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Finite quantum field theory and renormalization group

M. A. Green[1], **J. W. Moffat**[1,2,a]

[1] Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada
[2] Department of Physics and Astronomy, University of Waterloo, Waterloo, ON N2L 3G1, Canada

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Abstract Renormalization group methods are applied to a scalar field within a finite, nonlocal quantum field theory formulated perturbatively in Euclidean momentum space. It is demonstrated that the triviality problem in scalar field theory, the Higgs boson mass hierarchy problem and the stability of the vacuum do not arise as issues in the theory. The scalar Higgs field has no Landau pole.

1 Introduction

An alternative version of the standard model (SM), constructed using an ultraviolet finite quantum field theory with nonlocal field operators, was investigated in previous work [1,2]. In place of Dirac delta functions, $\delta(x)$, the theory uses distributions $\mathcal{E}(x)$ based on finite-width Gaussians. The Poincaré and gauge-invariant model adapts perturbative quantum field theory (QFT), with a finite renormalization, to yield finite quantum loops. For the weak interactions, $SU(2) \times U(1)$ is treated as an *ab initio* broken symmetry group with nonzero masses for the W and Z intermediate vector bosons and for left and right quarks and leptons. The model guarantees the stability of the vacuum. Two energy scales, Λ_M and Λ_H, were introduced; the rate of asymptotic vanishing of all coupling strengths at vertices not involving the Higgs boson is controlled by Λ_M, while Λ_H controls the vanishing of couplings to the Higgs. Experimental tests of the model, using future linear or circular colliders, were proposed. The present observations are consistent with $\Lambda_M \geq 10$ TeV. The Higgs boson mass hierarchy problem will be solved if future experiments confirm the prediction $\Lambda_H \lesssim 1$ TeV.

In the following, we will investigate the consequences of an application of renormalization group (RG) methods for the perturbative finite renormalizable model. We will concentrate on a nonlocal spin 0 scalar field $\phi = \phi_H$ Lagrangian model which is perturbatively formulated in Euclidean momentum space and might describe the Higgs boson field if nonlocality were fundamental. The ultraviolet finite theory resolves the Higgs mass hierarchy problem, the scalar field model triviality problem and removes the Landau pole singularity for the Higgs field.

[a] e-mail: jmoffat@perimeterinstitute.ca (corresponding author)

2 Scalar field theory

The Lagrangian we consider for a real scalar field $\phi \equiv \phi_H$ describing the Higgs boson in Euclidean space is

$$\mathcal{L}_H = \frac{1}{2}(-\phi\Box\phi + m_0^2\phi^2) + \frac{1}{4!}\lambda_0\phi^4. \tag{1}$$

Using the formalism of [3], we assume that the vacuum expectation of the bare field ϕ vanishes and write $\phi = Z^{1/2}\phi_r$, where ϕ_r is the renormalized field. Expressed as series expansions in powers of the physical coupling λ, mass m and energy scale Λ_H, the field strength renormalization constant Z and the bare parameters m_0 and λ_0 are given by:

$$Z = 1 + \delta Z(\lambda, m, \Lambda_H^2), \tag{2}$$

$$Zm_0^2 = m^2 + \delta m^2(\lambda, m, \Lambda_H^2), \tag{3}$$

$$Z^2\lambda_0 = \lambda + \delta\lambda(\lambda, m, \Lambda_H^2). \tag{4}$$

The propagator in Euclidean momentum space is given by

$$i\Delta_H(p) \equiv \frac{i\mathcal{E}^2(p)}{p^2 + m^2}, \tag{5}$$

where $\mathcal{E}(p)$ is the entire function:

$$\mathcal{E}(p) = \exp\left[-\left(\frac{p^2 + m^2}{2\Lambda_H^2}\right)\right]. \tag{6}$$

