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$O(d, d)$ symmetry in teleparallel dark energy

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Abstract An important characteristic of the dilaton cosmological model is the Gasperini–Veneziano duality transformation which follows from the existence of the $O(d, d)$ symmetry. In this study, we consider the equivalent dilaton theory in teleparallel dark energy with the $O(d, d)$ symmetry, while the equivalent teleparallel-duality transformation is presented. The classical solution of the field equations is derived. Finally, the Wheeler–DeWitt equation of quantum cosmology is discussed.

1 Introduction

A main property of the conformal field theory is the duality symmetry. In [1], Veneziano introduced the duality symmetry in string theory in order to solve problem in the early universe. This approach opened a new subject of study known as string cosmology. String cosmology is based on the existence of a scalar field $\phi(x^k)$, known as the dilaton field, coupled to gravity. The gravitational action integral is defined as [2–4]

$$S_{\text{dilaton}} = \int d^D x \sqrt{-g} e^{-2\phi} \left(R - 4g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \Lambda \right), \tag{1}$$

where we have assumed the antisymmetric tensor strength of the sigma model, constructed by the three-form axion fields to be zero. Λ is the cosmological constant term, and R is the Ricciscalar of the background space with metric tensor $g_{\mu\nu}$. The action integral (1) has many similarities with the Brans–Dicke theory [5]. Indeed, the Brans–Dicke theory for a fixed Brans–Dicke parameter is recovered after the change of variables $\phi(x^k) = -\frac{1}{2}\ln\psi(x^k)$ [6].

The main properties of string cosmology are summarized in [7]. Specifically, inflation in string cosmology follows naturally without impose any fine-tune potential, the electromagnetic perturbations can describe the galactic magnetic fields, while matter perturbations remain small to support the homogeneity of the universe, for more details we refer the reader to [8].

In the case of a D-dimensional spatially flat and homogeneous background space, the Lagrangian function for the dilaton field depends only the scale-factor $a(t)$ and on the scalar field $\phi(t)$. The Lagrangian function admits the scale-factor duality property, that is, the action integral (1) is invariant under the transformation [1]

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$$a(t) \rightarrow a^{-1}(t) \ , \ \phi(t) \rightarrow \phi(t) - (D-1)\ln a. \tag{2}$$

The scale-factor duality transformation has been generalized and for the case of anisotropic and inhomogeneous spacetimes in [2]. Nowadays, it is known as Gasperini–Veneziano duality property. The duality symmetry is a discrete transformation and an isometry should exist in the background space [3,4]. The fundamental origin for the Gasperini–Veneziano transformation is the $O(d, d)$ symmetry [2]. Furthermore, for the Hubble function in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) space, the duality transformation reads $H \rightarrow -H$. Therefore, under the second discrete transformation $t \rightarrow -t$, it follows, $\dot{H}(t) \rightarrow \dot{H}(-t)$ which leads to the string-driven pre-big bang cosmology [9]. Furthermore, it was found that the dilaton field (1) admits a large number of symmetries which were used to solve completely the Wheeler–DeWitt equation of quantum cosmology [10]. In the case of cosmological studies, the scale-factor duality transformation has been attributed to a local transformation which leave invariant the action integral (1), that is, a variational symmetry [6]. Moreover, the viability of the discrete transformation under conformal transformations is investigated in [11]. In addition, a new mathematical construction approach for the determination of discrete transformations is established in [11].

