that relates any Noether symmetry to a specific conservation law.

Indeed, if Z is the Noether symmetry for the dynamical system with Lagrangian $L(t, x^j, \dot{x}^k)$, then the following quantity

$$\Phi = \xi \left(\dot{x}^k \frac{\partial L}{\partial \dot{x}^k} - L \right) - \eta^k \frac{\partial L}{\partial \dot{x}^k} + f \tag{39}$$

is a conservation law, i.e., $\frac{d\Phi}{dt} = 0$.

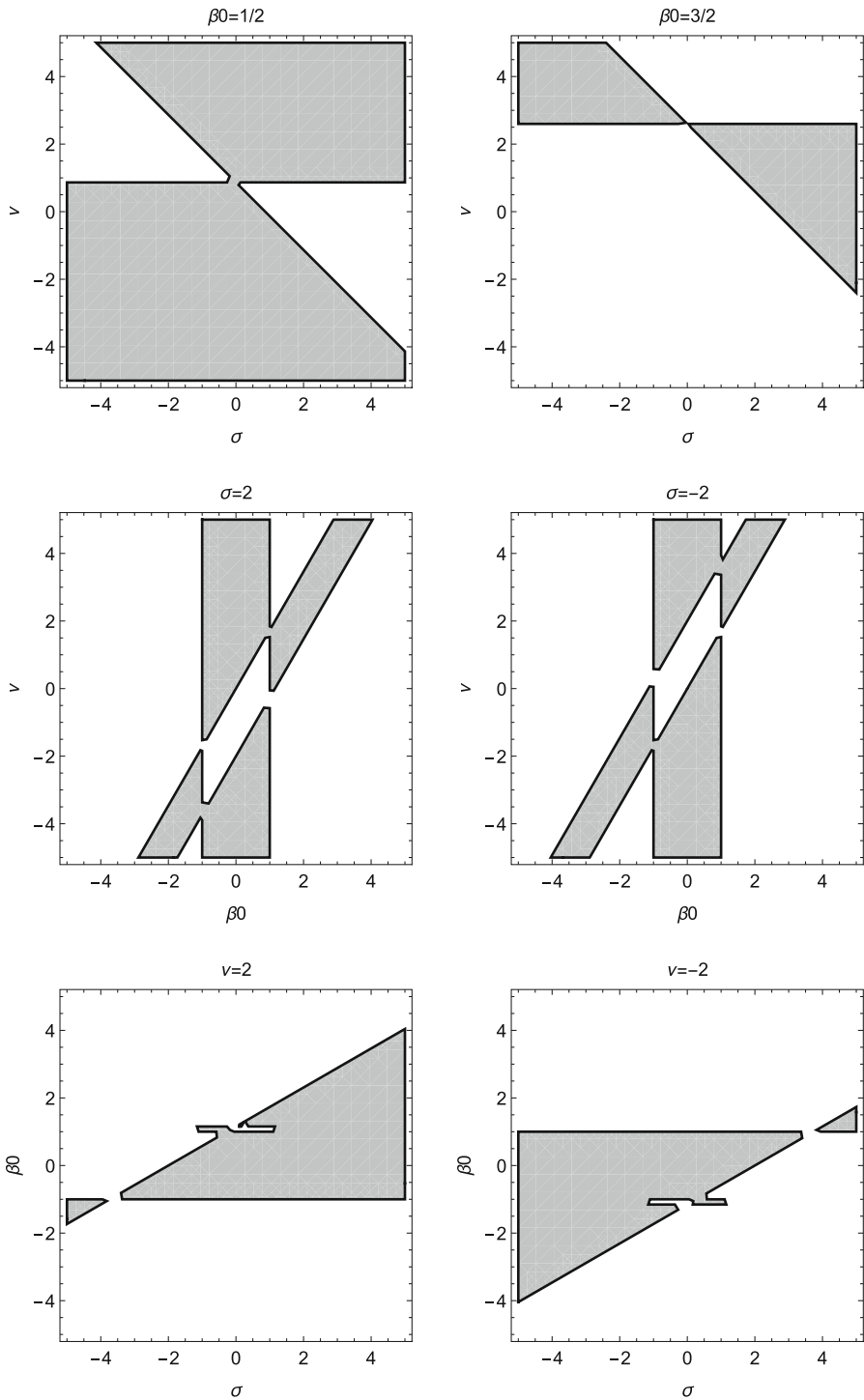


Fig. 1 Region in the spaces $\{\sigma, \nu\}$, $\{\beta_0, \nu\}$ and $\{\sigma, \beta_0\}$ in which the real parts for the eigenvalues $e^\pm(B_4)$ are negative and the de Sitter universe described by B_4 is an attractor

