



# Lie symmetries, optimal system, group-invariant solutions and dynamical behaviors of solitary wave solutions for a (3+1)-dimensional KdV-type equation

Sachin Kumar<sup>1,a</sup> , Dharmendra Kumar<sup>2</sup>, Abdul-Majid Wazwaz<sup>3</sup>

<sup>1</sup> Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India

<sup>2</sup> SGTB Khalsa College, University of Delhi, Delhi 110007, India

<sup>3</sup> Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA

Received: 1 December 2020 / Accepted: 4 May 2021

© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2021

**Abstract** Solitary waves are localized gravity waves that preserve their consistency and henceforth their visibility by the properties of nonlinear hydrodynamics. In this present work, numerous group-invariant solutions of the (3+1)-dimensional KdV-type equation are derived with the virtue of Lie symmetry analysis. Also, we obtain the corresponding infinitesimal generators, Lie point symmetries, geometric vector fields, commutator table and a one-dimensional optimal system of subalgebras. In addition, two-dimensional optimal system of subalgebra is also obtained using one-dimensional optimal system. Several interesting symmetry reductions and corresponding group-invariant solutions of the equation are obtained based on a one-dimensional optimal system of subalgebras. These group-invariant solutions include special functions like the Weierstrass Zeta function, *W*-shaped solitons, *M*-shaped solitons, bright-dark solitons, solitary waves and rogue waves which we furnish for the first time for this equation. The physical interpretation of the obtained solutions is discussed graphically based on numerical simulation through *Mathematica*. Furthermore, nonlocal conservation laws are studied via the Ibragimov approach for Lie point symmetries.

## 1 Introduction

Nonlinear evolution equations (NLEEs) and solitons are generally utilized to explain complex nonlinear physical phenomena in many emerging engineering areas, such as fiber optics, nonlinear dynamics, plasma physics, condensed matter, fluid dynamics, ion-acoustics, convective fluids and quantum field theory [1]. In view of the substantial role of solitons and nonlinear equations that play in these scientific fields, constructing exact analytic solutions for the NLEEs is of great value.

Lie symmetry analysis is a powerful method, that is highly used for solving nonlinear evolution equations (NLEEs) in many real-world physical problems in mathematical physics and other nonlinear wave phenomena. Because it is a strong method, it can be applied to various higher-order NLEEs, even if the equations are integrable or nonintegrable, linear or nonlinear [2, 3]. For a given NPDEs, there are many extensive applications in science and

<sup>a</sup> e-mail: [sachinambariya@gmail.com](mailto:sachinambariya@gmail.com) (corresponding author)

engineering from point symmetries, such as finding new solutions from relatively old ones [4], reducing dimensions of NPDEs by using similarity reductions and obtaining group-invariant solutions [5] and finding nonlocal conservation laws [6].

To obtain exact analytic solutions by applying Lie point symmetries, a one-dimensional optimal system is constructed. Then, our main goal is to analyze the dynamical behavior of the obtained solutions. An optimal system of symmetry algebra is studied through the characteristic equations and used methodically to classify the obtained symmetry subalgebras and group-invariant solutions. The concept of the optimal system was originated by Ovsiannikov [4]. This method was developed by Meleshko [7] and Ibragimov et al. [8]. However, Olver [2] formulated the optimal system by applying the adjoint table and the corresponding group classification. Nowadays, researchers are frequently following the modified method introduced by Olver [2].

In the mid-nineteenth century, John Scott Russell first investigated shallow-water solitary waves experimentally and noted their importance through nonlinear interactions [9, 10]. Both Boussinesq and Rayleigh established mathematically the existence of steady solitary waves on shallow water before KdV published their famous PDE, which was first originally derived by Boussinesq [11–13]. After this work, the theory of solitary waves remained almost untouched for 70 years until the mid-1960s when numerical studies by Zabusky and Kruskal [14] discovered the robust nature of soliton interactions, prompting an explosion of refined mathematical analysis on nonlinear PDEs. The history of solitary waves has been analyzed by Miles [15], and that of water waves more generally by Darrigol [16]. In 1895, Korteweg and de Vries (KdV) gave the first derivation of the NPDE

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1)$$

The evolution of small amplitude and long water waves down a canal of rectangular cross section is described by the KdV Eq. (1). This equation is characterized by the special waves which are known as solitons on shallow water surfaces [14]. Equation(1) has a number of connections with physical problems like shallow-water waves with weakly nonlinear restoring forces, ion-acoustic waves in plasma, acoustic waves on a crystal lattice and long internal waves in a density-stratified ocean.

The inverse scattering method and many other approaches were used to solve the KdV equation. Many other methods, such as the Hirota bilinear method, exp function method, Kudryashov simplest equation method, Darboux transformation method, the tanh method, and the Lie group of transformation method, were formally employed for solving this equation to make further progress and to obtain more results and conclusions [2, 3, 9, 17–28].

In this work, we study a (3+1)-dimensional KdV-type equation of the form

$$\Delta := u_t + 6u_x u_y + u_{xxy} + u_{xxxxz} + 60u_x^2 u_z + 10u_{xxx} u_z + 20u_x u_{xxz} = 0, \quad (2)$$

which was introduced by Lou [29] where five different types of multidromion solutions were obtained. In this equation,  $u = u(x, y, z, t)$ ,  $(x, y, z) \in \mathbb{R}^3$  and  $t > 0$ ,  $x$  is the direction of propagation while  $y$  and  $z$  are transverse variables. Further, Wazwaz [30] investigated one- and two-soliton solutions only with the help of a simplified form of Hirota's direct method established by Hereman and Nuseir [31]. In addition, the same problem was tackled by Ünsal [32] and obtained complexiton and interaction solutions by Hirota direct method. Liu et al. [33] constructed two homoclinic breather solutions and rogue wave solutions with the help of another method extended homoclinic test. Also, Mao et al. [34] used Bell polynomial approach to find Hirota's bilinear form equation, for exploring the rogue wave solution, the homoclinic breather wave solution, one-soliton solution and two-soliton solution. Wazwaz

introduced the concept of non-singular complexiton solutions for nonlinear partial differential equations in [35].

Motivated by the aforementioned references, the KdV Eq. (2) will be investigated by using the Lie symmetry approach. The prime objective of this paper is to obtain several symmetry reductions and numerous group-invariant solutions by using the Lie symmetry method. We study various exact closed-form solutions of the equation via the computerized symbolic calculations, including solitary waves solitons, single solitons, doubly solitons, multi-solitons, W-shaped solitons, M-shaped solitons and dark-bright solitons. Furthermore, the exact solutions of Eq. (2) are graphically analyzed through their profiles. This study reveals that waves that propagate in a certain medium are solitary waves specifically special M-shaped and W-shaped solitons. Eventually, we also discussed the physical interpretation of Eq. (2) via numerical simulation. Also, we depict the explicit conservation laws for the KdV equation.

The skeleton of the paper is organized as follows: In Sect. 2, we found the Lie point symmetries of Eq. (2) using Lie group analysis and all the geometric vector fields are presented. Also, group transformations are discussed in detail. In Sect. 3, an optimal system of a one-dimensional subalgebra of the Lie algebra  $L^{10}$  for Eq. (2) is constructed. We also obtain two-dimensional optimal system of symmetry subalgebra. In Sect. 4, we obtain the group-invariant forms and Lie symmetry reductions corresponding to the optimal system of subalgebra and their exact analytic solutions. Numerical simulation of obtained solution is discussed through *Mathematica 11.3*. Also, M-shaped and W-shaped soliton solutions are constructed in this section. In Sect. 5, adjoint equation and conservation laws are established using the nonlocal conservation theorem. The different dynamical wave structures of the established soliton solutions are addressed in Sect. 6. Finally, the concluding remarks are discussed in Sect. 7.

## 2 Lie symmetry analysis

In this section, we will utilize the powerful Lie symmetry method to obtain the numerous group-invariant solutions for the KdV type equation. If Eq. (2) is invariant under a one-parameter Lie group of transformations [2,3]:

$$\begin{aligned} \tilde{x} &= x + \epsilon \xi^1 + O(\epsilon^2), & \tilde{y} &= y + \epsilon \xi^2 + O(\epsilon^2), \\ \tilde{z} &= z + \epsilon \xi^3 + O(\epsilon^2), & \tilde{t} &= t + \epsilon \xi^4 + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon \eta + O(\epsilon^2), \end{aligned}$$

where  $\epsilon$  is a one-parameter with infinitesimal generator

$$\mathbf{V} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z} + \xi^4 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \tag{3}$$

where  $\xi^1, \xi^2, \xi^3, \xi^4$  and  $\eta$  are functions of independent variables, then the associated vector field given by Eq. (3) generates a Lie point symmetry of Eq. (2). Moreover, Eq. (3) must satisfy

$$\text{pr}^{(5)}\mathbf{V}(\Delta)|_{\Delta=0} = 0, \tag{4}$$

where  $\text{pr}^{(5)}\mathbf{V}$  denotes the fifth prolongation. So, applying  $\text{pr}^{(5)}\mathbf{V}$  to Eq. (2), then we obtain

$$\eta^t + 6\eta^x u_y + 6u_x \eta^y + \eta^{xy} + \eta^{xxxz} + 120\eta^x u_x u_z + 60u_x^2 \eta^z + 10\eta^{xxx} u_z$$

$$+ 10u_{xxx}\eta^z + 20\eta^x u_{xxz} + 20u_x \eta^{xxz} = 0, \tag{5}$$

with coefficients

$$\begin{aligned} \eta^t &= \mathcal{D}_t(\eta) - u_x \mathcal{D}_t(\xi^1) - u_y \mathcal{D}_t(\xi^2) - u_z \mathcal{D}_t(\xi^3) - u_t \mathcal{D}_t(\xi^4), \\ \eta^x &= \mathcal{D}_x(\eta) - u_x \mathcal{D}_x(\xi^1) - u_y \mathcal{D}_x(\xi^2) - u_z \mathcal{D}_x(\xi^3) - u_t \mathcal{D}_x(\xi^4), \end{aligned} \tag{6}$$

etc., where  $\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_z$  and  $\mathcal{D}_t$  denote the total derivatives for  $x, y, z$  and  $t$ , respectively. All the expressions of Eq. (6) into Eq. (4) are incorporated, and then by equating the same powers of  $u$  and its derivatives to zero, we get the desired system of determining equations:

$$\begin{aligned} \eta_t &= 0, \quad 2\eta_u = -\xi_t^4 + \xi_y^2, \quad 6\eta_x = \xi_t^2, \quad 6\eta_y = \xi_t^1, \\ 10\eta_z &= \xi_y^1, \quad \xi_u^4 = \xi_x^4 = \xi_y^4 = \xi_z^4 = \xi_{tt}^4 = 0, \\ \xi_u^1 &= 0, \quad 2\xi_x^1 = \xi_t^4 - \xi_y^2, \quad \xi_z^1 = \xi_{tt}^1 = \xi_{ty}^1 = \xi_{yy}^1 = 0, \\ \xi_u^2 &= \xi_x^2 = \xi_z^2 = \xi_{tt}^2 = \xi_{ty}^2 = \xi_{yy}^2 = 0, \\ \xi_t^3 &= \xi_u^3 = \xi_x^3 = 0, \quad 3\xi_y^3 = 10\xi_t^2, \quad \xi_z^3 = -\xi_t^4 + 2\xi_y^2, \end{aligned} \tag{7}$$

where  $\eta_t = \frac{\partial \eta}{\partial t}, \eta_x = \frac{\partial \eta}{\partial x}, \eta_u = \frac{\partial \eta}{\partial u}, \xi_x^1 = \frac{\partial \xi^1}{\partial x}, \xi_{tt}^4 = \frac{\partial^2 \xi^4}{\partial t^2}, \xi_{ty}^1 = \frac{\partial^2 \xi^1}{\partial t \partial y}, etc.$  Solving coupled partial differential equations given in Eq. (7) resulted in the following infinitesimal generators:

$$\begin{aligned} \xi^1 &= \frac{1}{4}(a_1 - a_5)x + a_8t + a_9y + a_{10}, & \xi^2 &= \frac{1}{2}(a_1 + a_5)y + \frac{3t}{10}a_4 + a_7, \\ \xi^3 &= a_4y + a_5z + a_6, & \xi^4 &= a_1t + a_3, \\ \eta &= \frac{1}{4}(a_5 - a_1)u + \frac{x}{20}a_4 + \frac{y}{6}a_8 + \frac{z}{10}a_9 + a_2, \end{aligned} \tag{8}$$

where  $a_i, (1 \leq i \leq 10)$  are arbitrary parameters. Hence, Lie algebra of vector fields of Eq. (2) is given as follows

$$\begin{aligned} \mathcal{V}_1 &= \frac{x}{4} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} - \frac{u}{4} \frac{\partial}{\partial u}, & \mathcal{V}_2 &= \frac{\partial}{\partial u}, \\ \mathcal{V}_3 &= \frac{\partial}{\partial t}, & \mathcal{V}_4 &= \frac{3t}{10} \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + \frac{x}{20} \frac{\partial}{\partial u}, \\ \mathcal{V}_5 &= \frac{-x}{4} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + \frac{u}{4} \frac{\partial}{\partial u}, & \mathcal{V}_6 &= \frac{\partial}{\partial z}, \\ \mathcal{V}_7 &= \frac{\partial}{\partial y}, & \mathcal{V}_8 &= t \frac{\partial}{\partial x} + \frac{y}{6} \frac{\partial}{\partial u}, \\ \mathcal{V}_9 &= y \frac{\partial}{\partial x} + \frac{z}{10} \frac{\partial}{\partial u}, & \mathcal{V}_{10} &= \frac{\partial}{\partial x}. \end{aligned} \tag{9}$$

To obtain the group transformation which is generated by the infinitesimal generator, we need to solve the following system of ordinary differential equations with the initial condition:

$$g_i : (x, y, z, t, u) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}), \tag{10}$$

which is generated by the generators of infinitesimal transformations  $\mathcal{V}_i$  for  $1 \leq i \leq 10$ . In order to get some exact solutions from known ones, we should find the Lie symmetry groups from the related symmetries. To get the Lie symmetry group, we should solve the following problems For this purpose, we need to solve the following system of ODEs

$$\frac{d}{d\epsilon}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) = \sigma(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}), \quad \text{when } (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u})|_{\epsilon=0} = (x, y, z, t, u),$$

where  $\epsilon$  is an arbitrary real parameter and

$$\sigma = \xi^1 u_x + \xi^2 u_y + \xi^3 u_z + \xi^4 u_t + \eta u. \tag{11}$$

So, we can obtain the Lie symmetry group

$$g : (x, y, z, t, u) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}). \tag{12}$$

According to different  $\xi^1, \xi^2, \xi^3, \xi^4$  and  $\eta$ , we have the following groups generated by each point symmetries given in the following form:

$$\begin{aligned} g_1 &:(x, y, z, t, u) \rightarrow (xe^\epsilon, ye^\epsilon, z, te^{4\epsilon}, ue^{-\epsilon}), \\ g_2 &:(x, y, z, t, u) \rightarrow (x, y, z, t, u + \epsilon), \\ g_3 &:(x, y, z, t, u) \rightarrow (x, y, z, t + \epsilon, u), \\ g_4 &:(x, y, z, t, u) \rightarrow (x, y + 6t\epsilon, z + 60t\epsilon^2 + 20y\epsilon, t, u + \epsilon x), \\ g_5 &:(x, y, z, t, u) \rightarrow (xe^{-\epsilon}, ye^{2\epsilon}, ze^{4\epsilon}, t, ue^\epsilon), \\ g_6 &:(x, y, z, t, u) \rightarrow (x, y, z + \epsilon, t, u), \\ g_7 &:(x, y, z, t, u) \rightarrow (x, y + \epsilon, z, t, u), \\ g_8 &:(x, y, z, t, u) \rightarrow (x + 6\epsilon t, y, z, t, u + \epsilon y), \\ g_9 &:(x, y, z, t, u) \rightarrow (x + 10\epsilon y, y, z, t, u + \epsilon z), \\ g_{10} &:(x, y, z, t, u) \rightarrow (x + \epsilon, y, z, t, u). \end{aligned}$$

The entries on the right side give the transformed point  $\exp(x, y, z, t, u) = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u})$ . The symmetry groups  $g_2, g_3, g_6, g_7$  and  $g_{10}$  demonstrate the space and time invariance of the equation. The well-known scaling symmetry turns up in  $g_1, g_4, g_5, g_8$  and  $g_9$ . We can obtain the corresponding new solutions by applying above groups  $g_i, 1 \leq i \leq 10$ .