In this piece of work, we are interested on the existence of discrete transformations, similar with the scale-factor duality transformation (2), in the case of teleparallel dark energy theory [12]. The aforementioned theory belongs to the so-called alternative/modified theories of gravity [13–22] which have been introduced in the last years by cosmologists in order to explain the cosmological observations [23,24]. Teleparallelism has drawn attention in the last years because it provides a systematic geometric description for the explanation of the cosmological observations. In the teleparallel equivalence of general relativity [25], instead of the torsion-less Levi–Civita connection, the curvatureless Weitzenböck connection is considered, where the corresponding dynamical fields are the four linearly independent vierbeins. Hence, the gravitational field are defined by the Weitzenböck tensor and its scalar T [26–28]. The teleparallel dark energy is the analogue of the scalar tensor theories, where a scalar field is introduced in teleparallel action integral, while the scalar field interacts with the scalar T for the Weitzenböck tensor. The theory is also known as scalar-torsion theory [29–32]. Under a conformal transformation [33], the theory can be related with the so-called $f(T, B)$ theory [34], which is a fourth-order theory of gravity, as an analogue of the equivalence of O' Hanlon theory with $f(R)$ gravity [35]. The evolution of the cosmological dynamics in teleparallel dark energy was studied before in [36–38]. Analysis of the cosmological observations with the teleparallel dark energy are presented in [39,40], while some other studies are given in [41,42]. From the latter studies, it is clear that the theory can play an important role for the description of the late-time and the early acceleration phases of the universe [40]. Moreover, from the analysis of the evolution for the matter perturbations in teleparallel dark energy, it was found that teleparallel dark energy theory is favoured with respect to the quintessence theory. Modified teleparallel theories of gravity are Lorentz violated theories. Nowadays, Lorentz violation has not been observed; however, Lorentz violation is a prediction for various models of quantum gravity, for a review see [43]. The motivation of this work is to define a teleparallel dark energy model which admits a discrete transformation and open the way for the teleparallel string cosmology. The plan of the paper is as follows.

In Sect. 2, we present the field equations for the teleparallel dark energy. In Sect. 3, we define the teleparallel dilaton model, which is invariant under a discrete transformation similar to the scale-factor duality transformation of the dilaton cosmological model. This new discrete transformation has its origin on the presence of the $O(d, d)$ symmetry. Moreover, the field equations are found to be superintegrable, and the analytic solution is expressed in

terms of exponential functions. Finally, in Sect. 4, we summarize our results, while we solve the Wheeler–DeWitt equation of quantum cosmology for the teleparallel dilaton model.

2 Teleparallel dark energy

In teleparallelism, the dynamical variables are the vierbein fields. They are defined by the requirement $g(e_i, e_j) = e_i.e_j = \eta_{ij}$ where $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ is the Lorentz metric in canonical form.

The metric tensor $g_{\mu\nu}(x^\kappa)$ in terms of coordinates is defined as

$$g_{\mu\nu} = \eta_{ij} h^i{}_\mu h^j{}_\nu, \tag{3}$$

where $e^i(x^\kappa) = h^i{}_\mu(x^\kappa) dx^i$ is the dual basis, in which $e^i(e_j) = \delta^i_j$.

In contrary to general relativity, the curvatureless teleparallel torsion tensor is the fundamental geometric object in teleparallelism, and is defined by the antisymmetric part of the affine connection coefficients as follows:

$$T_{\mu\nu}{}^\beta = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu} = h^\beta{}_i(\partial_\mu h^i{}_\nu - \partial_\nu h^i{}_\mu). \tag{4}$$

The gravitational Lagrangian for the teleparallel equivalent of general relativity is defined by the scalar $T = S_\beta{}^{\mu\nu} T^\beta{}_{\mu\nu}$ where $S_\beta{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}_\beta + \delta^\mu_\beta T^{\theta\nu}_\theta - \delta^\nu_\beta T^{\theta\mu}_\theta)$ and $K^{\mu\nu}_\beta$ is the tensor $K^{\mu\nu}_\beta = -\frac{1}{2}(T^{\mu\nu}_\beta - T^{\nu\mu}_\beta - T^{\mu\nu}_\beta)$. The latter tensor equals the difference of the Levi–Civita connection in the holonomic and the unholonomic frame.

As stated by the cosmological principle, the universe in large scales is homogeneous and isotropic described by the spatially flat FLRW metric

$$ds^2 = N^2(t) dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \tag{5}$$

Therefore, in order to recover such cosmological scenario, we assume the diagonal frame for the vierbein fields $h^i{}_\mu(t) = \text{diag}(1, a(t), a(t), a(t))$, where we calculate the scalar

$$T = 6H^2, \tag{6}$$

in which $H = \frac{1}{N}\frac{\dot{a}}{a}$ is the Hubble function.

