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First-order Darboux transformations for Dirac equations with arbitrary diagonal potential matrix in two dimensions

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Abstract We establish first-order Darboux transformations for the two-dimensional Dirac equation with diagonal matrix potential. The potential is allowed to depend arbitrarily on both variables. The systems we consider here include the scenario of a position-dependent mass as well as the massless case. Our Darboux transformations are more general than their existing counterparts (Pozdeeva and Schulze-Halberg in J Math Phys 51:113501, 2010).

1 Introduction

The synthesis of graphene [1] sparked a growing interest in Dirac materials [2,3]. These materials share the remarkable property that their low-energy electrons behave like massless relativistic particles. As such, the dynamics and in particular the confinement of these electrons is governed by the massless Dirac equation. While the two-dimensional version of the latter equation is sufficient when considering a monolayer of the Dirac material, multilayer structures require a model beyond the Dirac equation, see for example [4] for the bilayer case of graphene and [5] for the recently discovered twisted bilayer case. From a mathematical viewpoint, even for the monolayer situation the mathematical task of solving the governing two-dimensional Dirac equation is in general possible only by means of numerical approximations. A common scenario that has been studied thoroughly is the particular case of the system's potential depending on a single variable only. Under these circumstances, the Dirac equation can be decoupled into a scalar ordinary differential equation of second order that in very few cases allows for closed-form solutions. This method has been used to study Dirac systems coupled to magnetic fields [6–8], with purely electric fields [9,10], and including a position-dependent mass [11,12], among a large amount of other applications. We remark that the most general case of a Dirac equation with electromagnetic fields or a position-dependent mass can be obtained by assuming the potential to be of general matrix form, as it was done for example in [13]. The scarceness of Dirac systems that admit closed-form solutions motivates the search for methods that can generate such systems. A particularly useful class of methods is based on Darboux transformations. While upon its introduction

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the latter transformation was applicable to linear, second-order equations only [14–16], the formalism has been generalized to work with a wide variety of linear and nonlinear models, see [17] or [18] for an overview. These models include the Dirac equation, where Darboux transformations were established for both the one-dimensional and the two-dimensional case. In one dimension, Darboux transformations were first constructed for the Dirac equation as a conjugate mapping to the Schrödinger scenario [19]. Further and recent applications include a study of the Dirac Hamiltonian's most general supersymmetric structure exhibited in the time-dependent context [20], and in the stationary situation under the presence of coupling to scalar and pseudoscalar interactions [21]. The two-dimensional case was derived in [22], and applied in several scenarios, such as in Dirac systems coupled to magnetic fields [23,24], with position-dependent mass [25], for a linear potential [26], and through a dynamical systems approach [27], just to name a few examples. While in these applications the Dirac potential depends on a single variable, the purpose of the present work is to overcome this limitation. More precisely, we will adapt a first-order Darboux transformation for two-dimensional coupled Korteweg-de Vries equations [28] to the two-dimensional Dirac equation with diagonal matrix potential. We do not impose any restriction on the way the solutions or the potential depends on the two variables. In Sect. 2 the for us relevant results from [28] are summarized. The actual Darboux transformations are constructed in Sect. 3, while Sect. 4 is devoted to applications.

2 Preliminaries

In order to make this work self-contained, we will now briefly review partial results from [28]. Our starting point is the following equation in one dimension that can be related to coupled Korteweg-de Vries systems:

$$\chi'' - f\,\chi' + \lambda\, f\,\chi + e\,\chi - \lambda^2\,\chi = 0, \tag{1}$$

where the functions e, f, and the solution χ depend on the real variable x. Furthermore, the parameter λ is a complex-valued constant. In [28], Darboux transformations are defined for equations of the form (1). These transformations can be adapted to the Dirac equation [29] because after decoupling its form can be matched with (1). The latter equation can be generalized to a two-dimensional version by assuming all quantities in the equation to depend on two variables x and z. Furthermore, we make the formal replacements

$$' \rightarrow \frac{\partial}{\partial x} \qquad\qquad \lambda \;\rightarrow\; \frac{\partial}{\partial z}.$$

Upon implementation of these settings we can write down an initial and a transformed two-dimensional analogue of (1)

$$\frac{\partial^2 \chi}{\partial x^2} - \frac{\partial^2 \chi}{\partial z^2} - f\,\frac{\partial \chi}{\partial x} + f\,\frac{\partial \chi}{\partial z} + e\,\chi = 0 \tag{2}$$

$$\frac{\partial^2 \hat{\chi}}{\partial x^2} - \frac{\partial^2 \hat{\chi}}{\partial z^2} - \hat{f}\,\frac{\partial \hat{\chi}}{\partial x} + \hat{f}\,\frac{\partial \hat{\chi}}{\partial z} + \hat{e}\,\hat{\chi} = 0. \tag{3}$$

Note that for the sake of brevity we did not include arguments of the functions involved. In fact, all quantities depend on the two real variables x and z. The functions e, $\hat{e}$ and f, $\hat{f}$ are arbitrary, while χ and $\hat{\chi}$ stand for solutions of the respective equations. It turns out that we can connect the partner equations (2) and (3) by means of two Darboux transformations.

To this end, let u be a solution of the initial equation (2), note that we will refer to u as transformation function.

- We define a first Darboux transformation by means of

$$\hat{\chi} = \frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial z} - \frac{1}{u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\right)\chi. \tag{4}$$

This function is a solution to our equation (3), provided the contraints

$$\hat{e} = e - \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)\left[-\frac{1}{u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \tag{5}$$

$$\hat{f} = f + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\log\left[f - \frac{1}{u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\right)\right]. \tag{6}$$

are met.

- A second Darboux transformation is obtained by defining the function

$$\hat{\chi} = u\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z}\right)^{-1}\left(\frac{\partial \chi}{\partial x} - \frac{\partial \chi}{\partial z}\right) - \chi, \tag{7}$$

which solves equation (3) if the conditions

$$\hat{e} = e - \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)\left[e\, u\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z}\right)^{-1}\right] \tag{8}$$

$$\hat{f} = f + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)\log\left[u\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z}\right)^{-1}\right], \tag{9}$$

are satisfied. Observe that $\hat{e}$ vanishes if its initial counterpart e vanishes, and that this peculiar property does not occur in the first Darboux transformation.

In summary, both Darboux transformations (4)–(6) and (7)–(9) connect our two partner equations (2) and (3).