If  $u = f(x, y, z, t)$  is a known solution of Eq. (2), then by using above groups  $g_i, 1 \leq i \leq 10$  corresponding infinite new solutions  $u_i, 1 \leq i \leq 10$  can be obtained as follows

$$\begin{aligned} u^{(1)} &= e^\epsilon f_1(xe^{-\epsilon}, ye^{-\epsilon}, z, te^{-4\epsilon}), & u^{(6)} &= f_6(x, y, z - \epsilon, t), \\ u^{(2)} &= f_2(x, y, z, t) - \epsilon, & u^{(7)} &= f_7(x, y - \epsilon, z, t), \\ u^{(3)} &= f_3(x, y, z, t - \epsilon), & u^{(8)} &= f_8(x - 6t\epsilon, y, z, t) - \epsilon y, \\ u^{(4)} &= f_4(x, y - 6t\epsilon, z - 60t\epsilon^2 - 20y\epsilon, t) - \epsilon x, & u^{(9)} &= f_9(x - 10\epsilon y, y, z, t) - \epsilon z, \\ u^{(5)} &= e^{-\epsilon} f_5(xe^\epsilon, ye^{-2\epsilon}, ze^{-4\epsilon}, t), & u^{(10)} &= f_{10}(x - \epsilon, y, z, t). \end{aligned}$$

Similarity variables, similarity forms and group-invariant solutions associated with any vector field  $V$  in (3) can be accomplished by its characteristic equation

$$\frac{dx}{\xi^1} = \frac{dy}{\xi^2} = \frac{dz}{\xi^3} = \frac{dt}{\xi^4} = \frac{du}{\eta}. \tag{13}$$

By applying the Lie symmetry reductions through the characteristic equation, then we obtain numerous group-invariant solutions by allocating the appropriate values to arbitrary constants  $a_i (1 \leq i \leq 10)$ . Subsequently, we compute one-dimensional and two-dimensional optimal system of symmetry subalgebras to a (3+1)-dimensional KdV equation.

**Table 1** Commutation relations for Eq. (9)

*	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_1$	0	$\frac{1}{4}\mathcal{V}_2$	$-\mathcal{V}_3$	$\frac{1}{2}\mathcal{V}_4$	0	0	$-\frac{1}{2}\mathcal{V}_7$	$\frac{3}{4}\mathcal{V}_8$	$\frac{1}{4}\mathcal{V}_9$	$-\frac{1}{4}\mathcal{V}_{10}$
$\mathcal{V}_2$	$-\frac{1}{4}\mathcal{V}_2$	0	0	0	$\frac{1}{4}\mathcal{V}_2$	0	0	0	0	0
$\mathcal{V}_3$	$\mathcal{V}_3$	0	0	$\frac{3}{10}\mathcal{V}_7$	0	0	0	$\mathcal{V}_{10}$	0	0
$\mathcal{V}_4$	$-\frac{1}{2}\mathcal{V}_4$	0	$-\frac{3}{10}\mathcal{V}_7$	0	$\frac{1}{2}\mathcal{V}_4$	0	$-\mathcal{V}_6$	0	$\frac{3}{10}\mathcal{V}_8$	$-\frac{1}{20}\mathcal{V}_2$
$\mathcal{V}_5$	0	$-\frac{1}{4}\mathcal{V}_2$	0	$-\frac{1}{2}\mathcal{V}_4$	0	$-\mathcal{V}_6$	$-\frac{1}{2}\mathcal{V}_7$	$\frac{1}{4}\mathcal{V}_8$	$\frac{3}{4}\mathcal{V}_9$	$\frac{1}{4}\mathcal{V}_{10}$
$\mathcal{V}_6$	0	0	0	0	$\mathcal{V}_6$	0	0	0	$\frac{1}{10}\mathcal{V}_2$	0
$\mathcal{V}_7$	$\frac{1}{2}\mathcal{V}_7$	0	0	$\mathcal{V}_6$	$\frac{1}{2}\mathcal{V}_7$	0	0	$\frac{1}{6}\mathcal{V}_2$	$\mathcal{V}_{10}$	0
$\mathcal{V}_8$	$-\frac{3}{4}\mathcal{V}_8$	0	$-\mathcal{V}_{10}$	0	$-\frac{1}{4}\mathcal{V}_8$	0	$-\frac{1}{6}\mathcal{V}_2$	0	0	0
$\mathcal{V}_9$	$-\frac{1}{4}\mathcal{V}_9$	0	0	$-\frac{3}{10}\mathcal{V}_8$	$-\frac{3}{4}\mathcal{V}_9$	$-\frac{1}{10}\mathcal{V}_2$	$-\mathcal{V}_{10}$	0	0	0
$\mathcal{V}_{10}$	$\frac{1}{4}\mathcal{V}_{10}$	0	0	$\frac{1}{20}\mathcal{V}_2$	$-\frac{1}{4}\mathcal{V}_{10}$	0	0	0	0	0

### 3 Optimal system of Lie subalgebras

We construct invariant function of the symmetry algebra  $L^{10}$  [25,36,37] in this section. As Olver [2] said, the investigation of that kind of an invariant is important as it places restrictions on what stage, we can expect to simplify  $\mathbf{V}$ . The ten-dimensional Lie algebra is generated through the obtained symmetry generators, given by Eq. (9). Also, it is easy to verify that the symmetry generators obtained in Eq. (9) form a closed Lie algebra whose commutation relations are provided in Table 1. Infinitesimal generators given in Eq. (2) can be furnished as a linear combination of  $\mathcal{V}_i$  as

$$\mathbf{V} = a_1\mathcal{V}_1 + a_2\mathcal{V}_2 + a_3\mathcal{V}_3 + a_4\mathcal{V}_4 + a_5\mathcal{V}_5 + a_6\mathcal{V}_6 + a_7\mathcal{V}_7 + a_8\mathcal{V}_8 + a_9\mathcal{V}_9 + a_{10}\mathcal{V}_{10}. \tag{14}$$

We observed that commutator Table 1 is skew-symmetric where  $(i, j)$ th entry of Table 1 is given by  $[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_i \cdot \mathcal{V}_j - \mathcal{V}_j \cdot \mathcal{V}_i$ . Even, the generators  $\mathcal{V}_i, 1 \leq i \leq 10$  are linearly independent.

Then, taking any subgroup  $g = e^W (W = \sum_{j=1}^{10} b_j \mathcal{V}_j)$  to act on  $V$ , we have

$$Ad_{\exp(\epsilon \mathbf{W})}(\mathbf{V}) = e^{-\epsilon \mathbf{W}} \mathbf{V} e^{\epsilon \mathbf{W}}, \tag{15}$$

$$= V - \epsilon[W, V] + \frac{1}{2!}\epsilon^2[W, [W, V]] - \dots \tag{16}$$

$$\begin{aligned} &= (a_1\mathcal{V}_1 + \dots + a_n\mathcal{V}_n) - \epsilon[b_1\mathcal{V}_1 + \dots + b_n\mathcal{V}_n, a_1\mathcal{V}_1 + \dots + a_n\mathcal{V}_n] + O(\epsilon^2) \\ &= (a_1\mathcal{V}_1 + \dots + a_n\mathcal{V}_n) - \epsilon(\Theta_1\mathcal{V}_1 + \dots + \Theta_n\mathcal{V}_n), \end{aligned} \tag{17}$$

where  $\Theta = \Theta(a_1, \dots, a_n, b_1, \dots, b_n)$  can be obtained by using the commutator table. The commutation relations are given in Table 1. Putting  $V = \sum_{i=1}^{10} a_i \mathcal{V}_i$  and  $W = \sum_{j=1}^{10} b_j \mathcal{V}_j$  in Eq. (15) with

$$\begin{aligned} \Theta_1 &= 0, \\ \Theta_2 &= \frac{1}{60}(3a_4b_{10} - 15a_1b_2 + 15a_5b_2 - 3a_{10}b_4 + 15a_2(b_1 - b_5)) \end{aligned}$$

$$\begin{aligned}
 &+ 6a_9b_6 + 10a_8b_7 - 10a_7b_8 - 6a_6b_9), \\
 \Theta_3 &= -a_3b_1 + a_1b_3, \\
 \Theta_4 &= \frac{1}{2}(a_4b_1 - a_1b_4 + a_5b_4 - a_4b_5), \\
 \Theta_5 &= 0, \\
 \Theta_6 &= a_7b_4 - a_6b_5 + a_6b_6 + a_4b_7, \\
 \Theta_7 &= \frac{1}{10}(3a_4b_3 - 3a_3b_4 - 5a_7(b_1 + b_5) + 5a_1b_7 + 5a_5b_7), \\
 \Theta_8 &= \frac{1}{20}(6a_9b_4 + 5a_8(3b_1 + b_5) - 15a_1b_8 - 5a_5b_8 - 6a_4b_9), \\
 \Theta_9 &= \frac{1}{4}(a_9(b_1 + 3b_5) - (a_1 + 3a_5)b_9), \\
 \Theta_{10} &= \frac{1}{4}(a_1b_{10} - a_5b_{10} + 4a_8b_3 + a_{10}(-b_1 + b_5) + 4a_9b_7 - 4a_3b_8 - 4a_7b_9).
 \end{aligned}$$

For any  $b_j, 1 \leq j \leq 10$ , it requires

$$\Theta_1 \frac{\partial \phi}{\partial a_1} + \Theta_2 \frac{\partial \phi}{\partial a_2} + \Theta_3 \frac{\partial \phi}{\partial a_3} \cdots + \Theta_9 \frac{\partial \phi}{\partial a_9} + \Theta_{10} \frac{\partial \phi}{\partial a_{10}} = 0. \tag{18}$$

Collecting the coefficients of all  $b_i$  in the above equation, ten differential equations about  $\phi(a_1, a_2, \dots, a_{10})$  are obtained as

$$\begin{aligned}
 b_1 : a_{10} \frac{\partial \phi}{\partial a_{10}} + 2a_7 \frac{\partial \phi}{\partial a_7} + 4a_3 \frac{\partial \phi}{\partial a_3} &= a_9 \frac{\partial \phi}{\partial a_9} + 3a_8 \frac{\partial \phi}{\partial a_8} + 2a_4 \frac{\partial \phi}{\partial a_4} + a_2 \frac{\partial \phi}{\partial a_2}, \\
 b_2 : (a_1 - a_5) \frac{\partial \phi}{\partial a_2} &= 0, \\
 b_3 : a_8 \frac{\partial \phi}{\partial a_{10}} + \frac{3}{10} a_4 \frac{\partial \phi}{\partial a_7} + a_1 \frac{\partial \phi}{\partial a_3} &= 0, \\
 b_4 : 6a_9 \frac{\partial \phi}{\partial a_8} + 10(a_5 - a_1) \frac{\partial \phi}{\partial a_4} &= 6a_3 \frac{\partial \phi}{\partial a_7} + 20a_7 \frac{\partial \phi}{\partial a_6} + a_{10} \frac{\partial \phi}{\partial a_2}, \\
 b_5 : a_{10} \frac{\partial \phi}{\partial a_{10}} + 3a_9 \frac{\partial \phi}{\partial a_9} + a_8 \frac{\partial \phi}{\partial a_8} &= 2a_7 \frac{\partial \phi}{\partial a_7} + 4a_6 \frac{\partial \phi}{\partial a_6} + 2a_4 \frac{\partial \phi}{\partial a_4} + a_2 \frac{\partial \phi}{\partial a_2}, \\
 b_6 : 10a_6 \frac{\partial \phi}{\partial a_6} a_9 \frac{\partial \phi}{\partial a_2} &= 0, \\
 b_7 : 6a_9 \frac{\partial \phi}{\partial a_{10}} + 3(a_1 + a_5) \frac{\partial \phi}{\partial a_7} + 6a_4 \frac{\partial \phi}{\partial a_6} + a_8 \frac{\partial \phi}{\partial a_2} &= 0, \\
 b_8 : 12a_3 \frac{\partial \phi}{\partial a_{10}} + 3(3a_1 + a_5) \frac{\partial \phi}{\partial a_8} + 2a_7 \frac{\partial \phi}{\partial a_2} &= 0, \\
 b_9 : 20a_7 \frac{\partial \phi}{\partial a_{10}} + 5(a_1 + 3a_5) \frac{\partial \phi}{\partial a_9} + 6a_4 \frac{\partial \phi}{\partial a_8} + 2a_6 \frac{\partial \phi}{\partial a_2} &= 0, \\
 b_{10} : 5(a_1 - a_5) \frac{\partial \phi}{\partial a_{10}} + a_4 \frac{\partial \phi}{\partial a_2} &= 0.
 \end{aligned} \tag{19}$$