The gravitational action integral in teleparallel dark energy theory is defined to be

$$S = \frac{1}{16\pi G} \int d^4x e \left[F(\phi) \left(T + \frac{\omega}{2} \phi_{;\mu} \phi^\mu + V(\phi) \right) \right]. \tag{7}$$

where $e = \sqrt{-g}$, $F(\phi)$ is the coupling function, ω is a constant nonzero parameter, analogue of the Brans–Dicke parameter and $V(\phi)$ is the scalar field potential. We remark that we can always define new scalar field under the point transformation $d\psi = \sqrt{\omega F(\phi)} d\phi$, such that the action integral (7) to be written as follows:

$$S = \frac{1}{16\pi G} \int d^4x e \left[\hat{F}(\psi) T + \frac{1}{2} \psi_{;\mu} \psi^\mu + \hat{V}(\psi) \right]. \tag{8}$$

The field equations in this cosmological model admit a minisuperspace description. Indeed, by replacing scar T from (6) in (7) and assuming that the scalar field ϕ inherits the symmetries of the background space, i.e. $\phi = \phi(t)$, the point-like Lagrangian for the field equations is

$$L(a, \dot{a}, \phi, \dot{\phi}) = F(\phi) \left(\frac{1}{N} \left(6a\dot{a}^2 - \frac{\omega}{2} a^3 \dot{\phi}^2 \right) + Na^3 V(\phi) \right). \tag{9}$$

Hence, the gravitational field equations are derived by the variation of the Lagrangian function (9). Indeed, the field equations are

$$F(\phi)\left(6H^2 - \frac{\omega}{2N^2}\dot{\phi}^2 - V(\phi)\right) = 0, \tag{10}$$

$$\left(\frac{2}{N}\dot{H} + 3H^2\right) + \frac{1}{2}\left(\frac{\omega}{2N^2}\dot{\phi}^2 - V(\phi)\right) + 2(\ln F(\phi))_{,\phi}H\frac{\dot{\phi}}{N} = 0, \tag{11}$$

$$\omega\left(\frac{1}{N^2}\ddot{\phi} + \frac{3}{N}H\dot{\phi} - \frac{\dot{N}}{N^3}\dot{\phi}\right) + \left(\frac{\omega}{2N^2}\dot{\phi}^2 + V(\phi)\right) + V_{,\phi}(\phi) = 0. \tag{12}$$

We continue our analysis by assuming specific functional forms for the coupling function $F(\phi)$ and the potential $V(\phi)$ in which the field equations remain invariant under discrete transformations as that of the scale-factor duality transformation for the dilaton field. Without loss of generality in the following, we assume the lapse function to be constant, i.e. $N(t) = 1$. In this case, Eq. (10) can be seen as a the constraint equation of the Hamiltonian for the second-order differential equations (11) and (12).

3 $O(d, d)$ symmetry in teleparallel dark energy

For the unknown functions of the point-like Lagrangian (9), that is, the coupling function and the potential, we consider that they are $F(\phi) = e^{-2\phi}$ and $V(\phi) = \Lambda$. Hence, the point-like Lagrangian (9) reads

$$L\left(a, \dot{a}, \phi, \dot{\phi}\right) = e^{-2\phi}\left(6a\dot{a}^2 - \frac{\omega}{2}a^3\dot{\phi}^2 + a^3\Lambda\right). \tag{13}$$

This cosmological model shall be called as the teleparallel dilaton model or dilaton-tensor model. As we shall see in the following, for this specific selection of the free functions, the point-like Lagrangian (13) is invariant under the $O(d, d)$ symmetry. We continue with the construction of the discrete transformation and the derivation of the $O(d, d)$ symmetry for the field equations.