3 Darboux transformations for the Dirac equation

In this section we will adapt the construction from [28] to the case of the two-dimensional Dirac equation. Our starting point is the latter equation that can be written as

$$\left(\sigma_x\, p_x + \sigma_y\, p_y + V\right)\Phi = 0, \tag{10}$$

where σ_x, σ_y are the usual Pauli matrices, p_x, p_y stand for the momentum operators, $V = \text{diag}(V_{11}, V_{22})$ represents a diagonal 2×2 potential matrix, and Φ denotes a two-component solution. Upon substituting the explicit form of the momentum operators, we obtain

$$-\,i\,\sigma_x\,\frac{\partial \Phi}{\partial x} - i\,\sigma_y\,\frac{\partial \Phi}{\partial y} + V\,\Phi = 0. \tag{11}$$

Next, we split the solution into its component functions ϕ_1 and ϕ_2 via

$$\Phi = (\phi_1, \phi_2)^T\,. \tag{12}$$

We plug this setting into our Dirac equation (11), the components of which become

$$-\,i\,\frac{\partial \phi_1}{\partial x}+\frac{\partial \phi_1}{\partial y}+V_{22}\,\phi_2\;=\;0 \qquad\qquad -\,i\,\frac{\partial \phi_2}{\partial x}-\frac{\partial \phi_2}{\partial y}+V_{11}\,\phi_1\;=\;0. \tag{13}$$

In the particular case that the potential functions V_{11} and V_{22} depend on the variable x only, we can separate the y-dynamics off as a phase $\exp(i\,k\,y)$ in ϕ_1 and ϕ_2 for an arbitrary constant k_y. This is a common approach taken in the literature, as mentioned in Sect. 1. Decoupling of the system (13) then leads to a second-order equation in the variable x. Darboux transformations for this equation were defined and applied to the Dirac equation, see for example [25]. While the latter method relies on a potential that depends on a single variable, here we assume that the potential depends on both variables. Hence, in order to decouple our system (13), we must now substitute one of the equations into the other. We have to distinguish three cases here.

3.1 First case: $V_{11} \neq 0,\;\; V_{22} \neq 0$

We will now solve the left equation in (13) with respect to ϕ_2. This gives

$$\phi_2=\frac{1}{V_{22}}\left(i\,\frac{\partial \phi_1}{\partial x}-\frac{\partial \phi_1}{\partial y}\right), \tag{14}$$

Upon inserting this result into the right equation from (13) and collecting terms, we arrive at

$$\frac{\partial^2 \phi_1}{\partial x^2}+\frac{\partial^2 \phi_1}{\partial y^2}+\frac{1}{V_{22}}\left(-\frac{\partial V_{22}}{\partial x}+i\,\frac{\partial V_{22}}{\partial y}\right)\frac{\partial \phi_1}{\partial x}$$
$$-\frac{1}{V_{22}}\left(i\,\frac{\partial V_{22}}{\partial x}+\frac{\partial V_{22}}{\partial y}\right)\frac{\partial \phi_1}{\partial y}+\,V_{11}\,V_{22}\,\phi_1=0. \tag{15}$$

We will now demonstrate that equation (15) can be matched with the form (2). To this end, we first introduce the coordinate scaling

$$y\;=\;-\,i\,z \qquad\qquad \phi_1(x,y)\;=\;\chi(x,z). \tag{16}$$

By means of the chain rule, the derivatives in (15) transform as follows

$$\frac{\partial^2}{\partial y^2}\;=\;-\,\frac{\partial^2}{\partial z^2} \qquad\qquad \frac{\partial}{\partial y}\;=\;i\,\frac{\partial}{\partial z}. \tag{17}$$

We use these settings for implementing the coordinate switch (16) in our equation (15). For the sake of simplicity we will not change the name of our potential components V_{11} and V_{22}. We find

$$\frac{\partial^2 \chi}{\partial x^2}-\frac{\partial^2 \chi}{\partial z^2}-\frac{1}{V_{22}}\left(\frac{\partial V_{22}}{\partial x}+\frac{\partial V_{22}}{\partial z}\right)\frac{\partial \chi}{\partial x}+\frac{1}{V_{22}}\left(\frac{\partial V_{22}}{\partial x}+\frac{\partial V_{22}}{\partial z}\right)\frac{\partial \chi}{\partial z}+\,V_{11}\,V_{22}\,\chi=0, \tag{18}$$

note that both derivatives of V_{22} and χ with respect to the second variable are affected by the change of coordinate. Next, comparison of (18) with (2) brings up the following two conditions for matching.

$$e\;=\;V_{11}\,V_{22} \qquad\qquad f\;=\;\frac{1}{V_{22}}\left(\frac{\partial V_{22}}{\partial x}+\frac{\partial V_{22}}{\partial z}\right). \tag{19}$$

This system can be solved in closed form for the potential components. The result reads

$$V_{11} = \exp\left[-\int^{x} f(t, -x+z+t)\,\mathrm{d}t\right] \frac{e}{F(-x+z)}$$

$$V_{22} = \exp\left[\int^{x} f(t, -x+z+t)\,\mathrm{d}t\right] F(-x+z),$$

where F is an arbitrary function of its argument. Upon reverting the coordinate change (16), we can write

$$V_{11} = \exp\left[-\int^{x} f(t, -x+i\,y+t)\,\mathrm{d}t\right] \frac{e}{F(-x+i\,y)} \tag{20}$$

$$V_{22} = \exp\left[\int^{x} f(t, -x+i\,y+t)\,\mathrm{d}t\right] F(-x+i\,y). \tag{21}$$

Hence, if we choose the potential components in the form (20) and (21), our initial equation (2) matches its counterpart (18). Since the latter equation is linked to the Dirac equation (11) by means of (14) and (16), we can now adapt our Darboux transformations (4)–(6) and (7)–(9) to the Dirac context.

- *First Darboux transformation* We implement the coordinate change (16) in (4)-(6). The transformed solution is then obtained in the form

$$\hat{\phi}_1 = \frac{\partial \phi_1}{\partial x} - i\,\frac{\partial \phi_1}{\partial y} - \frac{1}{u}\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right)\phi_1. \tag{22}$$

  Similarly, the parameter functions $\hat{e}$ and $\hat{f}$ read

$$\hat{e} = e - \left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\left[-\frac{1}{u}\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right)\right] \tag{23}$$

$$\hat{f} = f + \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\log\left[f - \frac{1}{u}\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right)\right]. \tag{24}$$

  We can now determine the transformed Dirac potential matrix $\hat{V} = \mathrm{diag}(\hat{V}_{11}, \hat{V}_{22})$ from (20), (21) by replacing e and f with their transformed counterparts (23) and (24), respectively. The diagonal components of the latter matrix are given by

$$\hat{V}_{11} = \exp\left[-\int^{x} \hat{f}(t, -x+i\,y+t)\,\mathrm{d}t\right] \frac{\hat{e}}{G(-x+i\,y)} \tag{25}$$

$$\hat{V}_{22} = \exp\left[\int^{x} \hat{f}(t, -x+i\,y+t)\,\mathrm{d}t\right] G(-x+i\,y), \tag{26}$$

  note that G is an arbitrary function of its argument.
- *Second Darboux transformation* We now handle our second Darboux transformation (7)-(9) in the same way as its counterpart by implementing the coordinate change (16).