As per the following references [25, 26, 36], the general invariant function of the symmetry algebra  $L^{10}$  is

$$\phi(a_1, \dots, a_{10}) = F(a_1, a_5),$$

**Table 2** Adjoint table of Lie algebra for (3+1)-dimensional KdV equation

$Ad_g$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$
$\mathcal{V}_1$	$\mathcal{V}_1$	$\mathcal{V}_2 e^{-\frac{\epsilon}{4}}$	$\mathcal{V}_3 e^\epsilon$	$\mathcal{V}_4 e^{-\frac{\epsilon}{2}}$	$\mathcal{V}_5$
$\mathcal{V}_2$	$\mathcal{V}_1 + \frac{\epsilon}{4} \mathcal{V}_2$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5 - \frac{\epsilon}{4} \mathcal{V}_2$
$\mathcal{V}_3$	$\mathcal{V}_1 - \epsilon \mathcal{V}_3$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 - \frac{3\epsilon}{10} \mathcal{V}_7$	$\mathcal{V}_5$
$\mathcal{V}_4$	$\mathcal{V}_1 + \frac{\epsilon}{2} \mathcal{V}_4$	$\mathcal{V}_2$	$\mathcal{V}_3 + \frac{3\epsilon}{10} \mathcal{V}_7 + \frac{3\epsilon^2}{20} \mathcal{V}_6$	$\mathcal{V}_4$	$\mathcal{V}_5 - \frac{\epsilon}{2} \mathcal{V}_4$
$\mathcal{V}_5$	$\mathcal{V}_1$	$\mathcal{V}_2 e^{\frac{\epsilon}{4}}$	$\mathcal{V}_3$	$\mathcal{V}_4 e^{\frac{\epsilon}{2}}$	$\mathcal{V}_5$
$\mathcal{V}_6$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5 - \epsilon \mathcal{V}_6$
$\mathcal{V}_7$	$\mathcal{V}_1 - \frac{\epsilon}{2} \mathcal{V}_7$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 - \epsilon \mathcal{V}_6$	$\mathcal{V}_5 - \frac{\epsilon}{2} \mathcal{V}_7$
$\mathcal{V}_8$	$\mathcal{V}_1 + \frac{3\epsilon}{4} \mathcal{V}_8$	$\mathcal{V}_2$	$\mathcal{V}_3 + \epsilon \mathcal{V}_{10}$	$\mathcal{V}_4$	$\mathcal{V}_5 + \frac{\epsilon}{4} \mathcal{V}_8$
$\mathcal{V}_9$	$\mathcal{V}_1 + \frac{\epsilon}{4} \mathcal{V}_9$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 + \frac{3\epsilon}{10} \mathcal{V}_8$	$\mathcal{V}_5 + \frac{3\epsilon}{4} \mathcal{V}_9$
$\mathcal{V}_{10}$	$\mathcal{V}_1 - \frac{\epsilon}{4} \mathcal{V}_{10}$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 + \frac{\epsilon}{20} \mathcal{V}_2$	$\mathcal{V}_5 + \frac{\epsilon}{4} \mathcal{V}_{10}$
$Ad_g$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_1$	$\mathcal{V}_6$	$\mathcal{V}_7 e^{\frac{\epsilon}{2}}$	$\mathcal{V}_8 e^{-\frac{3\epsilon}{4}}$	$\mathcal{V}_9 e^{-\frac{\epsilon}{4}}$	$\mathcal{V}_{10} e^{\frac{\epsilon}{4}}$
$\mathcal{V}_2$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_3$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8 - \epsilon \mathcal{V}_{10}$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_4$	$\mathcal{V}_6$	$\mathcal{V}_7 + \epsilon \mathcal{V}_6$	$\mathcal{V}_8$	$\mathcal{V}_9 - \frac{3\epsilon}{10} \mathcal{V}_8$	$\mathcal{V}_{10} + \frac{\epsilon}{20} \mathcal{V}_2$
$\mathcal{V}_5$	$\mathcal{V}_6 e^\epsilon$	$\mathcal{V}_7 e^{\frac{\epsilon}{2}}$	$\mathcal{V}_8 e^{-\frac{\epsilon}{4}}$	$\mathcal{V}_9 e^{-\frac{3\epsilon}{4}}$	$\mathcal{V}_{10} e^{-\frac{\epsilon}{4}}$
$\mathcal{V}_6$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9 - \frac{\epsilon}{10} \mathcal{V}_2$	$\mathcal{V}_{10}$
$\mathcal{V}_7$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8 - \frac{\epsilon}{6} \mathcal{V}_2$	$\mathcal{V}_9 - \epsilon \mathcal{V}_{10}$	$\mathcal{V}_{10}$
$\mathcal{V}_8$	$\mathcal{V}_6$	$\mathcal{V}_7 + \frac{\epsilon}{6} \mathcal{V}_2$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_9$	$\mathcal{V}_6 + \frac{\epsilon}{10} \mathcal{V}_2$	$\mathcal{V}_7 + \epsilon \mathcal{V}_{10}$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$
$\mathcal{V}_{10}$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$

where  $F$  is an arbitrary function of two basic invariants  $a_1$  and  $a_5$  of Eq. (2). We form the adjoint matrix to get an optimal system of Eq. (2). The adjoint representation table of the ten-dimensional Lie algebra can be formulated in Table 2.

For  $F_i^\epsilon : g \rightarrow g$  defined by  $\mathbf{V} \rightarrow Ad(exp(\epsilon_i \mathcal{V}_i). \mathbf{V})$  is a linear map, for  $i = 1, 2, \dots, 10$ . The matrix  $M_i^\epsilon$  of  $F_i^\epsilon, i = 1, 2, \dots, 10$  with respect to basis  $\{\mathcal{V}_1, \dots, \mathcal{V}_{10}\}$  are given below:

$$M_1^\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{\epsilon_1}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{\epsilon_1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3\epsilon_1}{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\epsilon_1}{4}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_1}{4}} \end{pmatrix}, M_2^\epsilon = \begin{pmatrix} 1 & \frac{\epsilon_2}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon_2}{4} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$



$$\begin{aligned}
 M_3^\epsilon &= \begin{pmatrix} 1 & 0 & -\epsilon_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{3\epsilon_3}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\epsilon_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & M_4^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & \frac{\epsilon_4}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{3\epsilon_4^2}{20} & \frac{3\epsilon_4}{10} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\epsilon_4}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon_4}{10} & 1 \\ 0 & \frac{\epsilon_4}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 M_5^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{\epsilon_5}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{\epsilon_5}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\epsilon_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_5}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\epsilon_5}{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3\epsilon_5}{4}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\epsilon_5}{4}} \end{pmatrix}, & M_6^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\epsilon_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{\epsilon_6}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 M_7^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\frac{\epsilon_7}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\epsilon_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{\epsilon_7}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon_7}{6} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\epsilon_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & M_8^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\epsilon_8}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \epsilon_8 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & 0 & 0 & \frac{\epsilon_8}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 \\ 0 & \frac{\epsilon_8}{6} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
 M_9^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_9}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & \frac{3\epsilon_9}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & 0 & \frac{3\epsilon_9}{4} & 0 & 0 \\ 0 & \frac{\epsilon_9}{10} & 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \epsilon_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & M_{10}^\epsilon &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\epsilon_{10}}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_{10}}{20} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & 0 & 0 & \frac{\epsilon_{10}}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

Similarly, one can find other matrices; hence, using these ten matrices we obtain the adjoint group which is defined by the matrix

$$A = (A_{ij})_{10 \times 10} = Ad(e^{-\epsilon_1 \mathcal{V}_1})Ad(e^{-\epsilon_2 \mathcal{V}_2})Ad(e^{-\epsilon_3 \mathcal{V}_3})Ad(e^{-\epsilon_4 \mathcal{V}_4})Ad(e^{-\epsilon_5 \mathcal{V}_5})$$

$$Ad(e^{-\epsilon_6 \mathcal{V}_6})Ad(e^{-\epsilon_7 \mathcal{V}_7})Ad(e^{-\epsilon_8 \mathcal{V}_8})Ad(e^{-\epsilon_9 \mathcal{V}_9})Ad(e^{-\epsilon_{10} \mathcal{V}_{10}}) \tag{21}$$

given in Appendix I.

### 3.1 Optimal system of one-dimensional Lie subalgebras

In order to form the one-dimensional optimal system of Eq. (2), For the vectors,  $V = \sum_{i=1}^{10} a_i \mathcal{V}_i$  and  $W = \sum_{i=1}^{10} \beta_i \mathcal{V}_i$ , we apply the adjoint transformations equation for Eq. (2) is given by

$$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}) = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \cdot A. \tag{22}$$

The following cases are considered to classify the one-dimensional Lie subalgebras of the resulted nonclassical symmetries.

*Case 1*  $a_1 \neq 0, a_3 = 0, a_4 = 0, a_5 = 0, a_9 = 0$ . Adopt one representative element  $\mathcal{T}_1 = \mathcal{V}_1$ . Substituting  $\beta_1 = 1, \beta_i = 0, 2 \leq i \leq 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_9 = 0, \epsilon_2 = -\frac{4}{3}(3a_2 - a_7a_8), \epsilon_7 = 2a_7, \epsilon_8 = -\frac{4}{3}a_8, \epsilon_{10} = 4a_{10}$$

$\epsilon_6$  is an arbitrary constants.

Substituting  $a_1 = 0, a_5 = 0$  in Eq. (22) provides new invariants

$$\phi(a_2, a_2, a_4, a_6, a_7, a_8, a_9, a_{10}) = a_3a_9(a_4)^{\frac{3}{2}}. \tag{23}$$

Due to the Remark 2 in [36], we have following three instances as:  $a_3a_9(a_4)^{\frac{3}{2}} = 1, a_3a_9(a_4)^{\frac{3}{2}} = -1$  and  $a_3a_9(a_4)^{\frac{3}{2}} = 0$ . By virtue of second invariant, the following sub-cases are constructed and discussed below.

*Case 2*  $a_3 \neq 0, a_i = 0, i = 1, 2, 4, \dots, 9$ . Choose a representative element  $\mathcal{T}_2 = \mathcal{V}_3$ . Substituting  $\beta_3 = 1, \beta_i = 0, 1 \leq i \leq 10, i \neq 3$  into Eq. (22), we get

$$\epsilon_1 = 0, \epsilon_4 = 0, \epsilon_5 = 0, \epsilon_8 = -a_{10},$$

$\epsilon_2, \epsilon_3, \epsilon_6, \epsilon_7, \epsilon_9$  and  $\epsilon_{10}$  are arbitrary constant.

*Case 3*  $a_4 \neq 0, a_1 = 0, a_3 = 0, a_5 = 0, a_9 = 0$ . Adopt one representative element  $w = \mathcal{V}_4$ . Substituting  $\beta_4 = 1, \beta_i = 0, i = 1, 2, 3, 5, \dots, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_3 = \frac{a_{10}}{a_8}, \epsilon_4 = \frac{1}{3a_8}(10a_7a_8 - 3a_{10}), \epsilon_7 = a_6, \epsilon_9 = -\frac{10}{3}a_8,$$

$$\epsilon_{10} = -\frac{10}{3}(6a_2 - a_6a_8)$$

$\epsilon_2, \epsilon_6$  and  $\epsilon_8$  are arbitrary constants.

*Case 4*  $a_6 \neq 0, a_i = 0, i = 1, 3, 4, 5, 7, 8, 9, 10$ . Adopt one representative element  $w = \mathcal{V}_6$ . Substituting  $\beta_6 = 1, \beta_i = 0, i = 1, \dots, 5, 7, \dots, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_9 = -\frac{10a_2}{a_6}$$

$\epsilon_i, i = 2, 3, 4, 6, 7, 8, 10$  are arbitrary constants.

*Case 5*  $a_2 \neq 0, a_7 \neq 0$ . Adopt one representative element  $w = a_2 \mathcal{V}_2 + \mathcal{V}_7, a_2 \in \{-1, 0, 1\}$ . Substituting  $\beta_2 = \beta_7 = 1, \beta_i = 0, i = 1, 3, 4, 5, 6, 8, 9, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_8 = -\frac{3}{10}a_{10}\epsilon_4, \epsilon_9 = -a_{10}$$

$\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7$  and  $\epsilon_{10}$  are arbitrary constants.

Case 6  $a_8 \neq 0, a_i = 0, i = 1, \dots, 7, 9$ . Adopt one representative element  $w = \mathcal{V}_8$ . Substituting  $\beta_8 = 1, \beta_i = 0, i = 1, \dots, 7, 9, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = \epsilon_7 = 0, \epsilon_3 = a_{10},$$

$\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_8, \epsilon_9$  and  $\epsilon_{10}$  are arbitrary constants.

Case 7  $a_9 \neq 0, a_i = 0, i = 1, 3, 4, 5, 6, 7$ . Adopt one representative element  $w = \mathcal{V}_9$ . Substituting  $\beta_9 = 1, \beta_i = 0, i = 1, \dots, 8, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_4 = \frac{10}{3}a_8, \epsilon_6 = \frac{5}{3}(6a_2 + a_{10}a_8 - a_8^2\epsilon_3), \epsilon_7 = a_{10} - a_8\epsilon_3,$$

$\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_8, \epsilon_9$  and  $\epsilon_{10}$  are arbitrary constants.

Case 8  $a_6 \neq 0, a_7 \neq 0$ . Adopt one representative element  $w = a_6\mathcal{V}_6 + \mathcal{V}_7, a_6 \in \{-1, 0, 1\}$ . Substituting  $\beta_6 = \beta_7 = 1, \beta_i = 0, i = 1, \dots, 5, 8, 9, 10$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_8 = -\frac{3}{5}(10a_2 - a_{10}a_6) - \frac{3}{10}a_{10}\epsilon_4, \epsilon_9 = -a_{10}$$

$\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7$  and  $\epsilon_{10}$  are arbitrary constants.