We observe that under the scale-factor duality transformation (2), for $D = 4$, Lagrangian function (13) does not remain invariant. Thus, we should investigate for other forms for the discrete transformation.

However, we observe that under the change of variables

$$a \to \bar{a}^{p_1}e^{p_2\bar{\phi}}, \quad \phi \to p_4\bar{\phi} + p_3 \ln a \tag{14}$$

with

$$p_1 = \frac{1 + \kappa^2}{1 - \kappa^2}, \ p_2 = -\frac{4}{3(1 - \kappa^2)}, \ p_3 = \frac{3\kappa^2}{1 - \kappa^2}, \ p_4 = -\frac{1 + \kappa^2}{1 - \kappa^2}, \ \omega = \frac{4}{3\kappa^2}. \tag{15}$$

the Lagrangian function (13) becomes

$$L\left(a, \dot{a}, \phi, \dot{\phi}\right) = e^{-2\bar{\phi}}\left(6\bar{a}\left(\frac{d\bar{a}}{dt}\right)^2 - \frac{\omega}{2}\bar{a}^3\left(\frac{d\bar{\phi}}{dt}\right)^2 + a^3\Lambda\right). \tag{16}$$

Thus, the discrete transformation (14) with (15) is a symmetry for the teleparallel dilaton model. In the case of large values of κ, the discrete transformation (14) becomes $a \to \bar{a}^{-1}, \phi \to \bar{\phi} - 3 \ln \bar{a}$ which is the Gasperini–Veneziano scale-factor duality. Furthermore, for $\omega = \frac{8}{3}$, that is, $\kappa^2 = 1$, the discrete transformation does not exist.

In order to understand the origin of this discrete transformation, consider the point trans-formation

$$u(a, \phi; \kappa) = \frac{8}{3(1 + \kappa)} a^{\frac{3}{2}(1+\kappa)} \exp\left(-\frac{1 + \kappa}{\kappa}\phi\right), \qquad (17)$$

$$v(a, \phi; \kappa) = -\frac{1}{1 - \kappa} a^{\frac{3}{2}(1-\kappa)} \exp\left(\frac{1 - \kappa}{\kappa}\phi\right). \qquad (18)$$

Therefore, in the new variables, the point-like Lagrangian (13) becomes

$$L(u, v, \dot{u}, \dot{v}) = -\left(\dot{u}\dot{v} + \frac{3}{8}\left(1 - \kappa^2\right)\Lambda uv\right). \qquad (19)$$

Hence, the discrete transformation (14) in the new variables becomes $\{x \to \bar{y}, y \to \bar{x}\}$, which is the rotational symmetry in the two-dimensional plane, i.e. the origin of (14) is the $O(d, d)$ symmetry.

Moreover, in the new variables, from (19), the field equations reads

$$\dot{u}\dot{v} - \frac{3}{8}\left(1 - \kappa^2\right)\Lambda uv = 0, \qquad (20)$$

$$\ddot{u} - \frac{3}{8}\left(1 - \kappa^2\right)\Lambda u = 0, \qquad (21)$$

$$\ddot{v} - \frac{3}{8}\left(1 - \kappa^2\right)\Lambda v = 0. \qquad (22)$$

The latter system is the two-dimensional oscillator, a well-known superintegrable system. Hence, similarly with the dilaton field [10], the existence of $O(d, d)$ symmetry in the telepar-allel dark energy theory leads to a superintegrable system.

The closed-form solution is

$$u(t) = c_1 e^{\sqrt{\bar{\Lambda}}t} + c_2 e^{-\sqrt{\bar{\Lambda}}t}, \qquad (23)$$

$$v(t) = c_3 e^{\sqrt{\bar{\Lambda}}t} + c_4 e^{-\sqrt{\bar{\Lambda}}t}, \qquad (24)$$

with constraint $c_1 c_4 + c_2 c_3 = 0$ and $\bar{\Lambda} = \frac{3}{8}\left(1 - \kappa^2\right)\Lambda$. For large values of t and for $\bar{\Lambda} > 0$, the solution becomes asymptotically $u(t) \simeq e^{\sqrt{\bar{\Lambda}}t}$, $v(t) \simeq e^{\sqrt{\bar{\Lambda}}t}$, where it is clear that the final solution the scale factor is $a(t) \simeq e^{\Omega(\Lambda,\kappa)t}$, where $\Omega(\Lambda, \kappa)$ is a constant. We conclude that de Sitter inflation is natural in teleparallel dilaton theory.