The transformed solution is expressed as

$$\hat{\phi}_1 = u \left(\frac{\partial u}{\partial x} + i \, \frac{\partial u}{\partial y} \right)^{-1} \left(\frac{\partial \phi_1}{\partial x} + i \, \frac{\partial \phi_1}{\partial y} \right) - \phi_1. \tag{27}$$

The associated parameter functions $\hat{e}$ and $\hat{f}$ take the form

$$\hat{e} = e - \left(\frac{\partial}{\partial x} + i \, \frac{\partial}{\partial y} \right) \left[e \, u \left(\frac{\partial u}{\partial x} + i \, \frac{\partial u}{\partial y} \right)^{-1} \right] \tag{28}$$

$$\hat{f} = f + \left(\frac{\partial}{\partial x} - i \, \frac{\partial}{\partial y} \right) \log \left[u \left(\frac{\partial u}{\partial x} + i \, \frac{\partial u}{\partial y} \right)^{-1} \right]. \tag{29}$$

We plug these expressions into (25) and (26) in order to obtain the components of the transformed Dirac potential matrix.

In summary, our Darboux transformations are applied as follows: starting out from an initial Dirac equation (11) with diagonal matrix potential V, we determine the parameter functions e and f by means of the relations (19). If we want to apply the first Darboux transformation, we calculate the transformed solution and parameters via (22) and (23), (24), respectively. The potential matrix is then found through (25), (26), while the solution of the transformed Dirac equation

$$\left(\sigma_x \; p_x + \sigma_y \; p_y + \hat{V} \right) \hat{\Phi} = 0, \tag{30}$$

is obtained by first using the function (22) to determine the second component of the Dirac solution as

$$\hat{\phi}_2 = \frac{1}{\hat{V}_{22}} \left(i \, \frac{\partial \hat{\phi}_1}{\partial x} - \frac{\partial \hat{\phi}_1}{\partial y} \right). \tag{31}$$

The full solution of the transformed Dirac equation (30) is then given by

$$\hat{\Phi} = \left(\hat{\phi}_1, \hat{\phi}_2 \right)^T. \tag{32}$$

For applying the second Darboux transformation, we evaluate (27)–(29). In the next step, the entries of the transformed potential matrix are found from (25), (26), and the corresponding solution of the transformed Dirac equation is found by means of (31) and (32). Recall that these results are only valid if both potential entries V_{11}, V_{22} of the initial Dirac potential matrix do not vanish.

3.2 Second case: $V_{11} = 0, \;\; V_{22} \neq 0$

If the potential entry V_{11} equals zero, the system (13) changes in the second equation. We have

$$-\,i \, \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial y} + V_{22} \, \phi_2 \; = \; 0 \qquad\qquad -\,i \, \frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_2}{\partial y} \; = \; 0. \tag{33}$$

Since now the second equation only depends on the solution component ϕ_2, we can solve it. This gives solutions of the form

$$\phi_2 = k_2(i \; x + y), \tag{34}$$

where k_2 is an arbitrary function of its argument. Next, we plug (34) into the first equation of (33) and solve with respect to ϕ_1. The result reads

$$\phi_1 = k_1(-i\,x + y) - i \int^x V_{22}(t, i\,t - i\,x + y)\, k_2(2\,i\,t - i\,x + y)\, \mathrm{d}t. \tag{35}$$

Note that k_2 is given in (35), and that k_1 is an arbitrary function of its argument. Hence, the functions ϕ_1 and ϕ_2, as defined in (34) and (35), provide a solution (12) of our initial Dirac equation (11) for the case $V_{11} = 0$. Consequently, we can now apply our Darboux transformations in the same way as in the previous case, taking into account that solutions of our initial Dirac equation must have the form (34) and (35). Before we conclude this paragraph, let us briefly focus on the second Darboux transformation, given by formulas (25)-(29). More precisely, we will now show that application of the latter Darboux transformation in the present case $V_{11} = 0$ leads to the results $\hat{V}_{11} = 0$ and $\hat{\phi}_2 = 0$. In other words, the top left entry of the transformed potential matrix vanishes, and so does the transformed second solution component. In order to derive these results, we start out with the observation that the current setting $V_{11} = 0$ forces $e = 0$, see (19). Next, inspection of the transformed function $\hat{e}$ in (8) reveals that $e = 0$ implies $\hat{e} = 0$. Note that this implication does not hold for the first Darboux transformation. Now, formula (25) dictates that $\hat{V}_{11}$ vanishes. Let us now focus on the second component of the transformed solution that can be determined by means of (14). We will show that the term in parenthesis equals zero if ϕ_1 has the form (35). To this end, we note that a transformation function for our Darboux transformation must have the same form as (35), that is, it must read

$$u = k_u(-i\,x + y) - i \int^x V_{22}(t, i\,t - i\,x + y)\, k_2(2\,i\,t - i\,x + y)\, \mathrm{d}t, \tag{36}$$

where k_u is an arbitrary function of its argument. Now we plug the function (35) along with (36) into the transformed solution (27). For the sake of simplicity we introduce an abbreviation for the integral in (35) and (36) as

$$I = -i \int^x V_{22}(t, i\,t - i\,x + y)\, k_2(2\,i\,t - i\,x + y)\, \mathrm{d}t.$$

Upon implementing this in (27), we note that the terms in parenthesis cancel because we have

$$\frac{\partial k_1(-i\,x + y)}{\partial x} + i\,\frac{\partial k_1(-i\,x + y)}{\partial y} = \frac{\partial k_u(-i\,x + y)}{\partial x} + i\,\frac{\partial k_u(-i\,x + y)}{\partial y} = 0.$$

Consequently, the transformed solution component (27) takes the form

$$\hat{\phi}_1 = k_u(-i\,x + y) - k_1(-i\,x + y). \tag{37}$$

We can now calculate the second component by inserting (37) into (14). This gives immediately $\hat{\phi}_2 = 0$ because the term in parenthesis vanishes. As mentioned before, these findings are true for the second Darboux transformation only.