The gravitational field equations of the Einstein-æther scalar-tensor cosmology form a two-dimensional Hamiltonian constraint system. The constraint equation, which is the first modified Friedmann’s equation, can be seen as a conservation law. Thus, we should determine a second conservation laws in order to infer the integrability of the field equations for other forms of the scalar field potential. We prefer to work in the coordinates (X, Ψ) and the lapse function which we used to write the field equations (23)–(25). In these coordinates and for an arbitrary potential function, the point-like Lagrangian is

$$L(X, \dot{X}, \Psi, \dot{\Psi}) = \left(6\dot{X}^2 + \frac{1}{2}\dot{\Psi}^2 \right) - \hat{V}(\Psi) e^{6X}, \tag{40}$$

where $d\Psi = \sqrt{1 - 3\beta(\psi)}d\psi$ and $\hat{V}(\psi) = V(\psi) A(\psi) e^{-3 \int \frac{B(\psi)}{A(\psi)} d\psi}$.

At this point, it is important to mention that, while our original system was dependent upon three free functions, $A(\psi)$, $B(\psi)$ and $V(\psi)$, we were able to find a specific coordinate system in which only one function is essential, that is the scalar field potential $V(\psi)$. Functions $A(\psi)$ and $B(\psi)$ play a significant role in the physical properties of the solutions. However, that is not true of the dynamics described by (40). Function $A(\psi)$ has been eliminated by the field equations by selecting a specific lapse function, while $B(\psi)$ has been eliminated by a coordinate transformation on the field ψ . Moreover, because we are interested on time-independent conservation laws which are in-involution with the Hamiltonian, $\{\Phi, \mathcal{H}\} = 0$, where $\{, \}$ is the Poisson bracket and $\mathcal{H} = \dot{x}^k \frac{\partial L}{\partial \dot{x}^k} - L$, from the results of the symmetry analysis of [85] we search for Noether symmetries with $\xi(t, x^j) = 0$.

We omit the derivation of the Lie point symmetries and the unique real nonconstant scalar field potential $\hat{V}(\Psi)$ that we find in which the point-like Lagrangian (40) admits additional Noether symmetries in the exponential potential $\hat{V}(\Psi) = \hat{V}_0 \exp(-6\mu\Psi)$. The corresponding symmetry vector is $Z = \partial_X + \frac{1}{\mu} \partial_\Psi$ while the resulting Noetherian conservation law is $\Phi_0 = 12\dot{X} + \frac{1}{\mu} \dot{\Psi}$.

We define the new variable $X = \frac{u}{6} + \mu\Psi$. Thus, the point-like Lagrangian (40) is written as

$$L(u, \dot{u}, \Psi, \dot{\Psi}) = \dot{u}^2 + 12\mu\dot{u}\dot{\Psi} + 3(1 + 12\mu^2)\dot{\Psi}^2 - 6\hat{V}_0 e^u, \tag{41}$$

while the field equations are

$$\begin{aligned} \dot{u}^2 + 12\mu\dot{u}\dot{\Psi} + 3(1 + 12\mu^2)\dot{\Psi}^2 - 6\hat{V}_0 e^u &= 0, \\ \ddot{u} + 3\hat{V}_0(1 + 12\mu^2)e^u &= 0, \end{aligned} \tag{42}$$

$$\ddot{\Psi} - 6\hat{V}_0\mu e^u = 0. \tag{43}$$

Hence, the analytic solution is

$$u(t) = \ln\left(\frac{u_1}{6\hat{V}_0(1 + 12\mu^2)}\right) - 2 \ln\left(\cosh\left(\frac{\sqrt{u_1}}{2}t\right)\right), \tag{44}$$

$$\Psi(t) = \frac{2}{3(1 + 12\mu^2)^2} \left(6\mu(1 + 12\mu^2) \ln\left(\cosh\left(\frac{\sqrt{u_1}}{2}t\right)\right) - \frac{\sqrt{3}}{2} \sqrt{-u_1} (1 + 12\mu^2)t \right). \tag{45}$$