Finally, for the Hubble function $H(t)$, we find that under the discrete transformation (14) is transformed as $H(t) \to p_1 \bar{H}(t) + p_2 \dot{\phi}(t)$, $\bar{H}(t) = \frac{d}{dt}(\ln \bar{a})$, where it is clear that for small values of ω, $H(t) \to -\bar{H}(t)$.

4 Conclusions

In this study, we generalized the dilaton cosmological model in the context of teleparallel dark energy theory. For our new model, we developed that the main properties of the dilaton field, i.e. the de Sitter inflation, and the superintegrable property for the field equations, hold and for the teleparallel dilaton model. The two theories share a common property, they admit an isometry which is the $O(d, d)$ symmetry.

For the teleparallel dilaton model, we determined a discrete transformation for the dynami-cal variables of the field equations, i.e. the scale-factor $a(t)$ and the scalar field $\phi(t)$, in which

the field equations remain invariant. This transformation is more general than the Gasperini–Veneziano scale-factor duality transformation, while the Gasperini–Veneziano transformation is recovered in the case of the teleparallel dilaton field when a free parameter for the model is very small.

As far as the Hubble function is concerned, the discrete symmetry in terms of the Hubble function reads $H(t) \to p_1(\kappa) \bar{H}(t) + p_2(\kappa) \dot{\phi}(t)$. Thus, for specific values of κ, the sign of $H(t)$ can be changed such that under the second change of variables $t \to -t$, we are able to study the pre-big bang epoch for the universe in a similar way as in string cosmology. However, because of the presence of the nonzero parameter $p_2(\kappa)$, the behaviour in the pre-big bang epoch in the teleparallel model is different from that of the dilaton field.

Finally, because of the existence of the minisuperspace Lagrangian (13), we are able to write the Wheeler–DeWitt equation of quantum cosmology [44], similarly with the analysis presented in [10]. The Hamiltonian constraint for the teleparallel dilaton model is written as:

$$\mathcal{H} \equiv e^{2\phi} \left(\frac{p_a^2}{6a} - \frac{p_\phi^2}{2\omega a^3} - a^3 \Lambda \right) = 0 \tag{25}$$

where the Wheeler–DeWitt equation reads $\mathcal{W} \equiv \mathcal{H}\Psi$, that is,

$$\mathcal{W} \equiv e^{2\phi} \left(\frac{1}{6a} \frac{\partial^2}{\partial a^2} - \frac{1}{2\omega a^3} \frac{\partial^2}{\partial \phi^2} - a^3 \Lambda \right) \Psi(a, \phi) = 0 \tag{26}$$

In the variables $\{u, v\}$ defined by expressions (17), (18), the Wheeler–DeWitt equation is written in the simplest form

$$\mathcal{W} \equiv \left(\frac{\partial^2}{\partial u \partial v} - \bar{\Lambda} u v \right) \Psi(u, v) = 0. \tag{27}$$

Equation (27) admits the quantum operator the $\left(\frac{\partial}{\partial u^2} - \frac{\partial}{\partial v^2} - \bar{\Lambda} \left(u^2 - v^2 \right) \right) \Psi = Q_0 \Psi$, which is nothing else than the Schrödinger equation for the two-dimensional (hyperbolic) oscillator. Thus, we stop our discussion here.

We showed that the teleparallel dilaton model has important characteristics similar to the classical dilaton model. That makes the model of special interests for future studies. Furthermore, the generalization of the new discrete transformation in the case of a anisotropic and inhomogeneous background space should be investigated.

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