3.3 Third case: $V_{11} \neq 0,\ V_{22} = 0$

Upon implementing this setting in our Dirac system (13), we obtain

$$-i\,\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial y} = 0 \qquad -i\,\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_2}{\partial y} + V_{11}\,\phi_1 = 0. \tag{38}$$

We observe that the first equation depends on ϕ_1 only and therefore permits solution. We obtain

$$\phi_1 = k_1(-i\, x + y), \tag{39}$$

introducing a function k_1 that depends arbitrarily on its argument. Upon insertion of (39) into the second equation in (33) we can solve for ϕ_2. This yields

$$\phi_2 = k_2(i\, x + y) - i \int^x V_{11}(t, -i\, t + i\, x + y)\, k_1(-2\, i\, t + i\, x + y)\, \mathrm{d}t. \tag{40}$$

The functions (39) and (40) form a solution of our initial Dirac equation (11) for the case $V_{22} = 0$. Let us now proceed with application of our Darboux transformations. To this end, we first note that our approach leading to (15) does not work under the present setting $V_{22} = 0$ because it generates undefined expressions in the latter equation. In order to circumvent this problem, we consider the Dirac system (38). Solving the second equation with respect to ϕ_1 gives

$$\phi_1 = \frac{1}{V_{11}} \left(i\, \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial y} \right).$$

Insertion of this result into the first equation of (38) leads to

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} - \frac{1}{V_{11}} \left(\frac{\partial V_{11}}{\partial x} + i\, \frac{\partial V_{11}}{\partial y} \right) \frac{\partial \phi_2}{\partial x} + \frac{1}{V_{11}} \left(i\, \frac{\partial V_{11}}{\partial x} - \frac{\partial V_{11}}{\partial y} \right) \frac{\partial \phi_2}{\partial y} = 0. \tag{41}$$

In the next step we apply the change of coordinates

$$x \;=\; -\, i\, z \qquad\qquad \phi_2(x, y) \;=\; \chi(x, z). \tag{42}$$

In comparison to (16), this time we switch the coordinate x rather than y. Implementation of (42) in our equation (41) gives

$$-\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial y^2} + \frac{1}{V_{11}} \left(\frac{\partial V_{11}}{\partial z} + \frac{\partial V_{11}}{\partial y} \right) \frac{\partial \chi}{\partial z} - \frac{1}{V_{11}} \left(\frac{\partial V_{11}}{\partial z} + \frac{\partial V_{11}}{\partial y} \right) \frac{\partial \chi}{\partial y} = 0, \tag{43}$$

Now, in order to match this equation with the required form (2), we make the identifications

$$e \;=\; 0 \qquad\qquad f \;=\; \frac{1}{V_{11}} \left(\frac{\partial V_{11}}{\partial z} + \frac{\partial V_{11}}{\partial y} \right). \tag{44}$$

Comparison of (43) with (2) shows that both equations match, provided we switch the coordinates x and y. Consequently, our equations for the first Darboux transformation (4)-(6) take the following form after reverting the coordinate change (42):

$$\hat{\phi}_2 = \frac{\partial \phi_2}{\partial y} - i\, \frac{\partial \phi_2}{\partial x} - \frac{1}{u} \left(\frac{\partial u}{\partial y} - i\, \frac{\partial u}{\partial x} \right) \phi_2$$
$$\hat{e} = e - \left(\frac{\partial}{\partial y} + i\, \frac{\partial}{\partial x} \right) \left[-\frac{1}{u} \left(\frac{\partial u}{\partial y} - i\, \frac{\partial u}{\partial x} \right) \right]$$
$$\hat{f} = f + \left(\frac{\partial}{\partial y} - i\, \frac{\partial}{\partial x} \right) \log \left[f - \frac{1}{u} \left(\frac{\partial u}{\partial y} - i\, \frac{\partial u}{\partial x} \right) \right].$$

In the same way we obtain the form of the second Darboux transformation (7)-(9). Reversion of the coordinate change (42) leads to the following equations:

$$
\begin{aligned}
\hat{\phi}_2 &= u \left(\frac{\partial u}{\partial y} + i \, \frac{\partial u}{\partial x} \right)^{-1} \left(\frac{\partial \phi_2}{\partial y} + i \, \frac{\partial \phi_2}{\partial x} \right) - \phi_2 \\
\hat{e} &= e - \left(\frac{\partial}{\partial y} + i \, \frac{\partial}{\partial x} \right) \left[e \, u \left(\frac{\partial u}{\partial y} + i \, \frac{\partial u}{\partial x} \right)^{-1} \right] \\
\hat{f} &= f + \left(\frac{\partial}{\partial y} - i \, \frac{\partial}{\partial x} \right) \log \left[u \left(\frac{\partial u}{\partial y} + i \, \frac{\partial u}{\partial x} \right)^{-1} \right].
\end{aligned}
$$

It remains to determine the transformed entries of the Dirac potential matrix. We can obtain them by switching variables in (25) and (26). This gives

$$
\begin{aligned}
\hat{V}_{11} &= \exp \left[\int^{y} \hat{f}(-x + i \; y + t, t) \, \mathrm{d}t \right] G(-y + i \; x) \\
\hat{V}_{22} &= \exp \left[- \int^{y} \hat{f}(-x + i \; y + t, t) \, \mathrm{d}t \right] \frac{\hat{e}}{G(-y + i \; x)},
\end{aligned}
$$

These equations conclude the construction of our Darboux transformations for the case $V_{22} = 0$. Let us point out that if the second Darboux transformation is applied to the present case, we obtain $\hat{V}_{22} = \hat{\phi}_1 = 0$. This follows from an argument analogous to the one given at the end of the previous paragraph.

3.4 Comparison with former work

As mentioned in Sect. 1, Darboux transformations for the two-dimensional Dirac equation (11) were introduced previously, see for example [22]. While a precise characterization of the relation between the two types of Darboux transformations—presented in the aforementioned reference and here, respectively—is beyond the scope of this work, we will use an argument for a particular scenario to show that the present transformation type is more general than its counterpart from [22]. To this end, let us consider the particular case of Dirac equations (11), where the potential depends on a single variable, say x, only. As a consequence, the y-dependency of the two-component solution (12) can be split off as $\exp(i k_y y)$ for a real-valued constant k_y. Within this scenario, it is known that the formalism considered in [22] can be expressed through the standard Darboux (SUSY) transformation for the Schrödinger equation [30]. In other words, Darboux transformations for the Dirac and the Schrödinger equation are conjugate mappings that are connected through decoupling of the Dirac equation. This result essentially goes back to a finding for the one-dimensional case [19]. In contrast, the formalism introduced in this work reduces to Darboux transformations for a generalized type of Schrödinger equation with quadratically energy-dependent potential [31]. The latter Darboux transformations are more general than their standard (SUSY) counterparts, and contain them as a special case [28,29]. Consequently, the Darboux transformations for the two-dimensional Dirac equation considered here generalize the ones introduced in [22].