Case 9  $a_2 \neq 0, a_7 \neq 0, a_{10} \neq 0$ . Adopt one representative element  $w = \mathcal{V}_2 + \mathcal{V}_7 + \mathcal{V}_{10}$ . Substituting  $\beta_2 = \beta_7 = \beta_{10} = 1, \beta_i = 0, i = 1, 3, 4, 5, 6, 8, 9$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = \epsilon_9 = 0, \epsilon_8 = -6(a_2 - 1) - \frac{3}{10}\epsilon_4,$$

$\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7$  and  $\epsilon_{10}$  are arbitrary constants.

Case 10  $a_7 \neq 0, a_8 \neq 0, a_{10} \neq 0$ . Adopt one representative element  $w = \mathcal{V}_7 + \mathcal{V}_8 + \mathcal{V}_{10}$ . Substituting  $\beta_7 = \beta_8 = \beta_{10} = 1, \beta_i = 0, i = 1, 2, 3, 4, 5, 6, 9$  into Eq. (22), we get

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_8 = \frac{1}{10}(-60a_2 - 3a_{10}\epsilon_4 + 3\epsilon_3\epsilon_4 + 10\epsilon_7), \epsilon_9 = 1 - a_{10} + \epsilon_3$$

$\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7$  and  $\epsilon_{10}$  are arbitrary constants.

Case 11  $a_2 \neq 0, a_6 \neq 0$ . Adopt one representative element  $w = \mathcal{V}_2 + \mathcal{V}_6$ . Substituting  $\beta_2 = \beta_6 = 1, \beta_i = 0, i = 1, 3, 4, 5, 7, 8, 9, 10$  into Eq. (22) we obtain the solution

$$\epsilon_1 = \epsilon_5 = 0, \epsilon_8 = \frac{1}{10}(-60a_2 - 3a_{10}\epsilon_4 + 3a_8\epsilon_3\epsilon_4 + 10a_8\epsilon_7), \epsilon_9 = -a_{10} + a_8\epsilon_3$$

$\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7$  and  $\epsilon_{10}$  are arbitrary constants.

Similarly, we can find values of  $\epsilon_i$ 's for other members of optimal system. We concluded that the one-dimensional optimal system of subalgebra for KdV-type equation is as follows:

- (1)  $\mathcal{T}_1 = \mathcal{V}_1$
- (2)  $\mathcal{T}_2 = \mathcal{V}_3$
- (3)  $\mathcal{T}_3 = \mathcal{V}_4$
- (4)  $\mathcal{T}_4 = \mathcal{V}_6$
- (5)  $\mathcal{T}_5 = a_2\mathcal{V}_2 + \mathcal{V}_7,$
- (6)  $\mathcal{T}_6 = \mathcal{V}_8$
- (7)  $\mathcal{T}_7 = \mathcal{V}_b$
- (8)  $\mathcal{T}_8 = a_6\mathcal{V}_6 + \mathcal{V}_7,$
- (9)  $\mathcal{T}_9 = \mathcal{V}_2 + \mathcal{V}_7 + \mathcal{V}_{10}$
- (10)  $\mathcal{T}_{10} = \mathcal{V}_7 + \mathcal{V}_8 + \mathcal{V}_{10}$
- (11)  $\mathcal{T}_{11} = \mathcal{V}_2 + \mathcal{V}_6$

### 3.2 Optimal system of two-dimensional Lie subalgebras

Further, in this section we construct an optimal system of two-dimensional Lie subalgebras with the help of one-dimensional optimal system [4].

Let  $\langle W \rangle$  be a one-dimensional Lie subalgebra from previous section and consider the problem of finding all two-dimensional Lie subalgebras containing  $\langle W \rangle$ . To construct such optimal system, we must find all possible  $V \in \mathfrak{g}$  such that  $\langle V, W \rangle$  is a two-dimensional Lie subalgebra. As required,  $\langle V, W \rangle$  to be a two-dimensional Lie subalgebra, there must be some constants  $\lambda$  and  $\mu$  such that

$$[W, V] = \lambda W + \mu V \tag{24}$$

which are to be determined. Here, we must find  $V$  such that  $\langle V, W \rangle$  is a two-dimensional vector space, and  $\lambda$  and  $\mu$  satisfy (24). This gives number of algebraic equations whose solutions will give the two-dimensional Lie subalgebras including  $\langle W \rangle$ .

To illustrate, we compute all two-dimensional Lie subalgebras which contain the one-dimensional Lie subalgebra  $\langle V_1 \rangle$ . Let  $V = a_2V_2 + \dots + a_9V_9 + a_{10}V_{10}$  be a second basis element of a desired two-dimensional Lie subalgebra. Using Table 1, we obtain

$$[V_1, V] = \frac{1}{4}a_2V_2 - a_3V_3 + \frac{1}{2}a_4V_4 - \frac{1}{2}a_7V_7 + \frac{3}{4}a_8V_8 + \frac{1}{4}a_9V_9 - \frac{1}{4}a_{10}V_{10} \tag{25}$$

which provides the left-hand side of Eq. (24). The problem here is to find  $a_2, \dots, a_{10}$  not simultaneously zero and  $\lambda$  and  $\mu$  such that

$$\begin{aligned} \lambda = 0, \quad \mu a_2 = \frac{1}{4}a_2, \quad \mu a_3 = -a_3, \quad \mu a_4 = \frac{1}{2}a_4, \\ \mu a_5 = 0, \quad \mu a_6 = 0, \quad \mu a_7 = -\frac{1}{2}a_7, \quad \mu a_8 = \frac{3}{4}a_8, \quad \mu a_9 = \frac{1}{4}a_9, \quad \mu a_{10} = -\frac{1}{4}a_{10}. \end{aligned} \tag{26}$$

*Case 1:* When  $\mu = 0$  then,  $a_2 = a_3 = a_4 = a_7 = a_8 = a_9 = a_{10} = 0$ . Any  $a_5$  and  $a_6$ , not simultaneously zero, which is to be chosen arbitrarily. This gives a required member of two-dimensional Lie subalgebras  $\langle a_5V_5 + a_6V_6, V_1 \rangle$ ,  $(a_5, a_6) \in \mathbb{R}^2 \setminus (0, 0)$ .

*Case 2:* When  $\mu \neq 0$  then,  $a_5 = a_6 = 0$ . As  $V \neq 0$ ,  $\mu = \frac{1}{4}$ , is the only choice and  $a_2, a_3, a_4, a_7, a_8, a_9, a_{10}$  which is to be chosen arbitrarily. This gives a required member of two-dimensional Lie subalgebras  $\langle a_2V_2 + a_3V_3 + a_4V_4 + a_7V_7 + a_8V_8 + a_9V_9 + a_{10}V_{10}, V_1 \rangle$ ,  $(a_2, a_3, a_4, a_7, a_8, a_9, a_{10}) \in \mathbb{R}^7 \setminus (0, 0, 0, 0, 0, 0, 0)$ .

Secondly, we compute all two-dimensional Lie subalgebras which contain the one-dimensional Lie subalgebra  $\langle V_6 \rangle$ . Let  $V = a_1V_1 + \dots + a_5V_5 + a_7V_7 + a_8V_8 + a_9V_9 + a_{10}V_{10}$  be a second basis element of a desired two-dimensional Lie subalgebra. Using Table 1, we obtain

$$[V_6, V] = a_5V_6 + a_9\frac{1}{10}V_2 \tag{27}$$

which provides the left-hand side of Eq. (24). The problem here is to find  $a_1, a_2, \dots, a_{10}$  not simultaneously zero and  $\lambda$  and  $\mu$  such that

$$\begin{aligned} \lambda = a_5, \quad \mu a_2 = \frac{1}{10}a_9, \quad \mu a_1 = 0, \quad \mu a_3 = 0, \\ \mu a_4 = 0, \quad \mu a_5 = 0, \quad \mu a_7 = 0, \quad \mu a_8 = 0, \quad \mu a_9 = 0, \quad \mu a_{10} = 0. \end{aligned} \tag{28}$$

*Case 3:* When  $\mu = 0$  then,  $a_9 = 0$ . Any  $a_1, a_3, a_4, a_5, a_7, a_8, a_9$  and  $a_{10}$  is not simultaneously zero, which is to be chosen arbitrarily. This gives a required member of two-dimensional Lie subalgebras  $\langle a_1 V_1 + \dots + a_5 V_5 + a_7 V_7 + a_8 V_8 + a_{10} V_{10}, V_6 \rangle$ ,  $(a_1, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}) \in \mathbb{R}^8 \setminus (0, 0, 0, 0, 0, 0, 0, 0)$ .

*Case 4:* When  $\mu \neq 0$  then,  $a_1 = a_3 = a_4 = a_5 = a_7 = a_8 = a_9 = a_{10} = 0$ . As  $V \neq 0, \mu = 1$  is the only choice and  $a_2$ , which is to be chosen arbitrarily. This gives a required member of two-dimensional Lie subalgebras  $\langle a_2 V_2, V_6 \rangle, a_2 \in \mathbb{R} \setminus (0)$ .

This process, applied to all the one-dimensional Lie subalgebras from previous section, computes the two-dimensional Lie subalgebras which contain one-dimensional Lie subalgebras.

### 4 Group-invariant solutions

In this section, we restrict our study to the one-dimensional optimal system of Lie subalgebras computed in previous section. We derive several Lie symmetry reductions and corresponding group-invariant solutions with the help of one-dimensional optimal system of subalgebras.

#### 4.1 Subalgebra $\mathcal{T}_1 = \mathcal{V}_1$ :

For the infinitesimal generator

$$\mathcal{V}_1 = \frac{x}{4} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} - \frac{u}{4} \frac{\partial}{\partial u}, \tag{29}$$

Thus, Eq. (13) becomes

$$\frac{dx}{\frac{x}{4}} = \frac{dy}{\frac{y}{2}} = \frac{dz}{0} = \frac{dt}{t} = \frac{du}{\frac{u}{4}}. \tag{30}$$

which gives

$$u(x, y, z, t) = \mathcal{F}(X, Y, Z), \tag{31}$$

where  $\mathcal{F}(X, Y, Z)$  is similarity function in which similarity variables  $X, Y$  and  $Z$  can be expressed as

$$X = xt^{-\frac{1}{4}}, Y = yt^{-\frac{1}{2}}, Z = z. \tag{32}$$

Using Eq. (31) into Eq. (2), we get the following (2+1)-dimensional nonlinear reduced equation with variable coefficients as first reduction of the equation given as

$$\begin{aligned} &\mathcal{F} - 4(10\mathcal{F}_Z(6\mathcal{F}_X^2 + \mathcal{F}_{XXX}) + 20\mathcal{F}_X\mathcal{F}_{XXZ} + \mathcal{F}_{XXXXZ} + \mathcal{F}_{XXY}) \\ &+ 2\mathcal{F}_Y(Y - 12\mathcal{F}_X) + X\mathcal{F}_X = 0, \end{aligned} \tag{33}$$

where  $\mathcal{F}_X = \frac{\partial \mathcal{F}}{\partial X}, \mathcal{F}_Y = \frac{\partial \mathcal{F}}{\partial Y}$ , etc. To solve Eq. (33), we obtain new set of infinitesimal given as

$$\xi_X = -\frac{X}{4}A_1, \xi_Y = \frac{Y}{2}A_1, \xi_Z = A_1Z + A_2, \eta_{\mathcal{F}} = A_1\frac{\mathcal{F}}{4}, \tag{34}$$

where  $A_1$  and  $A_2$  are arbitrary constants.

4.1.1 For  $A_1 \neq 0, A_2 = 0$  in Eq. (34)

With the help of characteristic Eq. (13), then function  $F$  can be written as

$$\mathcal{F} = Z^{\frac{1}{4}}G(r, s), \quad (35)$$

through  $r = XZ^{\frac{1}{4}}$  and  $s = \frac{Y}{\sqrt{Z}}$ . Putting the similarity form in Eq. (33), we have following reduced equation

$$60rG_r^3 + G(10G_{rrr} + 60G_r^2 - 1) + G_r(24G_s + 60G_{rr} - 40sG_{rrs} + r(30G_{rrr})) + 4G_{rrs} + 5G_{rrrr} + rG_{rrrr} - 2s(G_s(1 + 60G_r^2 + 10G_{rrr}) + G_{rrrs}) = 0. \quad (36)$$

We could not find generators of Eq. (36) because of high nonlinearity. Hence, this equation can be solved numerically.

4.1.2 For  $A_1 = 0, A_2 \neq 0$  in Eq. (34)

Using characteristic equations for this subcase, we obtain the similarity variables as follows

$$r = X \text{ and } s = Y \text{ with } \mathcal{F} = G(r, s). \quad (37)$$

By substituting  $F$  in Eq. (33), we have

$$G + 2G_s(s - 12G_r) + rG_r - 4G_{rrs} = 0 \quad (38)$$

Again, we can find infinitesimals for Eq. (38) given as

$$\xi_r = -\frac{r}{2}b_1, \quad \xi_s = b_1s, \quad \xi_G = \frac{G}{2}b_1, \quad (39)$$

where  $b_1$  is an arbitrary constant. Then, established characteristic equation for Eq. (39) is

$$\frac{dr}{-\frac{r}{2}b_1} = \frac{ds}{b_1s} = \frac{dG}{\frac{G}{2}b_1}. \quad (40)$$

We obtain the similarity variable as  $w = r\sqrt{s}$ , and the similarity form is given by

$$G(r, s) = \frac{1}{r}R(w). \quad (41)$$

Putting  $G$  in Eq. (38), we get

$$w^2R''' - R'(-6wR' + 6R + w^2) = 0, \quad (42)$$

which is a highly nonlinear ODE. where  $'$  denotes the derivative with respect to  $w$ . One particular result is given below

$$R(w) = \frac{1}{6}w^2, \quad (43)$$

Using Eqs. (43), (41), (37) in Eq. (31), one obtains

$$u(x, y, z, t) = \frac{xy}{6t}. \quad (44)$$

4.2 **Subalgebra**  $\mathcal{T}_2 = \mathcal{V}_3 = \frac{\partial}{\partial t}$ :

Solving characteristic equations in this case yields

$$u(x, y, z, t) = \mathcal{F}(X, Y, Z), \tag{45}$$

with  $X = x, Y = y, Z = z$ . On inserting Eq. (45) into Eq. (2), reduced equation is

$$6\mathcal{F}_X \mathcal{F}_Y + 20\mathcal{F}_X \mathcal{F}_{XXZ} + \mathcal{F}_{XXY} + 10\mathcal{F}_Z(6\mathcal{F}_X^2 + \mathcal{F}_{XXX}) + \mathcal{F}_{XXXXZ} = 0. \tag{46}$$

Then, the solution form for Eq. (46) is

$$\mathcal{F}(X, Y, Z) = c_1 \tanh(c_1 X - 4c_3 c_1^2 Y + c_3 Z + c_4) + c_2. \tag{47}$$

with  $c_i$  ( $1 \leq i \leq 4$ ) being arbitrary constants. Hence, Eq. (47) gives

$$u(x, y, z, t) = c_1 \tanh(c_1 x - 4c_3 c_1^2 y + c_3 z + c_4) + c_2, \tag{48}$$

Applying Lie symmetry method, Eq. (46) admits infinitesimals as

$$\begin{aligned} \xi_X &= \frac{X}{2}(A_3 - A_1) + Y A_5 + A_6, & \xi_Y &= A_3 Y + A_4, \\ \xi_Z &= A_1 Z + A_2, & \eta_{\mathcal{F}} &= \frac{1}{2}(A_1 - A_3)\mathcal{F} + \frac{Z}{10}A_5 + A_7, \end{aligned} \tag{49}$$

with  $\xi_X, \xi_Y, \xi_Z$  and  $\eta_{\mathcal{F}}$  and  $A_i$  ( $1 \leq i \leq 7$ ) being arbitrary constants. Moreover, some particular cases are discussed below.