Evaluating the one-loop self-energy graph gives a constant shift to the Higgs boson bare self-energy [3]:

$$-i\Sigma_0 = \frac{-iZ^{-2}\lambda}{32\pi^2}m^2\,\Gamma\left(-1, \frac{m^2}{\Lambda_H^2}\right), \tag{7}$$

where $\Gamma(n, z)$ is the incomplete gamma function:

$$\Gamma(n, z) = \int_z^\infty dt\, t^{n-1}\exp(-t) = (n-1)\Gamma(n-1, z) + z^{n-1}\exp(-z). \tag{8}$$

Setting $n = 0$ in (8) gives:

$$\Gamma(0, z) = E_1(z) = \int_z^\infty dt\frac{\exp(-t)}{t} = -\ln(z) - \gamma - \sum_{n=1}^\infty \frac{(-z)^n}{nn!}, \tag{9}$$

$$\Gamma(-1, z) = -\Gamma(0, z) + \frac{\exp(-z)}{z}. \tag{10}$$

The renormalized one-loop self-energy $\Sigma_R(p^2)$ can then be written in the form:

$$\Sigma_R(p^2) = \delta Z(p^2 + m^2) + \delta m^2 + \frac{Z^{-1}\lambda}{32\pi^2}m^2\Gamma\left(-1, \frac{m^2}{\Lambda_H^2}\right) + \mathcal{O}(\lambda^2). \tag{11}$$

The renormalized mass and field strength are given by

$$\delta m^2 = -\frac{\lambda}{32\pi^2}m^2\Gamma\left(-1, \frac{m^2}{\Lambda_H}\right) + \mathcal{O}(\lambda^2), \tag{12}$$

$$\delta Z = \mathcal{O}(\lambda^2). \tag{13}$$

The expansion of the one-loop Higgs boson self-energy mass correction for $m \ll \Lambda_H$ is

$$\delta m^2 = \frac{\lambda}{32\pi^2}\left[-\Lambda_H^2 + m^2 \ln\left(\frac{\Lambda_H^2}{m^2}\right) + m^2(1-\gamma) + \mathcal{O}\left(\frac{m^2}{\Lambda_H^2}\right)\right] + \mathcal{O}(\lambda^2). \quad (14)$$

The one-loop vertex correction is given by

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2}\int_0^{1/2} dx\, \Gamma\left(0, \frac{1}{1-x}\frac{m^2}{\Lambda_H^2}\right) + \mathcal{O}(\lambda^3). \quad (15)$$

For $m \ll \Lambda_H$, this can be expanded for the Higgs boson to give

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2}\left[\frac{1}{2}\ln\left(\frac{\Lambda_H^2}{m^2}\right) + \frac{1}{2}(\ln(2) - 1 - \gamma) + \mathcal{O}\left(\frac{m^2}{\Lambda_H^2}\right)\right] + \mathcal{O}(\lambda^3). \quad (16)$$

3 Callan–Symanzik equation and running of λ

Let us consider the Callan–Symanzik equations [4–7] satisfied with our energy (length) scales Λ_i playing the roles of finite renormalization scales. In finite QFT theory, the equations for the regularized amplitudes $\Gamma^{(n)}(x-x')$ are

$$\left[\Lambda_i \frac{\partial}{\partial \Lambda_i} + \beta(g_i)\frac{\partial}{\partial g_i} - 2\gamma(g_i)\right]\Gamma^{(n)} = 0, \quad (17)$$

where g_i are the running coupling constants associated with diagram vertices. The correlation functions will satisfy this equation for the nth-order $\Gamma^{(n)}$ for the Gell-Mann–Low functions $\beta(g_i)$ and the anomalous dimensions in nth-loop order.