4 Applications

In this section we will present two examples that demonstrate how our Darboux transformations can be used to generate solvable Dirac equations in two dimensions. In these examples our initial and transformed Dirac potentials depend on both variables.

4.1 Exponential-type potential

We start out from our Dirac equation (11) that we equip with the matrix potential V, given by

$$V = \begin{pmatrix} 30 \ \exp(y) \ \mathrm{sech}(x)^2 & 0 \\ 0 & \exp(-y) \end{pmatrix}. \tag{45}$$

Clearly, the diagonal entries of this matrix read

$$V_{11} = 30 \ \exp(y) \ \mathrm{sech}(x)^2 \qquad V_{22} = \exp(-y). \tag{46}$$

Note that the numerical factor was chosen in order to shorten notation in subsequent calculations. In general, the choice of the potential matrix (45) is motivated mainly by the fact that it admits elementary solutions of the Dirac equation, which keeps calculations transparent and the results of the Darboux transformations manageable. Now, as a first preparation for performing our Darboux transformations, we must determine a solution of our initial Dirac equation (11) for the matrix potential (45). This can be done by substituting the latter potential into the decoupled form (15) of our equation. This yields

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} - i \ \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial y} + 30 \ \mathrm{sech}(x)^2 \ \phi_1 = 0.$$

We observe that this equation can be solved by separation of variables. For our purposes it is sufficient to use this method for obtaining a particular solution that has the form

$$\phi_1 = \exp\left(\frac{i \ x}{2}\right) P_5^{\frac{1}{2} \ \sqrt{4 \ k-1}} [\tanh(x)] \left\{ c_1 \ \exp\left[-\left(\frac{1}{2} + \frac{1}{2} \ \sqrt{4 \ k - 1}\right) y\right] + \right.$$
$$\left. + c_2 \ \exp\left[-\left(\frac{1}{2} - \frac{1}{2}\sqrt{4 \ k - 1}\right) y\right]\right\}, \tag{47}$$

where P stands for a Legendre function of the first kind [32]. The function (47) is the first component of the solution (12) to our initial Dirac equation (11) for the potential (45). The remaining component is calculated via (14), we omit to state it here due to its excessive length. Next, in order to apply any of the two Darboux transformations introduced in Sect. 3, we must first determine the functions e and f that are defined in (19). Substitution of our diagonal potential entries (46) gives the results

$$e = 30 \ \mathrm{sech}(x)^2 \qquad f = i. \tag{48}$$

Note that for the sake of convenience we calculated these quantities using the y-coordinate, while in (19) they are expressed through the z-coordinate. Recall that the two coordinates are related to each other by means of the scaling (16) that connects first derivatives according to (17). Next, we choose our transformation function u as the following particular case of our solution (47)

$$u = (\phi_1)_{|c_1=1, c_2=0, k=n^2+1/4} = \exp\left(\frac{i \ x}{2} - \frac{y}{2} - i \ n \ y\right) P_5^n \ [\tanh(x)]. \tag{49}$$

At this point we are ready to perform our Darboux transformations.

First Darboux transformation The first Darboux transformation is given by equations (22)-(26). In order to construct the potential matrix $\hat{V}$ in our transformed Dirac equation, we calculate the quantities $\hat{e}$ and $\hat{f}$ according to (23) and (24), respectively. We obtain by substitution of (19) and (49)

$$\hat{e} = \frac{1}{P_5^n\,[\tanh(x)]^2}\Big\{-(n-6)^2\;P_6^n\,[\tanh(x)]^2 + (n-7)\;(n-6)\;P_7^n\,[\tanh(x)]\,P_5^n\,[\tanh(x)] + 36\;P_5^n\,[\tanh(x)]^2\,\mathrm{sech}(x)^2 + (n-6)\;P_6^n\,[\tanh(x)]\,P_5^n\,[\tanh(x)]\;\;\tanh(x)\Big\}. \tag{50}$$

Since the explicit form of $\hat{f}$ is rather long, we omit to state it here. Instead, let us now give particular cases for $n = 5$ and $n = 4$. In these two cases, the functions $\hat{e}$ and $\hat{f}$ become elementary. For $n = 5$ these expressions simplify to

$$\hat{e}_{|n=5} \;=\; 25\;\mathrm{sech}(x)^2 \qquad\qquad \hat{f}_{|n=5} \;=\; 1 + i - \tanh(x).$$

Substitution of $n = 4$ into (50) and into the corresponding function (24) yields

$$\hat{e}_{|n=4} = 25\;\mathrm{sech}(x)^2 - \frac{1}{\sinh(x)^2}$$

$$\hat{f}_{|n=4} = i - 4 - \frac{24}{\mathrm{cotanh}(x) - 5} - \mathrm{cotanh}(x) - \tanh(x).$$

In the next step we plug the functions $\hat{e}$ and $\hat{f}$ into the potential matrix entries (25) and (26). Since the corresponding expressions are very long in their general form, we state particular cases only. Let us first set $n = 5$, after simplifications we arrive at

$$\left(\hat{V}_{11}\right)_{|n=5} = 25\;\exp(y-x)\;\mathrm{sech}(x) \tag{51}$$

$$\left(\hat{V}_{22}\right)_{|n=5} = \exp(x-y)\;\mathrm{sech}(x), \tag{52}$$

where the arbitrary function G was chosen suitably in order to absorb complex terms. Figure 1 shows graphs of these potential entries.