4.2.1 Case  $A_2 \neq 0$

Corresponding Lagrange’s equations are given below

$$\frac{dX}{0} = \frac{dY}{0} = \frac{dZ}{1} = \frac{d\mathcal{F}}{0}. \tag{50}$$

By solving Eq. (50), we obtain

$$\mathcal{F} = \mathcal{G}(r, s), \tag{51}$$

through  $r = X$  and  $s = Y$ . Thus, we have

$$6\mathcal{G}_s \mathcal{G}_r + \mathcal{G}_{rrs} = 0. \tag{52}$$

Substituting  $\mathcal{G}(r, s) = H(\zeta)$  where  $\zeta = ar + bs$  with  $a$  and  $b$  being constants in Eq. (52), we obtain an ordinary differential equation in  $H$  as

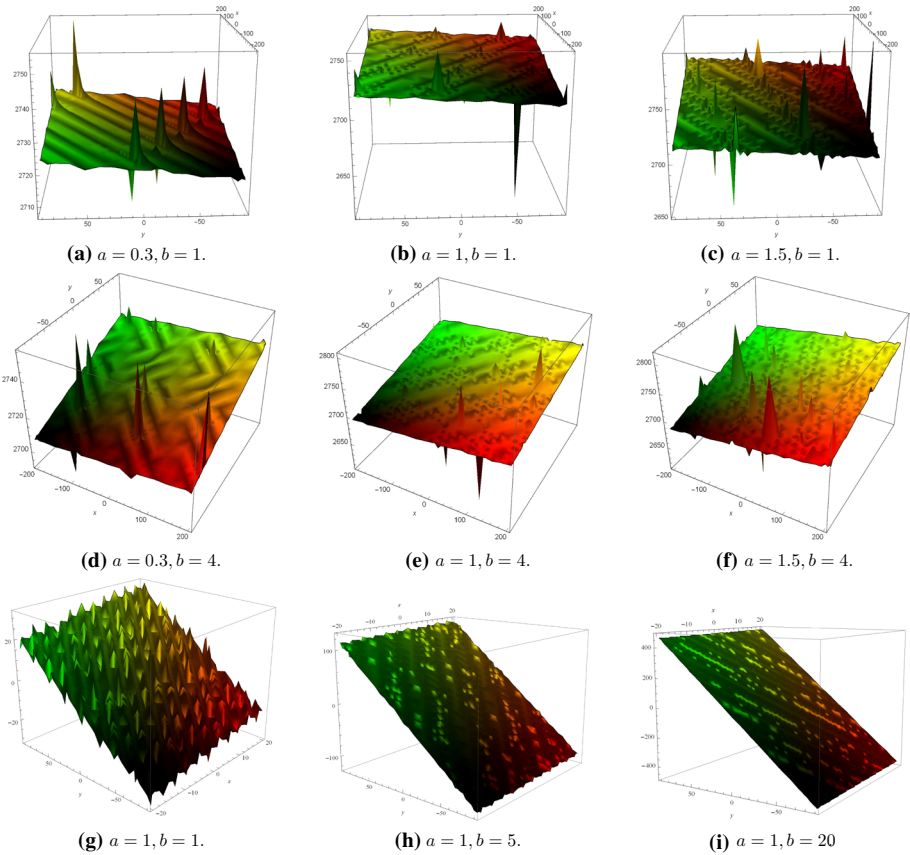
$$6H'^2 + aH''' = 0. \tag{53}$$

The general solution of (53) is given as

$$H(\zeta) = a(-a^{-1})^{-\frac{1}{3}} \text{WeierstrassZeta} \left[ (-a^{-1})^{\frac{1}{3}}(\zeta) + c_1; \{0, c_2\} \right] + c_3. \tag{54}$$

with  $c_1, c_2$  and  $c_3$  being arbitrary constants. Hence, using Eqs. (54) and (51), we obtain WeierstrassZeta function solution for governing KdV

$$u(x, y, z, t) = a(-a^{-1})^{-\frac{1}{3}} \text{WeierstrassZeta} \left[ (-a^{-1})^{\frac{1}{3}}(ax + by + c_1); \{0, c_2\} \right] + c_3. \tag{55}$$



**Fig. 1** The physical structures of Lump-type solitons and multi-solitons profiles for (55) with parameters  $c_1 = 154, c_2 = 0.003, c_3 = 2717$  for (a)–(f) and  $c_1 = 1, c_2 = 0.3, c_3 = 1$  for (g)–(i)

### 4.2.2 Case $A_3 \neq 0$

Lagrange’s system for this case is

$$\frac{dX}{\frac{X}{2}} = \frac{dY}{Y} = \frac{dZ}{0} = \frac{d\mathcal{F}}{-\frac{\mathcal{F}}{2}}. \tag{56}$$

By solving Eq. (56), we obtain

$$\mathcal{F} = \frac{\mathcal{G}(r, s)}{\sqrt{Y}}, \tag{57}$$

through similarity variables  $r = X/\sqrt{Y}$  and  $s = Z$ . Substituting the value of  $\mathcal{F}$  into Eq. (46), we obtain

$$2\mathcal{G}_{rrrr} - 3\mathcal{G}_{rr} - 2\mathcal{G}_r(3\mathcal{G} + 3(r - 20\mathcal{G}_s)\mathcal{G}_r - 20\mathcal{G}_{rrs}) - (r - 20\mathcal{G}_s)\mathcal{G}_{rrr} = 0. \tag{58}$$

Again, Eq. (58) admits infinitesimal generators as

$$\xi_r = -\frac{r}{2}b_1, \quad \xi_s = b_1s + b_2, \quad \xi_{\mathcal{G}} = \frac{\mathcal{G}}{2}b_1, \tag{59}$$



where  $b_1$  and  $b_2$  are arbitrary constants. Using generators (59), solution  $G(r, s)$  takes the form as

$$\mathcal{G}(r, s) = R(w) \sqrt{s}, \tag{60}$$

where  $w = r\sqrt{s}$ . Inserting Eq. (60) into Eq. (58), we have

$$w R^{(5)} + 5R^{(4)} - 3R'' + 6R'[(R + wR')(10R' - 1) + 10R''] + 10R + w(30R' - 1)R''' = 0. \tag{61}$$

Eq. (61) is a highly nonlinear ODE and cannot be solved easily. Assuming  $R(w)$  as polynomial with constants, we get

$$R(w) = \frac{1}{10}w + \gamma, \tag{62}$$

where  $\gamma$  is the arbitrary constant. Ultimately, group-invariant solution is

$$u(x, y, z, t) = \frac{xz}{10y} + \gamma \sqrt{\frac{z}{y}}. \tag{63}$$

#### 4.2.3 Case $A_4 \neq 0$

By solving characteristic equations, we obtain

$$\mathcal{F} = \mathcal{G}(r, s), \tag{64}$$

with  $r = X$  and  $s = Z$ . On inserting the value of  $\mathcal{F}$  solution (64) into Eq. (46), we get

$$20\mathcal{G}_r \mathcal{G}_{rrs} + 10\mathcal{G}_s (6\mathcal{G}_r^2 + \mathcal{G}_{rrr}) + \mathcal{G}_{rrrs} = 0. \tag{65}$$

Similarity transformation method (STM) provides the following generators infinitesimal

$$\xi_r = b_1r + b_2, \quad \xi_s = f(s), \quad \xi_G = -G b_1 + b_3, \tag{66}$$

where real parameters  $b_1, b_2$  and  $b_3$  are real arbitrary constants. Using Eq. (66), we obtain the solution  $\mathcal{G}(r, s)$  as

$$\mathcal{G}(r, s) = \frac{b_3r + R(w)}{b_1r + b_2}, \quad \text{with } w = \frac{b_1r + b_2}{b_1s}. \tag{67}$$

Inserting Eq. (67) into Eq. (65), we obtain ordinary differential equation as

$$b_1^4 w^4 R^{(5)} + 20b_1^2 w^2 R^{(3)} (b_1 w R' - b_1 R + b_2 b_3) + 10R' b_1^3 w^3 R^{(3)} - 30b_1^3 w^2 R' R'' - 60R' (b_1 R - b_1 w R') (b_1^2 + b_1 w R' - b_1 R + 2b_2 b_3) + 60b_2 b_3 (b_1^2 + b_2 b_3) R' = 0. \tag{68}$$

Eq. (68) is a complex nonlinear ordinary differential equation. General solution is not easy to obtain, but one solution is given below

$$R(w) = \gamma w + \frac{b_2 b_3}{b_1} + b_1, \tag{69}$$

where real parameters  $\gamma$  are the arbitrary constant. Using Eqs. (69), (67), (64) in Eq. (45), corresponding rational function solution  $u$  is obtained as

$$u(x, y, z, t) = \frac{\gamma}{b_1 z} + \frac{b_1}{b_1 x + b_2} + \frac{b_3}{b_1}. \tag{70}$$

#### 4.2.4 Case $A_5 \neq 0$

The Lagrange’s equations read as

$$\frac{dX}{Y} = \frac{dY}{0} = \frac{dZ}{0} = \frac{d\mathcal{F}}{\frac{Z}{10}} \tag{71}$$

which produces

$$\mathcal{F} = \frac{XZ}{10Y} + \mathcal{G}(r, s), \tag{72}$$

with variables  $r = Y$  and  $s = Z$ . Thus, one obtains

$$s\mathcal{G}_s + r\mathcal{G}_r = 0. \tag{73}$$

The general solution is given as

$$\mathcal{G}(r, s) = f\left(\frac{s}{r}\right). \tag{74}$$

Hence, using Eqs. (74), (72) in Eq. (45), the invariant solution is

$$u(x, y, z, t) = \frac{xz}{10y} + f\left(\frac{z}{y}\right). \tag{75}$$

with  $f$  being arbitrary function of  $y$  and  $z$ .

#### 4.3 Subalgebra $\mathcal{T}_3 = \mathcal{V}_4$ :

Using characteristic equation for  $\mathcal{V}_4 = \frac{3t}{10} \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + \frac{x}{20} \frac{\partial}{\partial u}$ , we obtain

$$u(x, y, z, t) = \mathcal{F}(X, Y, T) + \frac{xy}{6t}, \text{ through } X = x, Y = -\frac{5y^2}{3t} + z, T = t.$$

On inserting the similarity solution in Eq. (2), reduction equation is

$$X\mathcal{F}_X + T(\mathcal{F}_T + 20\mathcal{F}_X\mathcal{F}_{XXY} + 10\mathcal{F}_Y(6\mathcal{F}_X^2 + \mathcal{F}_{XXX}) + \mathcal{F}_{XXXXY}) = 0. \tag{76}$$

Also, infinitesimals for Eq. (76) are

$$\xi_X = \frac{X}{4}(A_1 - A_2) + A_4T, \quad \xi_Y = A_2T + A_3, \quad \xi_T = A_1T, \quad \eta_{\mathcal{F}} = \frac{\mathcal{F}}{4}(A_2 - A_1) + A_5, \tag{77}$$

where  $A_i$  ( $1 \leq i \leq 5$ ) are arbitrary constants. For completeness, reduced equations and invariant solutions for subcases are furnished in Table 3.

#### 4.4 Subalgebra $\mathcal{T}_4 = \mathcal{V}_6$ :

With the help of Lagrange system for  $\mathcal{T}_4 = \frac{\partial}{\partial z}$  we found invariants as  $X = x, Y = y, T = t$  and  $u = \mathcal{F}(X, Y, T)$ . By putting  $u$  in Eq. (2), we get

$$\mathcal{F}_{XXY} + 6\mathcal{F}_X\mathcal{F}_Y + \mathcal{F}_T = 0. \tag{78}$$

The solution of Eq. (78) is

$$\mathcal{F} = c_1 \tanh\left(c_1X - \frac{c_3Y}{4c_1^2} + c_3T + c_4\right) + c_2. \tag{79}$$

**Table 3** Deductions for Eq. (2) in subalgebra  $\mathcal{T}_3 = \mathcal{V}_4$

Subcase	Similarity variables	Reduced equations	Group-invariant solutions
$A_1 \neq 0$ , and other $A_i$ 's are zero	$\mathcal{F} = T^{-\frac{1}{4}} \mathcal{G}(r, s)$ $r = XT^{-\frac{1}{4}}, s = Y$	$\mathcal{G} + \mathcal{G}_r(-3r - 80\mathcal{G}_{rrs})$ $-40\mathcal{G}_s(6\mathcal{G}_r^2 + \mathcal{G}_{rr}) - 4\mathcal{G}_{rrrs} = 0$	$u = \frac{xy}{6t} - \frac{x\sqrt{3b_1tz - 5b_1y^2 + 3b_2t}}{6\sqrt{5}\sqrt{-b_1t}}$
$A_2 \neq 0$ , and other $A_i$ 's are zero	$\mathcal{F} = Y^{-\frac{1}{4}} \mathcal{G}(r, s)$ $r = XY^{\frac{1}{4}}, s = T$	$4r\mathcal{G}_r + s[4\mathcal{G}_s + 5(2\mathcal{G}(6\mathcal{G}_r^2 + \mathcal{G}_{rr})$ $+ 6\mathcal{G}_r(2\mathcal{G}_{rr} + r(2\mathcal{G}_r^2 + \mathcal{G}_{rr}))$ $+ \mathcal{G}_{rrr}) + r\mathcal{G}_{rrrr}] = 0$	$u = \frac{xy}{6t} - \frac{x\sqrt{5(5y^2 - 3tz)}}{30t}$
$A_3 \neq 0$ , and other $A_i$ 's are zero	$\mathcal{F} = \mathcal{G}(r, s)$ $r = X, s = T$	$s\mathcal{G}_s + r\mathcal{G}_r = 0$	$u = \frac{xy}{6t} + f\left(\frac{y}{t}\right)$
$A_4 \neq 0$ , and other $A_i$ 's are zero	$\mathcal{F} = \mathcal{G}(r, s)$ $r = Y, s = T$	$\mathcal{G}_s = 0$	$u = \frac{xy}{6t} + f\left(z - \frac{5y^2}{3t}\right)$

with  $c_1, c_2, c_3$  and  $c_4$  being arbitrary constants. Using Eq. (79), kink-type solution is obtained as

$$u(x, y, z, t) = c_1 \tanh\left(c_1x - \frac{c_3y}{4c_1^2} + c_3t + c_4\right) + c_2. \tag{80}$$

Moreover, by using wave transformation  $w = aX + bY - cT$  where  $a, b$  and  $c$  are constants in Eq. (78), putting  $F(x, y, t) = H(w)$ , we get

$$a^2bH^{(3)}(w) + 6abH'(w)^2 - cH'(w) = 0. \tag{81}$$

Solving Eq. (81), we obtain

$$H(w) = \frac{cw}{6ab} + a_1. \tag{82}$$

Eventually, solution of Eq. (2)

$$u(x, y, z, t) = \frac{c(ax + by - ct)}{6ab} + a_1, \tag{83}$$

where  $a_1$  is an arbitrary constant.