For the Higgs field, the RG equation is given by

$$\left[\Lambda_H \frac{\partial}{\partial \Lambda_H} + \beta(\lambda)\frac{\partial}{\partial \lambda} - 2\gamma(\lambda)\right]\Gamma^H = 0. \quad (18)$$

where the coupling λ runs with Λ_H. Neglecting the anomalous dimension term $\gamma(\lambda)$ and replacing the measured Higgs mass m by the RG scaling mass μ yields the equation:

$$\beta(\lambda) = -\frac{d\lambda}{d\ln\left(\frac{\Lambda_H}{\mu}\right)}. \quad (19)$$

We obtain from (15) the Higgs field β function:

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} I(\mu^2/\Lambda_H^2) + \mathcal{O}(\lambda^3), \quad (20)$$

where

$$I(\mu^2/\Lambda_H^2) = \int_0^{1/2} dx\, \Gamma\left(0, \frac{1}{1-x}\frac{\mu^2}{\Lambda_H^2}\right). \quad (21)$$

Using the identities $\Gamma(0, y) = E_1(y) = -\mathrm{Ei}(-y)$ yields:

$$I(\mu^2/\Lambda_H^2) = -\int_0^{1/2} dx\, \mathrm{Ei}\left(-\frac{1}{1-x}\frac{\mu^2}{\Lambda_H^2}\right)$$
$$= \frac{1}{2}\left(\exp\left(\frac{-2\mu^2}{\Lambda_H^2}\right) + \left(1 + \frac{2\mu^2}{\Lambda_H^2}\right)\mathrm{Ei}\left(\frac{-2\mu^2}{\Lambda_H^2}\right)\right)$$

$$-\exp\left(\frac{-\mu^2}{\Lambda_H^2}\right)-\left(1+\frac{\mu^2}{\Lambda_H^2}\right)\mathrm{Ei}\left(\frac{-\mu^2}{\Lambda_H^2}\right). \tag{22}$$

We have $\lambda = \lambda_0 + \delta\lambda$ and

$$\frac{d\lambda}{d\left(\frac{\Lambda_H}{\mu}\right)} = \frac{d\delta\lambda}{d\ln\left(\frac{\Lambda_H}{\mu}\right)} = -\beta(\lambda). \tag{23}$$

From (20), we obtain

$$\frac{d\lambda}{\lambda^2} = -\frac{3}{16\pi^2} dI(\mu^2/\Lambda_H^2). \tag{24}$$

Integrating this equation, we get

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + J(\mu^2/\Lambda_H^2), \tag{25}$$

where

$$J(\mu^2/\Lambda_H^2) = \frac{3}{16\pi^2}\int \frac{d\Lambda_H}{\Lambda_H} I(\mu^2/\Lambda_H^2). \tag{26}$$

Evaluating the integral for $J(\mu^2/\Lambda_H^2)$, using $x = \frac{\mu^2}{\Lambda_H^2}$, gives

$$\begin{aligned} J(x) = \frac{3}{128\pi^2}\big(&-2\exp(-2x)+4\exp(-x)+\pi^2-(2+4x)\mathrm{Ei}(-2x)+(4+4x)\mathrm{Ei}(-x)\\ &+4x\,{}_3\mathrm{F}_3(1,1,1;2,2,2;-2x)-4x\,{}_3\mathrm{F}_3(1,1,1;2,2,2;-x)\\ &-\ln(2)^2-\ln(4)\gamma+2(\gamma-\ln(2))\ln(x)+\ln(x)^2\big), \end{aligned} \tag{27}$$

where ${}_p\mathrm{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is a generalized hypergeometric function.

From (25), we obtain:

$$\lambda = \frac{\lambda_0}{1+\lambda_0 J(\mu^2/\Lambda_H^2)}, \tag{28}$$

or

$$\lambda_0 = \frac{\lambda}{1-\lambda J(\mu^2/\Lambda_H^2)}. \tag{29}$$

We can compare (25) with the equation obtained in SM:

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{3}{16\pi^2}\ln\left(\frac{\Lambda_C}{\mu}\right), \tag{30}$$

or

$$\lambda = \frac{\lambda_0}{1+\frac{3\lambda_0}{16\pi^2}\ln\left(\frac{\Lambda_C}{\mu}\right)}, \tag{31}$$

and

$$\lambda_0 = \frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\ln\left(\frac{\Lambda_C}{\mu}\right)}. \tag{32}$$