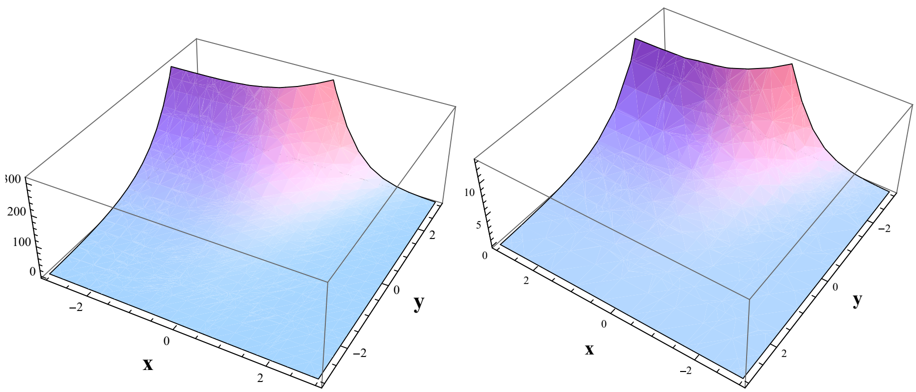

Fig. 1 Graphs of the potential entries (51) (left plot) and (52) (right plot)

Next we take $n = 4$, resulting in

$$\left(\hat{V}_{11}\right)_{|n=4} = \frac{4\ \exp(y)\left[3\ \exp(2\,x) - 2\right]}{1 - \exp(4\,x)}$$
$$\left(\hat{V}_{22}\right)_{|n=4} = \frac{4\ \exp(2\,x - y)\left[2\ \exp(2\,x) - 3\right]}{1 - \exp(4\,x)}.$$

As before, the function G absorbs complex-valued terms. We omit to show graphs of these functions. Let us now proceed to determining a solution of the transformed Dirac equation (30) for the potential matrix calculated above. To this end, we substitute (47) and the associated component (14) along with our transformation function (49) into the general solution forms (22) and (31). Since the resulting expressions are very long, we restrict ourselves to give a special case for $n = 5$ and $k = 101/4$. The solution components read

$$\hat{\phi}_1 = \exp\left[\frac{i\,x}{2} + \left(-\frac{1}{2} + 5\,i\right)y\right]\mathrm{sech}(x)^5$$
$$\hat{\phi}_2 = \frac{1}{1 + \exp(2\,x)}\ \exp\left[\left(1 + \frac{i}{2}\right)x + \left(\frac{1}{2} + 5\,i\right)y\right]\mathrm{sech}(x)^4.$$

These functions form the two-component solution (32) of the transformed Dirac equation (30) for the diagonal potential matrix $\hat{V}$, the diagonal entries are given by (51), (52). Solutions are visualized as probability densities $|\hat{\Phi}|^2 = |\hat{\phi}_1|^2 + |\hat{\phi}_2|^2$ in Fig. 2.

Second Darboux transformation Our second Darboux transformation consists of equations (25)-(29). In order to apply it, we first calculate the functions $\hat{e}$ and $\hat{f}$ from (23) and (24), respectively, by substitution of (19) and (49). Since the resulting expressions are very long, we will state particular cases only. Evaluation of (23) and (24) for $n = 5$ gives the following elementary functions

$$\hat{e}_{|n=5} = 24\ \mathrm{sech}(x)^2 \qquad\qquad \hat{f}_{|n=5} = 1 + i + \tanh(x).$$

In the same way we can generate another simple case if we substitute $n = 4$. This gives

$$\hat{e}_{|n=4} = 24\ \mathrm{sech}(x)^2 - \frac{24}{[\cosh(x) + 5\ \sinh(x)]^2}$$

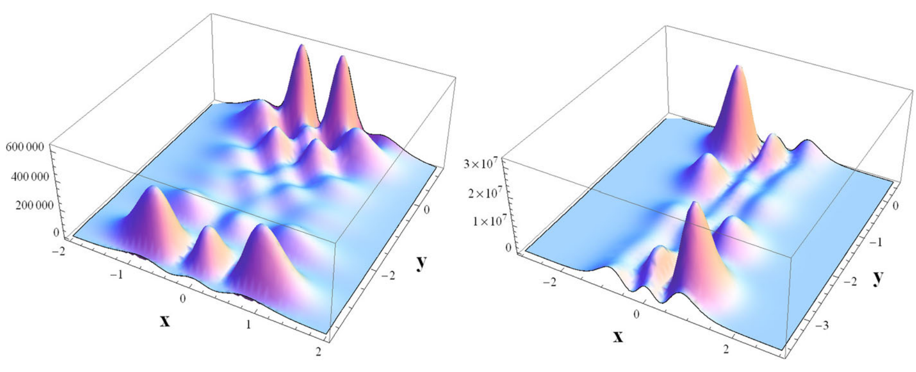

Fig. 2 Graphs of probability densities associated with the initial solution (47), (14) (left plot), and its transformed counterpart (22), (31) (right plot). Parameter settings are $k = 37/4$, and $c_1 = c_2 = 1$

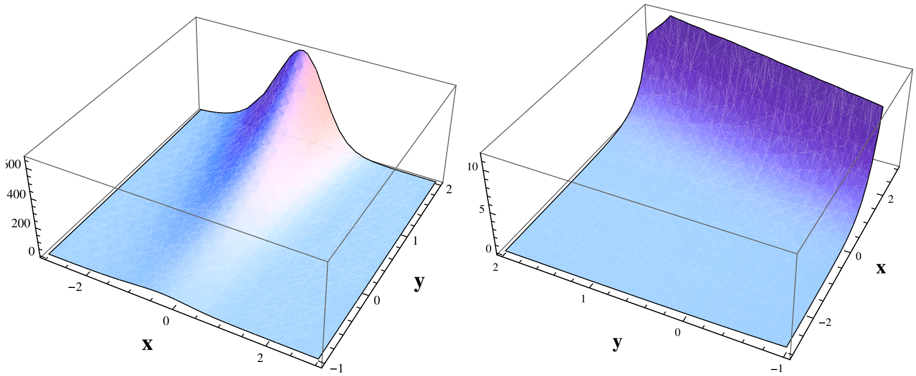

Fig. 3 Graphs of the potential entries (53) (left plot) and (54) (right plot)

$$\hat{f}_{|n=4} = i - 4 + \frac{24\ \sinh(x)}{\cosh(x) + 5\ \sinh(x)} + \text{cotanh}(x) + \tanh(x).$$

We can now determine the entries of the transformed potential matrix $\hat{V}$ that enters in the Dirac equation (30). To this end, we evaluate (25) and (26) for the present settings. For the sake of brevity we state the explicit form of the particular case $n = 5$ only. We get

$$\left(\hat{V}_{11}\right)_{|n=5} = 96\ \exp(x + y)\ \text{sech}(x)$$
$$\left(\hat{V}_{22}\right)_{|n=5} = \exp(x - y)\ \cosh(x).$$

Note that the arbitrary function G was chosen suitably in order to render the potential matrix entries real-valued. Let us present one more case for the setting $n = 4$. We obtain

$$\left(\hat{V}_{11}\right)_{|n=4} = \frac{240\ \exp(2\,x + y)\left[-3 + 7\ \exp(2\,x)\right]}{\left[1 + \exp(2\,x)\right]^3\left[-2 + 3\ \exp(2\,x)\right]} \tag{53}$$

$$\left(\hat{V}_{22}\right)_{|n=4} = \frac{\exp(-y)\left[\exp(4\,x) - 1\right]}{6\ \exp(2\,x) - 4}, \tag{54}$$

where as before we chose G chosen such as to absorb complex terms. Graphs of the functions (53) and (54) are shown in Fig. 3.