**4.5 Subalgebra  $\mathcal{T}_5 = a_2\mathcal{V}_2 + \mathcal{V}_7$ :**

**4.5.1 For  $a_2 = 0$**

Subalgebra reduces to  $\mathcal{V}_7 = \frac{\partial}{\partial y}$ , and using its characteristic equation the corresponding invariant form is

$$u(x, y, z, t) = \mathcal{F}(X, Z, T), \tag{84}$$

with  $X = x, Z = z, T = t$ . On inserting solution in Eq. (2), we found the following reduction equation

$$\mathcal{F}_{XXXXZ} + 20\mathcal{F}_{XXZ}\mathcal{F}_X + 10\mathcal{F}_Z(6\mathcal{F}_X^2 + \mathcal{F}_{XXX}) + \mathcal{F}_T = 0. \tag{85}$$

The solution for Eq. (85) is

$$\mathcal{F}(X, Z, T) = c_2 + c_1 \tanh \left( c_1 X - \frac{c_3}{16c_1^4} Z + c_3 T + c_4 \right). \tag{86}$$

where  $c_1, c_2, c_3$  and  $c_4$  are arbitrary constants. Using Eq. (86) in Eq. (84), the kink-type solution of Eq. (2) is

$$u(x, y, z, t) = c_2 + c_1 \tanh \left( c_1 x - \frac{c_3}{16c_1^4} z + c_3 t + c_4 \right). \tag{87}$$

Again, new set of infinitesimals for Eq. (85) are

$$\begin{aligned} \xi_X &= \frac{X}{4}(A_1 - A_3) + A_5, & \xi_Z &= A_3 Z + A_4, & \xi_T &= A_1 T + A_2, \\ \eta_{\mathcal{F}} &= \frac{\mathcal{F}}{4}(A_3 - A_1) + A_6, \end{aligned} \tag{88}$$

where  $A_i$  ( $1 \leq i \leq 6$ ) are arbitrary constants.

When  $A_1 = 0, A_3 = 0$  in Eq. (88)

Characteristic equations are given by

$$\frac{dX}{A_5} = \frac{dZ}{A_4} = \frac{dT}{A_2} = \frac{d\mathcal{F}}{A_6}. \tag{89}$$

Using Eqs. (89), the solution with similarity variables is given by

$$\mathcal{F} = \left( \frac{A_6}{A_2} \right) T + \mathcal{G}(r, s), \text{ with the help of } r = X - \left( \frac{A_5}{A_2} \right) T, s = Z - \left( \frac{A_4}{A_2} \right) T. \tag{90}$$

Substituting the above invariant form in Eq. (85), the reduced equation is given as

$$A_6 - \mathcal{G}_r (A_5 - 20A_2 \mathcal{G}_{rrs}) + \mathcal{G}_s (-A_4 + 10A_2 (6\mathcal{G}_r^2 + \mathcal{G}_{rrr}) + A_2 \mathcal{G}_{rrrs}) = 0. \tag{91}$$

Again, new set of infinitesimal generators are

$$\xi_r = b_2, \quad \xi_s = b_1, \quad \eta_{\mathcal{G}} = b_3, \tag{92}$$

where  $b_1, b_2$  and  $b_3$  are integral constants and group-invariant solution is

$$\mathcal{G} = \left( \frac{b_3}{b_1} \right) s + R(w), \text{ where } w = r - \left( \frac{b_2}{b_1} \right) s. \tag{93}$$

Using the value of  $\mathcal{G}$  in Eq. (91), we have

$$\begin{aligned} A_2 b_2 R^{(5)} + (10A_2 (R^{(3)} + 6R'^2) - A_4)(b_3 - b_2 R') \\ - R'(20A_2 b_2 R^{(3)} + A_5 b_1) + A_6 b_1 = 0. \end{aligned} \tag{94}$$

Unfortunately, the solution for Eq. (94) is extremely difficult to solve analytically, and we can assume one particular solution given as

$$R(w) = b w + c, \tag{95}$$

where  $b$  and  $c$ . Here,  $b$  in terms of other constants is given as follows

$$b = -\frac{\sqrt[3]{5A_8 + \frac{1}{216}\sqrt{A_7}}}{30A_2b_2} + \frac{5^{2/3}(-20A_2b_3^2 - A_4b_2^2 + A_5b_1b_2)}{30b_2\sqrt[3]{A_8 + \frac{\sqrt{A_7}}{1080}}} + \frac{b_3}{3b_2},$$

where

$$A_7 = 29160000A_2^4(40A_2b_3^3 - 3b_2b_3(2A_4b_2 + A_5b_1) + 9A_6b_1b_2^2) + (-3600A_2^2b_3^2 - 180A_2b_2(A_4b_2 - A_5b_1))^3,$$

$$A_8 = -200A_2^3b_3^3 + 15A_2^2b_2(2A_4b_2b_3 + A_5b_1b_3 - 3A_6b_1b_2).$$

Hence, using Eqs. (95), (90) in (84), solution for Eq. (2) is given as

$$u(x, y, z, t) = \frac{A_2(b_1(bx + c) + z(b_3 - bb_2)) + t(A_4bb_2 - A_4b_3 - A_5bb_1 + A_6b_1)}{A_2b_1}. \tag{96}$$

#### 4.5.2 For $a_2 = 1$

Thus, corresponding subalgebra reduces to  $\mathcal{V}_2 + \mathcal{V}_7 = \frac{\partial}{\partial y} + \frac{\partial}{\partial u}$ , and using its characteristic equation, the similarity solution is

$$u(x, y, z, t) = y + \mathcal{F}(X, Z, T), \tag{97}$$

where the similarity variables  $X = x, Z = z, T = t$ . Putting above value of  $u$  in (2), one obtain

$$\mathcal{F}_{XXXXZ} + 10\mathcal{F}_{XXX}\mathcal{F}_Z + 2\mathcal{F}_X(3 + 30\mathcal{F}_X\mathcal{F}_Z + 10\mathcal{F}_{XXZ}) + \mathcal{F}_T = 0. \tag{98}$$

The solution for Eq. (98) is given by

$$\mathcal{F} = c_5 + c_2 \tanh(Xc_2 + Zc_3 - 2Tc_2(3 + 8c_2^2c_3) + c_4). \tag{99}$$

Hence, with the help of Eq. (99) in (97), one gets

$$u(x, y, z, t) = y + c_5 + c_2 \tanh(xc_2 + zc_3 - 2tc_2(3 + 8c_2^2c_3) + c_4). \tag{100}$$

The case for  $a_2 = 0$  is already discussed when we take subalgebra  $\mathcal{V}_7$ . Similarly, we do for  $a_2 = -1$  to obtain solution of Eq. (2).

#### 4.6 Subalgebra $\mathcal{T}_6 = \mathcal{V}_8$ :

Using the characteristic equation for  $\mathcal{V}_8 = t \frac{\partial}{\partial x} + \frac{y}{6} \frac{\partial}{\partial u}$ , we get

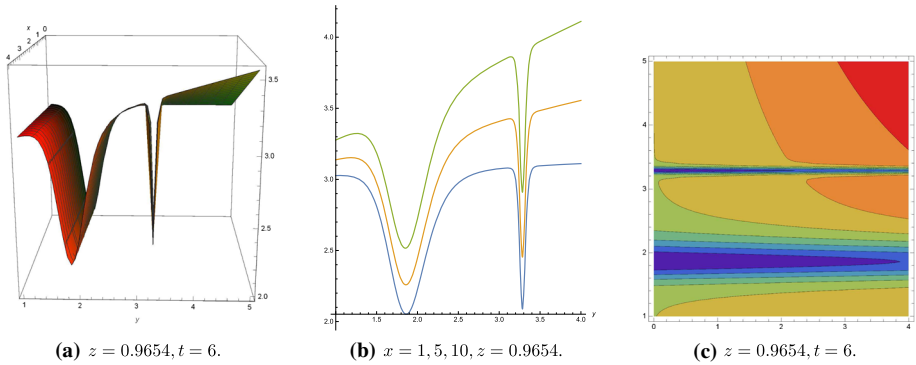
$$u(x, y, z, t) = \frac{xy}{6t} + \mathcal{F}(Y, Z, T), \tag{101}$$

with  $Y = y, Z = z, T = t$ . Making the use of  $u$  in (2), we have

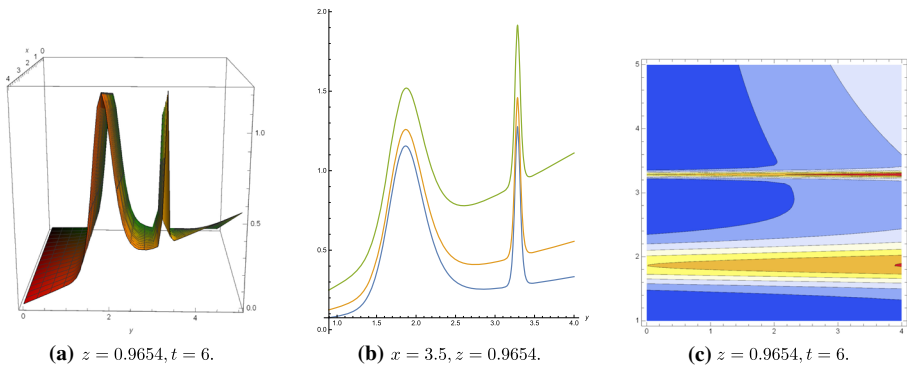
$$3T^2\mathcal{F}_T + 5Y^2\mathcal{F}_Z + 3YT\mathcal{F}_Y = 0. \tag{102}$$

By solving Eq. (102), we obtain

$$\mathcal{F}(Y, Z, T) = g\left(\frac{T}{Y}, \frac{3ZT - 5Y^2}{3T}\right). \tag{103}$$



**Fig. 2** The physical structure of W-shaped soliton profile for (104) presents doubly soliton, traveling wave along the  $x$ -axis for  $g\left(\frac{t}{y}, \frac{3zt-5y^2}{3t}\right) = \tanh^2\left(\frac{3tz-5y^2}{3y}\right) + \tanh^2\left(\frac{2t^3}{y^3} + \frac{2(3tz-5y^2)}{t}\right) + \tanh^2\left(\frac{(3tz-5y^2)^2}{9t^2} + \frac{2t^2}{y^2} + 1\right)$



**Fig. 3** The physical structure of M-shaped soliton profile for (104) shows doubly soliton, and 2D plot exhibits solitary waves for  $g\left(\frac{t}{y}, \frac{3zt-5y^2}{3t}\right) = \operatorname{sech}^2\left(\frac{3tz-5y^2}{3y}\right) + \operatorname{sech}^2\left(\frac{2t^3}{y^3} + \frac{2(3tz-5y^2)}{t}\right) + \operatorname{sech}^2\left(\frac{(3tz-5y^2)^2}{9t^2} + \frac{2t^2}{y^2} + 1\right)$

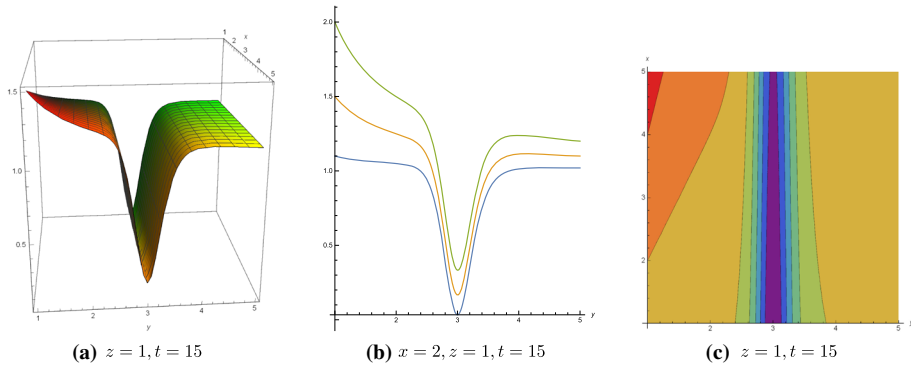
Hence, using Eq. (103) in Eq. (101), we get the desired group-invariant solution

$$u(x, y, z, t) = \frac{xy}{6t} + g\left(\frac{t}{y}, \frac{3zt - 5y^2}{3t}\right). \tag{104}$$

Choosing two suitable values of arbitrary function  $g(\cdot, \cdot)$  in the form of tanh function via Fig. 2 and sech function via Fig. 3 is demonstrated.