In the SM, the $\lambda\phi^4$ model is renormalizable and produces finite scattering amplitudes and cross sections, but renormalization theory demands that the cutoff Λ_C must be taken to infinity, $\Lambda_C \to \infty$ [8,9]. Then, from (30), the renormalized coupling constant $\lambda = 0$. This is known as the triviality problem [10–15]. This result holds even in the limit $\lambda_0 \to \infty$:

$$\frac{1}{\lambda} \sim \frac{3}{16\pi^2}\ln\left(\frac{\Lambda_C}{\mu}\right). \tag{33}$$

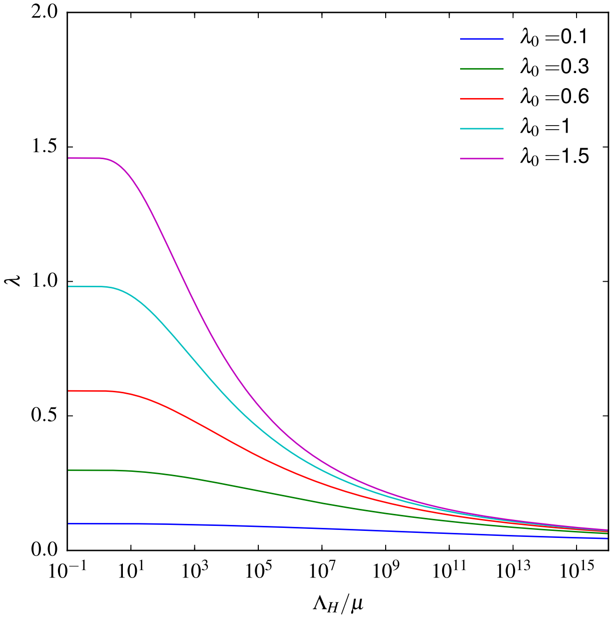

Fig. 1 Running of λ versus Λ_H/μ for finite QFT

In the earlier paper [16], it was demonstrated that the triviality problem for the scalar field field could be resolved in the finite QFT theory. Because $\Lambda_H = 1/\ell_H$ is a fundamental constant to be measured, it cannot be taken to infinity as in the case of infinite renormalization theory. Thus, we cannot take the limit $\ell_H \to 0$ corresponding to the δ-function limit. From Fig. 1, we observe that when we choose $\Lambda_H \lesssim 1$ TeV, the Higgs mass hierarchy problem is resolved, for we have $\delta m^2/m^2 \sim \mathcal{O}(1)$ where $m = 125$ GeV. From Fig. 1, we observe that for $\Lambda_H > \frac{1}{2}\mu$, we avoid a Landau pole and, in particular, for $700 < \Lambda_H < 1$ TeV, we resolve the triviality problem for the scalar Higgs field and the Higgs mass fine-tuning hierarchy problem.

Choosing an energy μ_0 above Λ_H as a measurement probe of the running of λ is attempting to make a measurement within the finite Gaussian distribution length size ℓ_H [1] and is prohibited within the perturbation approximations we have assumed. The results obtained for the running of λ are for a single Higgs particle interacting with another Higgs particle. This cannot describe a fully realistic situation, for the Higgs coupling to other particles such as the top quark (the top quark-Higgs coupling $\lambda_t \sim \mathcal{O}(1)$) may play an important role.

The above one-loop calculations have employed a perturbative formulation in Euclidean momentum space and rely on analytic continuation to obtain corresponding Lorentzian results. At tree level, the theory is completely equivalent to the classical field theory with the same Lagrangian, provided that spacetime Fourier transforms exist. Although loop diagrams should be finite at all levels, convergence of the quantum perturbative formalism has not been demonstrated. Because the Fourier transform to momentum space is generally not well-defined in curved spacetime, no claims can be made about whether or how the theory might be applied in the context of an expanding universe; the energy scales Λ_M, Λ_H may well-depend on emergent and evolving properties of the classical universe (e.g., entropy density), thus bridging the gap between quantum and classical. Whether a nonperturbative quantum formulation can be developed remains to be determined.

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