In the last step we will determine a solution to our transformed Dirac equation (30). To this end, we substitute our solution (47) and transformation function (49) into (27) and the remaining component (31). For $n = 5$ and $k = 101/4$ we obtain the explicit form

$$\hat{\phi}_1 = \exp\left[\frac{i\,x}{2} + \left(-\frac{1}{2} + 5\,i\right)y\right]\text{sech}(x)^4\left[\cosh(x) + \sinh(x)\right]$$
$$\hat{\phi}_2 = \frac{1}{\left[1 + \exp(2\,x)\right]^6}\ \exp\left[\left(7 + \frac{i}{2}\right)x + \left(\frac{1}{2} + 5\,i\right)y\right]\text{sech}(x)^8.$$

Figure 4 shows graphs of probability densities $|\hat{\Phi}|^2 = |\hat{\phi}_1|^2 + |\hat{\phi}_2|^2$ associated with solutions of the initial and transformed Dirac equations.

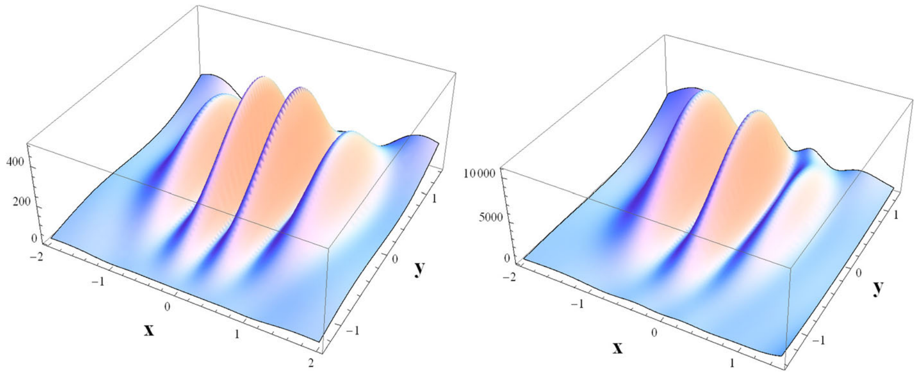

Fig. 4 Graphs of probability densities associated with the initial solution (47), (14) (left plot), and its transformed counterpart (22), (31) (right plot). Parameter settings are $k = 5/4$, and $c_1 = c_2 = 1$

4.2 Radially symmetric potential

In our next example we consider the initial Dirac equation (11) for the matrix potential

$$V = \begin{pmatrix} \kappa - x^2 - y^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{55}$$

where κ is a constant. We can read the diagonal entries off our potential matrix as

$$V_{11} = \kappa - x^2 - y^2 \qquad V_{22} = 1. \tag{56}$$

As in the previous example, we observe that our potential depends on both variables. We substitute the entries (56) into the second-order equation (15) resulting from decoupling (11). This gives

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + (\kappa - x^2 - y^2)\, \phi_1 = 0.$$

Using the technique of variable separation, we find the following solution to our equation

$$\begin{aligned} \phi_1 = &\left\{ c_1 \exp\left(-\frac{x^2}{2}\right) H_{\frac{\kappa_1 - 1}{2}}(x) + c_2 \exp\left(\frac{x^2}{2}\right) H_{\frac{-\kappa_1 - 1}{2}}(i\ x) \right\} \\ &\times \left\{ c_3 \exp\left(-\frac{y^2}{2}\right) H_{\frac{\kappa_2 - 1}{2}}(y) + c_4 \exp\left(\frac{y^2}{2}\right) H_{\frac{-\kappa_2 - 1}{2}}(i\ y) \right\}, \end{aligned} \tag{57}$$

where H stands for the Hermite function [32]. Furthermore, we introduce arbitrary constants c_1, c_2, c_3, c_4, and κ_1, κ_2. The last two constants must satisfy the constraint $\kappa = \kappa_1 + \kappa_2$. Recall that (57) gives the first component of the solution (12) to our initial Dirac equation (11), while the second component is found by means of (14). For the sake of brevity we omit to state the explicit form of this component here. Next, we will determine the functions e and f by means of (19). Insertion of our potential entries from (56) gives

$$e = \kappa - x^2 - y^2 \qquad f = 0. \tag{58}$$

Before we can proceed with our Darboux transformations, we need to define a transformation function. We will take a particular case of (57), given by

$$u = (\phi_1)_{c_1=c_4=0, c_2=c_3=1, \kappa_1=-\kappa_2=-1} = \exp\left(\frac{x^2-y^2}{2}\right). \tag{59}$$

We have now gathered all data in order to perform our Darboux transformations. In the following we will consider the first Darboux transformation (22)-(26) only. We start out by calculating the quantities $\hat{e}$ and $\hat{f}$ from (23) and (24), respectively. Substitution of the initial data (58) and the transformation function (59) gives

$$\hat{e} = \kappa - x^2 - y^2 \qquad \hat{f} = \frac{2}{x + i\, y}. \tag{60}$$

We observe that $\hat{e}$ equals its initial counterpart e from (58). We can now determine the entries of the transformed Dirac matrix potential by plugging (60) into (25) and (26). We obtain

$$\hat{V}_{11} = -\frac{(2\,C - x + i\,y)\,(-\kappa + x^2 + y^2)}{(x + i\,y)\,G(i\,x + y)} \tag{61}$$

$$\hat{V}_{22} = \frac{(x + i\,y)\,G(i\,x + y)}{2\,C - x + i\,y}, \tag{62}$$

where C is a constant of integration from (26), and G is an arbitrary function of its argument. While in the general form the potential entries (61) and (62) are complex-valued, we can choose G such that it absorbs the imaginary parts of both entries. Note that in the previous example we did this without actually showing the explicit form of G. Here we set

$$G(x) = x^2 - 2\,i\,C\,x,$$

which renders the entries of our transformed Dirac potential in the real-valued form

$$\hat{V}_{11} = \frac{\kappa}{x^2 + y^2} - 1 \qquad \hat{V}_{22} = x^2 + y^2. \tag{63}$$