#### 4.7 Subalgebra $\mathcal{T}_7 = \mathcal{V}_9$ :

For  $\mathcal{V}_9 = y \frac{\partial}{\partial x} + \frac{z}{10} \frac{\partial}{\partial u}$ , using characteristic equation we get



**Fig. 4** The physical structure of single soliton profile for (108) with  $g\left(\frac{z}{y}, \frac{3zt-5y^2}{3z}\right) = \tanh\left[\frac{3tz-5y^2}{3y}\right]^2$  shows perspective view of the real part of the dark-bright soliton solution, the wave propagation pattern of the wave along the  $y$ -axis

$$u(x, y, z, t) = \mathcal{F}(Y, Z, T) + \frac{xz}{10y}, \tag{105}$$

with invariants  $Y = y, Z = z, T = t$ .

Putting the above invariant solution in Eq. (2), we have

$$\mathcal{F}_T + \frac{3Z}{5Y^2}(Z\mathcal{F}_Z + Y\mathcal{F}_Y) = 0. \tag{106}$$

which has the general solution

$$\mathcal{F} = g\left(\frac{Z}{Y}, \frac{3ZT - 5Y^2}{3Z}\right). \tag{107}$$

Hence, using Eq. (107) in Eq. (105), we get the desired group-invariant solution is

$$u(x, y, z, t) = \frac{xz}{10y} + g\left(\frac{z}{y}, \frac{3zt - 5y^2}{3z}\right). \tag{108}$$

Some particular solutions are shown in Figs. 4 and 5.

4.8 *Subalgebra*  $\mathcal{T}_8 = a_6\mathcal{V}_6 + \mathcal{V}_7$ :

For  $a_6 = 1$ ,  $\mathcal{T}_8$  becomes  $\mathcal{V}_6 + \mathcal{V}_7 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$  and using its characteristic equation, we get the desired invariant solution as  $u(x, y, z, t) = \mathcal{F}(X, Y, T)$ , with  $X = x, Y = y - z$  and  $T = t$ . Thus, the reduction equation is

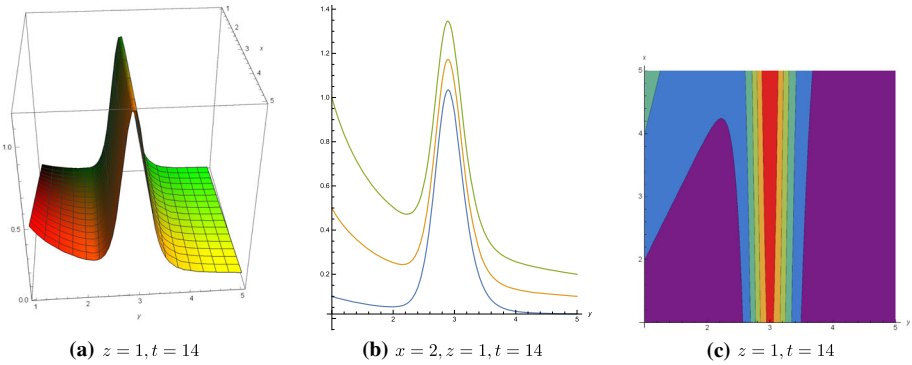
$$\mathcal{F}_T + (1 - 20\mathcal{F}_X)\mathcal{F}_{XXY} - 2\mathcal{F}_Y(-3\mathcal{F} - X + 30\mathcal{F}_X^2 + 5\mathcal{F}_{XXX}) - \mathcal{F}_{XXXXY} = 0 \tag{109}$$

By solving Eq. (109), we found the value of  $\mathcal{F}$  as

$$\mathcal{F} = c_1 + \frac{1}{\sqrt{5}} \tanh\left(\frac{1}{2\sqrt{5}}X + \frac{4}{25}Tc_3 - Yc_3 - c_2\right) \tag{110}$$

with  $c_1, c_2$  and  $c_3$ . Hence, using Eq. (110), the group-invariant solution in this case is

$$u(x, y, z, t) = c_1 + \frac{1}{\sqrt{5}} \tanh\left(\frac{1}{2\sqrt{5}}x + \frac{4}{25}c_3t - (y - z)c_3 - c_2\right) \tag{111}$$



**Fig. 5** The physical structure of single-soliton profile for (108) with  $g\left(\frac{z}{y}, \frac{3zt-5y^2}{3z}\right) = \text{sech}\left[\frac{3tz-5y^2}{3y}\right]^2$  shows perspective view of the real part of the bright-dark soliton solution, the wave propagation pattern of the wave along the  $y$ -axis

Following similar procedure we reduce Eq. (2) to ODEs and hence obtain solution for the case  $a_6 = -1$ .

4.9 **Subalgebra**  $\mathcal{T}_9 = \mathcal{V}_2 + \mathcal{V}_7 + \mathcal{V}_{10}$ :

In this case subalgebra is given by

$$\mathcal{V}_2 + \mathcal{V}_7 + \mathcal{V}_{10} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial u},$$

and using corresponding characteristic equations, we have  $u(x, y, z, t) = y + \mathcal{F}(X, Z, T)$ , with  $X = x - y, Z = z$  and  $T = t$ . Putting the value of  $u$  into (2), we have

$$\mathcal{F}_T + (-6 + 60F_Z)\mathcal{F}_X^2 + \mathcal{F}_X(6 + 20\mathcal{F}_{XXZ}) + (10\mathcal{F}_Z - 1)\mathcal{F}_{XXX} + \mathcal{F}_{XXXXZ} = 0 \quad (112)$$

Solving Eq. (112), we get

$$\mathcal{F}(X, Z, T) = 10c_2c_1 + \frac{1}{10c_2} \tanh\left(\frac{1}{20c_2}X + Zc_2 - \frac{(750c_2^2 - 1)}{2500c_2^3}T + c_4\right) \quad (113)$$

where  $c_1, \dots, c_4$  are arbitrary constants. Hence, one obtains

$$u(x, y, z, t) = 10c_2c_1 + \frac{1}{10c_2} \tanh\left(\frac{1}{20c_2}(x - y) + zc_2 - \frac{(750c_2^2 - 1)}{2500c_2^3}t + c_4\right) \quad (114)$$

Moreover, generators of Eq. (112) are



$$\begin{aligned} \xi_x &= \frac{1}{4}(18t + x)b_1 + \frac{1}{4}(6t - x)b_3 + b_5, \quad \xi_z = b_3z + b_4, \quad \xi_t = b_1t + b_2, \\ \eta_y &= \frac{1}{40}(-10b_1 + 10b_3)u + \frac{1}{40}zb_1 + \frac{3}{40}b_3z + b_6, \end{aligned} \tag{115}$$

where  $b_1, \dots, b_6$  are arbitrary constants. The characteristic equation is

$$\begin{aligned} \frac{dX}{\frac{1}{4}(18t + x)b_1 + \frac{1}{4}(6t - x)b_3 + b_5} &= \frac{dZ}{b_3z + b_4} = \frac{dT}{b_1t + b_2} \\ &= \frac{dF}{\frac{1}{40}(-10b_1 + 10b_3)u + \frac{1}{40}zb_1 + \frac{3}{40}b_3z + b_6}. \end{aligned} \tag{116}$$

Let  $b_3 = b_1$ , then group-invariant is

$$\mathcal{F}(X, Z, T) = G(r, s) + \frac{1}{10} \left( Z - \frac{(b_4 - 10b_6) \log(b_1Z + b_4)}{b_1} \right), \tag{117}$$

where

$$r = \frac{(6b_2 - b_5) \log(b_1T + b_2)}{b_1} - 6T + X, \quad s = \frac{b_1Z + b_4}{b_1^2T + b_1b_2}. \tag{118}$$

Reduced (1+1)-dimensional equation is given as

$$\begin{aligned} (b_1sG_r(20G_{rrs} + 6b_2 - b_5) + b_1sG_{rrrs} - 6(b_4 - 10b_6)G_r^2 - (b_4 - 10b_6)G_{rr}) \\ - b_1sG_s(b_1s - 10(6G_r^2 + G_{rr})) = 0. \end{aligned} \tag{119}$$

Under the condition  $b_6 = \frac{b_4}{10}$ , then above equation can be recast as

$$b_1s(G_s(60G_r^2 + 10G_{rr} - b_1s) + G_r(20G_{rrs} + 6b_2 - b_5) + G_{rrrs}) = 0. \tag{120}$$

Using infinitesimals, we write characteristic equation as

$$\frac{dr}{d_1} = \frac{ds}{0} = \frac{dG}{d_2}. \tag{121}$$

Then, we obtained  $G(r, s) = \frac{dr}{d_1} + R(w)$ , where  $w = s$ . We get the desired ODE as

$$\frac{b_1d_2s(6b_2 - b_5)}{d_1} - b_1sR'(s) \left( b_1s - \frac{60d_2^2}{d_1^2} \right) = 0. \tag{122}$$

The primitive is

$$R(s) = e_1 + \frac{d_2(6b_2 - b_5) \log(b_1d_1^2s - 60d_2^2)}{b_1d_1}. \tag{123}$$

Using back substitution, we obtain group-invariant solution is

$$\begin{aligned} u(x, y, z, t) = a_1 - \frac{b_5d_2 \ln \left( \frac{d_1^2(b_1z+b_4)}{b_1t+b_2} - 60d_2^2 \right)}{b_1d_1} + \frac{6b_2d_2 \ln \left( \frac{d_1^2(b_1z+b_4)}{b_1t+b_2} - 60d_2^2 \right)}{b_1d_1} \\ - \frac{b_5d_2 \ln(b_1t + b_2)}{b_1d_1} + \frac{6b_2d_2 \ln(b_1t + b_2)}{b_1d_1} - \frac{6d_2t}{d_1} + \frac{d_2x}{d_1} - \frac{d_2y}{d_1} + y + \frac{z}{10}. \end{aligned} \tag{124}$$

4.10 **Subalgebra**  $\mathcal{T}_{10} = \mathcal{V}_7 + \mathcal{V}_8 + \mathcal{V}_{10}$ :

For the infinitesimal generator

$$\mathcal{V}_7 + \mathcal{V}_8 + \mathcal{V}_{10} = (t + 1) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{y}{6} \frac{\partial}{\partial u},$$

ultimately gives  $u(x, y, z, t) = \mathcal{F}(X, Z, T)$ , where  $Y = x - y(1 + t)$ ,  $Z = z$  and  $T = t$ . Putting the above value of  $u$  into governing equation, we have

$$\mathcal{F}_T + 20\mathcal{F}_X \mathcal{F}_{XXZ} + \mathcal{F}_{XXXZ} - (1 + T - 10\mathcal{F}_Z)(6\mathcal{F}_X^2 + \mathcal{F}_{XX}) = 0. \tag{125}$$

Infinitesimals of Eq. (125) are

$$\begin{aligned} \xi_X &= \frac{2}{3} \tilde{b}_1 X + \tilde{b}_4, & \xi_Z &= -\frac{5}{3} \tilde{b}_1 Z + \tilde{b}_3, & \xi_T &= \tilde{b}_1 T + \tilde{b}_2, \\ \eta_{\mathcal{F}} &= \frac{1}{30} (-3Z - 20\mathcal{F}) \tilde{b}_1 + \frac{1}{10} Z \tilde{b}_2 + \tilde{b}_5, \end{aligned} \tag{126}$$

with  $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4$  and  $\tilde{b}_5$  being the arbitrary constants.

Case I: If  $\tilde{b}_1 \neq 0$ , then using Eq. (126)

$$\frac{dX}{\frac{2}{3}X + b_1} = \frac{dZ}{-\frac{5}{3}Z + b_2} = \frac{dT}{T + b_3} = \frac{d\mathcal{F}}{\frac{1}{30}(-3Z - 20\mathcal{F}) + \frac{1}{10}Zb_3 + b_4}, \tag{127}$$

where  $b_1 = \frac{\tilde{b}_4}{\tilde{b}_1}, b_2 = \frac{\tilde{b}_3}{\tilde{b}_1}, b_3 = \frac{\tilde{b}_2}{\tilde{b}_1}, b_4 = \frac{\tilde{b}_5}{\tilde{b}_1}$ , then similarity forms of original equation yield

$$\mathcal{F}(X, Z, T) = (3b_2 - 5Z)^{2/5} G(r, s) + \frac{1}{20} (3b_2(b_3 - 1) + 2(-b_3Z + 15b_4 + Z)), \tag{128}$$

with similarity variable

$$r = \frac{3b_1 + 2X}{2(b_3 + T)^{2/3}} \text{ and } s = -\frac{1}{5} (3b_2 - 5Z)(b_3 + T)^{5/3}. \tag{129}$$

The second reduction in this case

$$\begin{aligned} &- 60 \times 5^{2/5} s^{2/5} G_r (2G_{rr} + 5sG_{rrs}) \\ &+ 3 \times 5^{2/5} s^{2/5} \left( -50sG_s - 20G - (5s)^{3/5} \right) \left( 6 \times 5^{2/5} s^{2/5} G_r^2 + G_{rrr} \right) \\ &+ 5s (2rG_r - 5sG_s) - 15sG_{rrrs} - 6G_{rrrr} = 0. \end{aligned} \tag{130}$$

Clearly, one nontrivial solution of Eq. (130)

$$G(r, s) = c, \text{ a constant}, \tag{131}$$

Eventually, group-invariant solution of Eq. (2)

$$u(x, y, z, t) = \frac{1}{20} (3b_2(b_3 - 1) + 2(-b_3z + 15b_4 + z)) + c(3b_2 - 5z)^{2/5} + \frac{y^2}{12}. \tag{132}$$

which presents a parabolic wave profile in  $y - z$  plane.

Case II: If  $\tilde{b}_1 = 0$ , then using Eq. (126) then we obtain

$$\frac{dX}{b_4} = \frac{dZ}{b_3} = \frac{dT}{b_2} = \frac{d\mathcal{F}}{\frac{1}{10}Zb_2 + b_5}, \tag{133}$$

then similarity forms of KdV-type equation yield

$$\mathcal{F}(X, Z, T) = G(r, s) + \frac{\frac{b_2 Z^2}{2} + 10b_5 Z}{10b_3}, \tag{134}$$

where similarity variables  $s = Z - \frac{b_3 T}{b_2}$  and  $r = X - \frac{b_4 Z}{b_3}$ .