This is the analytic solution for the model the dynamics of which we studied in Sect. 4 when $\beta(\psi) = \beta_0$, $\Psi \simeq \psi$ such that $\mu = \mu(\lambda, \nu)$ and function $A(\psi)$ was the exponential function.

5.1 Nonlocal symmetries

The point-like Lagrangian (40) is in a form similar to that of the (phantom) minimally coupled scalar field Ψ in general relativity where X plays the role of the scale factor, for the lapse function $N = X^3$. Thus, if we select to work with nonlocal symmetries, we can obtain the results of [86] and end with the following Proposition 3.

Proposition 3 *The gravitational field equations of the Einstein-æther scalar-tensor with arbitrary potential functions $A(\psi)$, $B(\psi)$ and $V(\psi)$ are integrable according to the existence of nonlocal symmetries, consequently upon the existence of nonlocal conservation laws, because of the existence of the constraint equation.*

The proof of Proposition 3 is straightforward, by applying the same procedure with that of [86], thus we omit it. The result summarized in this proposition 3 is very important because if we elect to solve the field equations numerically for any functional form of the three unknown functions, we know that the numerical solutions correspond to actual solutions.

6 Conclusions

In this work, we introduced an Einstein-æther scalar-tensor cosmological model inspired by the Einstein-æther scalar field model proposed in [40] in which the æther coupling functions depend upon the scalar field. In our proposed model, we assumed that the scalar field is defined in the Jordan frame and that it is coupled to gravity. In the case of a spatially flat FLRW background space, the field equations describe the evolution of the scale factor and the scalar field, while there are three unknown functions that should be defined. These functions are related to the scalar field potential, the scalar-tensor coupled function to gravity, and the æther coefficient functions. The field equations are described by a minisuperspace and a point-like Lagrangian. This is an important observation because important techniques from Analytic Mechanics can be applied.

We investigated the integrability property of the field equations and the existence of analytic solutions. We were able to define a new set of variables in the minisuperspace and eliminate two of the three unknown functions of the model. We end with a dynamical system in which only the scalar field potential drives the dynamics. The other two unknown functions have been eliminated by a change in the lapse function of the background space and by a coordinate transformation. These functions play a significant role in the physical properties of the cosmological solution, but they do not affect the dynamics in the new variables.

To understand the evolution of the physical variables, we performed a detailed study of the field equations by investigating the stationary points in the dimensionless variables of the H -normalization approach. For exponential functional forms of the coupled variables and exponential scalar field potential, we found that there exist three stationary points which in general describe scaling solutions. The effective cosmological fluid in two of the stationary points depends only upon the kinetic part of the scalar field, while the scalar field potential contributes in the third point. These solutions can be seen as analogs in the Jordan frame of the Lorentz-violation inflationary solutions which were found in [40] defined in the Einstein frame. Moreover, for an arbitrary potential function, we found that the stationary points belong to the three families described by the exponential potential with the addition of a new stationary point which provides a de Sitter universe.

Finally, we performed a symmetry classification for the field equations according to the admitted Noether point symmetries, inspired by Ovsiannikov's classification scheme. We

found that for the scalar field potential, the field equations admit additional conservation laws and the cosmological solution can be written in a closed-form expression. In addition, this closed-form solution can describe the generic evolution which was found by the analysis of the stationary points in the H -normalization. That is a very interesting result. We can relate the analysis of the stationary points with an actual solution for the dynamical system.

From this analysis on the extension of the Einstein-aether scalar field model in the Jordan frame, we can conclude that the cosmological model can be physically viable since it can provide important eras in the cosmological evolution. In a future study, we plan to investigate further the physical properties of the model as also to investigate the effects of the conformal transformation in the physical properties of the theory.

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