We can now find a solution to our transformed Dirac equation (30) for the potential $\hat{V}$ featuring the diagonal entries (61) and (62). This solution is obtained by substituting our first solution component (57), its counterpart (14), and the transformation function (59) into (22) and (31). Since the resulting components of (32) have a very long explicit form, we restrict ourselves to show the particular case obtained for $\kappa_1 = -\kappa_2 = -1$. It is given by

$$\hat{\phi}_1 = -\frac{1}{2}\exp\left(-\frac{x^2+y^2}{2}\right)\left\{c_4\,\exp\left(x^2+y^2\right)\left[2\,c_2 + c_1\sqrt{\pi}\,\mathrm{erfc}(x)\right] + c_1\left[2\,c_3 + c_4\sqrt{\pi} - i\,c_4\sqrt{\pi}\,\mathrm{erfi}(y)\right]\right\} \tag{64}$$

$$\hat{\phi}_2 = \frac{1}{2\,G(i\,x+y)}\exp\left(-\frac{x^2+y^2}{2}\right)(-2\,i\,C + i\,x + y)\left\{c_4\,\exp\left(x^2+y^2\right) \times\left[2\,c_2 + c_1\sqrt{\pi}\,\mathrm{erfc}(x)\right] - c_1\left[2\,c_3 + c_4\sqrt{\pi} - i\,c_4\sqrt{\pi}\,\mathrm{erfi}(y)\right]\right\}, \tag{65}$$

where erfc and erfi stand for the complementary and the imaginary error function, respectively [32]. We observe that the probability density $|\hat{\Phi}|^2 = |\hat{\phi}_1|^2 + |\hat{\phi}_2|^2$ associated with the

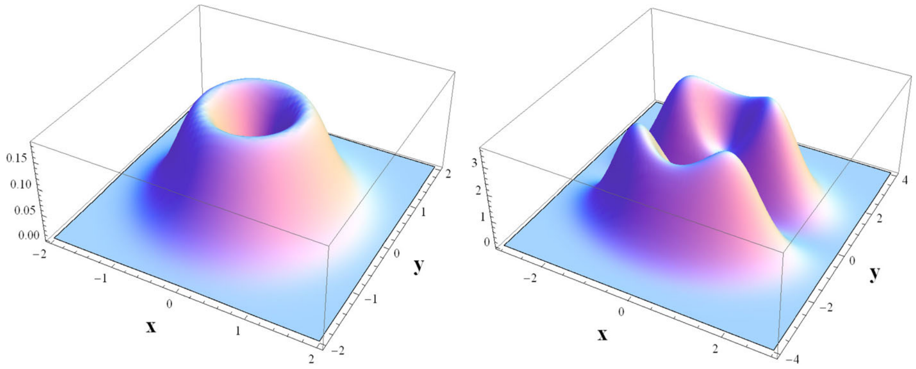

Fig. 5 Left plot: graphs of probability densities associated with the transformed solution components (66), (67) for $G = 1$ and $C = 0$. Right plot: graphs of probability densities associated with transformed solution components generated for the settings in (59) except $\kappa_2 = 3$. We also have $C = 0$ and $G = 1/x$

components (64) and (65) is in general unbounded except if we choose $c_4 = 0$. In this case the above components read

$$\hat{\phi}_1 = -\exp\left(-\frac{x^2+y^2}{2}\right) \tag{66}$$

$$\hat{\phi}_2 = -\frac{1}{G(i\,x+y)}\,\exp\left(-\frac{x^2+y^2}{2}\right)(-2\,i\,C+i\,x+y), \tag{67}$$

where for the sake of simplicity we also chose $c_2 = 0$. The left plot in Fig. 5 shows a graph of the probability density associated with these functions, where $C = 0$ and $G = 1$. We can construct different transformed potentials and the associated solutions of our Dirac equation (30) by modifying our transformation function (59). As an example, if we choose the same parameters in the latter function except for setting $\kappa_2 = 3$, we obtain a probability density that is visualized in the right plot of Fig. 5. Since the expressions generated by our Darboux transformation are very long, we omit to show them here.

4.3 The position-dependent mass scenario

We will now demonstrate that a two-dimensional Dirac system with a diagonal matrix potential can be interpreted as featuring a position-dependent mass and a scalar potential function. To this end, let us recall that the corresponding two-dimensional Dirac equation reads

$$\left(\sigma_x\,p_x+\sigma_y\,p_y+m\,\sigma_z+U\right)\Phi = 0, \tag{68}$$

where σ_z stands for the third Pauli matrix, m is the position-dependent mass function, and U represents a scalar potential. After substituting the momentum operators and summarizing terms, we can rewrite (68) in our form (11) with the following matrix potential

$$V = \begin{pmatrix} U+m & 0 \\ 0 & U-m \end{pmatrix}. \tag{69}$$

Consequently, any Dirac system with position-dependent mass m and scalar potential U can be written in standard form (11) with diagonal matrix potential (69). We can also reverse this statement by considering our Dirac equation (11) and imposing the constraint

$$\begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix} = \begin{pmatrix} U+m & 0 \\ 0 & U-m \end{pmatrix}. \tag{70}$$

Upon solving this equation with respect to the position-dependent mass m and the scalar potential U we obtain

$$m = \frac{V_{11} - V_{22}}{2} \qquad U = \frac{V_{11} + V_{22}}{2}. \tag{71}$$

Therefore, any diagonal matrix potential can be seen as a combination of a position-dependent mass and a scalar potential according to (71). As an example, let us consider the transformed potential entries in (63). Upon substituting them into (71) and choosing κ_1, κ_2 such that $\kappa = \kappa_1 + \kappa_2 = 0$, we find the following position dependent mass and scalar potential functions

$$m = \frac{x^2 + y^2 + 1}{2} \qquad U = \frac{1 - x^2 - y^2}{2}.$$

Instead of a scalar potential U we could also include the more general case of a matrix potential, which would generate additional free parameters in our equation (70).

5 Concluding remarks

The Darboux transformations introduced in the present work yield more general results than their formerly defined counterparts [22], and they overcome the common restriction of the Dirac potential depending on a single variable only. Despite these advantages, there are several technical issues that require further refinement in future research. For example, the potential matrix cannot have off-diagonal elements in order for our decoupled Dirac equation (15) to match the required form (2). The entries of the transformed potential matrix are determined through integration, see for example (25) and (26), where the integrals are hard to find in closed form because their integrand f is typically expressed through special functions. Also, at this point we do not have conditions for reality of the transformed potential or for regularity of the transformed solutions. A further point concerns solutions of the initial Dirac equation: due to the two-dimensional nature of the decoupled equation (15) it is difficult to find seed solutions for the generation of new solvable systems. Finally, a higher-order version of our Darboux transformations should be established. As mentioned above, this is subject to future research.

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