The second reduction of Eq. (2) produces

$$\begin{aligned} & b_2 \left( (6G_r^2 + G^{(3,0)}(r, s)) (10b_3 G_s - 10b_4 G_r + b_2 s - b_3 + 10b_5) \right. \\ & \quad + 20G_r (b_3 G_{rrs} - b_4 G_{rrr}) + b_3 G_{rrrs} \\ & \quad \left. - b_2 b_4 G_{rrrr} - b_3^2 G_s = 0. \right. \end{aligned} \tag{135}$$

Infinitesimals of Eq (135) are

$$\xi_r = d_1, \quad \xi_s = 0, \quad \eta_G = d_2, \tag{136}$$

where  $d_1$  and  $d_2$  are arbitrary constants. In order to find corresponding third reduction using Eq. (136), we obtain the correspond group-invariant form

$$G(r, s) = \frac{d_2 r}{d_1} + R(w), \tag{137}$$

with  $w = s$ , which leads to ordinary differential equation given as

$$\frac{6b_2 d_2^2 \left( b_2 s + 10b_3 R'(s) - b_3 - \frac{10b_4 d_2}{d_1} + 10b_5 \right)}{d_1^2} - b_3^2 R'(s) = 0. \tag{138}$$

The solution of (138) is

$$R(w) = \frac{6b_2 d_2^2 \left( \frac{1}{2} b_2 d_1 w^2 - b_3 d_1 w - 10b_4 d_2 w + 10b_5 d_1 w \right)}{b_3 d_1 (b_3 d_1^2 - 60b_2 d_2^2)} + c_1. \tag{139}$$

Eventually, the group-invariant solution of (2) is

$$\begin{aligned} u(x, y, z, t) = & \frac{6b_2 d_2^2 \left( -10b_4 d_2 \left( z - \frac{b_3 t}{b_2} \right) + 10b_5 d_1 \left( z - \frac{b_3 t}{b_2} \right) + \frac{1}{2} b_2 d_1 \left( z - \frac{b_3 t}{b_2} \right)^2 - b_3 d_1 \left( z - \frac{b_3 t}{b_2} \right) \right)}{b_3 d_1 (b_3 d_1^2 - 60b_2 d_2^2)} \\ & + \frac{\frac{b_2 z^2}{2} + 10b_5 z}{10b_3} + \frac{d_2 \left( -\frac{b_4 z}{b_3} - (t + 1)y + x \right)}{d_1} + c_1 + \frac{y^2}{12}. \end{aligned} \tag{140}$$

which exhibits also a parabolic wave profile in  $x, y, z, t$ .

*Remark* The obtained group-invariant solutions including the WeierstrassZeta function, M-shaped solitons, W-shaped solitons, multi-solitons and rational function solutions which are entirely different compared to other researchers works [29,32–34]. Also, the different dynamical structures of these solutions have rich localized structures due to the existence of free parameters in the infinitesimals.

4.11 **Subalgebra**  $\mathcal{T}_{11} = \mathcal{V}_2 + \mathcal{V}_6$ :

For the infinitesimal generator

$$\mathcal{V}_2 + \mathcal{V}_6 = \frac{\partial}{\partial z} + \frac{\partial}{\partial u},$$

we have following characteristic equation

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{du}{1}. \tag{141}$$

The similarity form of the solution of Eq. (2)

$$u(x, y, z, t) = \mathcal{F}(X, Y, T), \tag{142}$$

with similarity variables  $X = x, Y = y$  and  $T = t$ . Inserting the value of  $u$  into (2), we have

$$\mathcal{F}_T + 6\mathcal{F}_X\mathcal{F}_Y + 60\mathcal{F}_X^2 + \mathcal{F}_{XXY} + 10\mathcal{F}_{XXX} = 0. \tag{143}$$

To solve Eq. (143), we obtain new set of infinitesimal generators given as

$$\begin{aligned} \xi_X &= -\frac{X}{2}(X - 10Y)A_1 + \frac{1}{2}(-X + 30Y)A_4 + A_6T + A_7, \\ \xi_Y &= A_3T + A_4Y + A_5, \\ \xi_T &= A_1T + A_2, \\ \eta_{\mathcal{F}} &= \frac{1}{6}(-3A_1 + 3A_4)\mathcal{F} + \frac{1}{6}(X - 20Y)A_3 + \frac{1}{6}YA_6 + A_8, \end{aligned} \tag{144}$$

where  $A_1, A_2, A_3, A_4, A_5, A_6, A_7$  and  $A_8$  are arbitrary constants.

For  $A_2, A_5, A_7$  and  $A_8$  being nonzero and taking all other constants zero, we obtain following characteristic equations

$$\frac{dX}{1} = \frac{dY}{1} = \frac{dT}{1} = \frac{d\mathcal{F}}{1}. \tag{145}$$

The function  $\mathcal{F}$  can be written as

$$\mathcal{F}(X, Y, T) = T + G(r, s), \tag{146}$$

through  $r = X - T, s = Y - T$ . Putting the similarity form in Eq. (143), we have following reduced equation

$$1 - G_s - G_r + 6G_sG_r + 60G_r^2 + G_{rs} + 10G_{rr} = 0. \tag{147}$$

Again we can find infinitesimals for Eq. (143) given as

$$\frac{dr}{\frac{1}{3}(-r + 40s)B_1 + B_3} = \frac{ds}{B_1s + B_2} = \frac{dG}{\frac{1}{9}(-r + s + 3G)B_1 + B_4}, \tag{148}$$

where  $B_1, B_2, B_3, B_4$  are arbitrary constants. Let us take  $B_1 = 0$ , then the desired characteristic equations are given as

$$\frac{dr}{B_3} = \frac{ds}{B_2} = \frac{dG}{B_4}. \tag{149}$$

We obtain the similarity variable as  $w = r - \frac{B_3}{B_2}s$ , and the similarity form is given by

$$G(r, s) = \frac{B_2}{B_4}s + R(w), \tag{150}$$

Putting  $G$  in Eq. (147), we obtain

$$B_2 - B_4 + R'(-B_2 + B_3 + 6B_4 + (60B_2 - 6B_3)R') + (10B_2 - B_3)R''' = 0 \tag{151}$$

Primitives for Eq. (151) are

$$R(w) = \alpha + \frac{(B_2 - B_3 - 6B_4 + \sqrt{\xi})w}{2(60B_2 - 6B_3)},$$

$$R(w) = \alpha + \frac{(B_2 - B_3 - 6B_4 - \sqrt{\xi})w}{2(60B_2 - 6B_3)} \tag{152}$$

where  $\xi = -239B_2^2 + 22B_2B_3 + B_3^2 + 228B_2B_4 - 12B_3B_4 + 36B_4^2$  and  $\alpha$  is an arbitrary constant. Using Eqs (152), (150), (146) in (142), we obtain

$$u(x, y, z, t) = \alpha + t + z + \frac{B_4}{B_2}(y - t) + \frac{(B_2 - B_3 - 6B_4 + \sqrt{\xi})w}{2(60B_2 - 6B_3)}(B_2(t - x) + B_3(y - t)), \tag{153}$$

$$u(x, y, z, t) = \alpha + t + z + \frac{B_4}{B_2}(y - t) + \frac{(B_2 - B_3 - 6B_4 - \sqrt{\xi})w}{2(60B_2 - 6B_3)}(B_2(t - x) + B_3(y - t)). \tag{154}$$

In summary, the physical interpretation of the resulting soliton solutions is illustrated by various three-dimensional, two-dimensional and contour graphs through numerical simulation. The constructed group-invariant solutions involve many arbitrary constants and arbitrary functions, thereby exhibiting rich physical structures and including the existing solutions in the literature.

### 5 Adjoint equation and conservation laws

#### 5.1 Necessary preliminaries and adjoint equation

For a given differential equation, there is a close connection between Lie symmetries and conservation laws as established by Noether’s theorem. To derive conservation laws of Eq. (2), we use the following Theorem proved by Ibragimov [6,38].

**Theorem 1** Any symmetry (Lie point, Lie-Bäcklund, nonlocal symmetry)

$$X = \xi^i(x, u, u_{(1)}, \dots, u_{(s)}) \frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \dots, u_{(s)}) \frac{\partial}{\partial u}, \tag{155}$$

of Eq. (2) provides a conservation law  $\mathcal{D}_i(C_i) = 0$  for the system consisting of Eq. (2) and the adjoint equation

$$F^*(x, u, v, u_{(1)}, \mathcal{V}_{(1)} \dots u_{(s)}, \mathcal{V}_{(s)}) = \frac{\delta(pF)}{\delta u} = 0. \tag{156}$$

The conserved vector is given by

$$C^i = \xi^i \mathcal{L} + W \left[ \frac{\partial \mathcal{L}}{\partial u_i} - \mathcal{D}_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + \mathcal{D}_j \mathcal{D}_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) - \dots \right] \tag{157}$$

$$+ \mathcal{D}_j(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}} - \mathcal{D}_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) \dots \right] + \mathcal{D}_j \mathcal{D}_k(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}} \dots \right], \tag{158}$$

where  $W$  and  $\mathcal{L}$  are defined as

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = p(x)F(x, u, u_{(1)}, \dots, u_{(s)}). \tag{159}$$

For a conserved vector, the following conservation equation holds:

$$\mathcal{D}_x(C^x) + \mathcal{D}_y(C^y) + \mathcal{D}_z(C^z) + \mathcal{D}_t(C^t) = 0 \tag{160}$$

where  $C^x = C^x(t, x, y, z, u, \dots)$ ,  $C^y = C^y(t, x, y, z, u, \dots)$ ,  $C^z = C^z(t, x, y, z, u, \dots)$ ,  $C^t = C^t(t, x, y, z, u, \dots)$ .

A formal Lagrangian for three-dimensional KdV-type equation is

$$\mathcal{L} = p(x, y, z, t)[u_t + 6u_x u_y + u_{xxy} + u_{xxxz} + 60u_x^2 u_z + 10u_{xxx} u_z + 20u_x u_{xxz}]. \tag{161}$$

Here,  $p(x, y, z, t)$  is a new dependent variable also known as adjoint variable. According to Eq. (161), we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_x} &= -p_x(6u_y + 120u_z u_x + 20u_{xxz})p(120u_x u_{xz} + 6u_{xy} + 120u_z u_{xx} + 20u_{xxxz}), \\ \frac{\partial \mathcal{L}}{\partial u_y} &= -6p_y u_x - 6p u_{xy}, \quad \frac{\partial \mathcal{L}}{\partial u_z} = -p_z(60u_x^2 + 10u_{xxx}) - p(120u_x u_{xz} + 10u_{xxxz}), \\ \frac{\partial \mathcal{L}}{\partial u_t} &= p_t, \quad \frac{\partial \mathcal{L}}{\partial u_{xxz}} = -20u_{xz} p_{xx} - 40p_{xz} u_{xx} \\ &\quad - 20u_x p_{xxz} - 40p_x u_{xxz} - 20p_z u_{xxx} - 20p u_{xxxz}, \\ \frac{\partial \mathcal{L}}{\partial u_{xxy}} &= -p_{xy}, \quad \frac{\partial \mathcal{L}}{\partial u_{xxx}} = -30u_{xz} p_{xx} - 30p_x u_{xxz} - 10u_z p_{xxx} \\ &\quad - 10p u_{xxxz}, \quad \frac{\partial \mathcal{L}}{\partial u_{xxxz}} = -p_{xxxz} \end{aligned} \tag{162}$$

The adjoint equation for (3+1)-dimensional KdV-type equation is given by

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = 0, \tag{163}$$

where

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &= \frac{\partial \mathcal{L}}{\partial u} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_x} - \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_y} - \mathcal{D}_z \frac{\partial \mathcal{L}}{\partial u_z} - \mathcal{D}_t \frac{\partial \mathcal{L}}{\partial u_t} - \mathcal{D}_x \mathcal{D}_x \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_{xxy}} - \mathcal{D}_x \mathcal{D}_x \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxx}} \\ &\quad - \mathcal{D}_x \mathcal{D}_x \mathcal{D}_z \frac{\partial \mathcal{L}}{\partial u_{xxz}} - \mathcal{D}_x \mathcal{D}_x \mathcal{D}_x \mathcal{D}_z \frac{\partial \mathcal{L}}{\partial u_{xxxz}}, \end{aligned} \tag{164}$$

where  $\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_z$  and  $\mathcal{D}_t$  denote the total differentiation with respect to  $x, y, z$  and  $t$ , respectively. Substituting Eq. (164) into Eq. (163), the adjoint equation for the (3+1)-dimensional KdV-type equation is expressed by

$$\begin{aligned} F^* &= -p_t - 6p_y u_x - 6p u_{xy} - 50u_{xz} p_{xx} - 40p_{xz} u_{xx} - 20u_x p_{xxz} \\ &\quad - 70p_x u_{xxz} - p_x(6u_y + 120u_z u_x + 20u_{xxz}) - p_{xy} - 10u_z p_{xxx} - 20p_z u_{xxx} \\ &\quad - p_z(60u_x^2 + 10u_{xxx}) - 30p u_{xxxz} - p(120u_x u_{xz} + 10u_{xxxz}) \\ &\quad - p(120u_x u_{xz} + 6u_{xy} + 120u_z u_{xx} + 20u_{xxxz}) - p_{xxxz}. \end{aligned} \tag{165}$$

Clearly, when  $p = u$ , we obtain

$$\begin{aligned} &-u_t - 12u_y u_x - 180u_z u_x^2 - 240u u_x u_{xz} - 12u u_{xy} - 120u u_z u_{xx} \\ &- 90u_{xz} u_{xx} - 110u_x u_{xxz} - u_{xxy} - 40u_z u_{xxx} - 60u u_{xxxz} - u_{xxxz} = 0. \end{aligned} \tag{166}$$

It is easily obtain that on substituting  $u$  instead of  $p$  in adjoint equation (2) is not recovered. Thus, (3+1)-dimensional KdV-type equation is not self-adjoint”.

### 5.2 Conservation laws

Using Lagrangian in Eq. (161) and Theorem 1 for Eq. (2), the general form of conservation laws is given as

$$\begin{aligned}
 C^x &= \xi^1 \mathcal{L} + W \left( \frac{\partial \mathcal{L}}{\partial u_x} + \mathcal{D}_{xx} \frac{\partial \mathcal{L}}{\partial u_{xxx}} + \mathcal{D}_{xz} \frac{\partial \mathcal{L}}{\partial u_{xxz}} + \mathcal{D}_{xy} \frac{\partial \mathcal{L}}{\partial u_{xy}} + \mathcal{D}_{xxxz} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) \\
 &\quad + W_x \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxx}} - \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_{xxy}} - \mathcal{D}_z \frac{\partial \mathcal{L}}{\partial u_{xxz}} - \mathcal{D}_{xxz} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) \\
 &\quad + W_y \left( \frac{\partial \mathcal{L}}{\partial u_{xy}} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) + W_z \left( \frac{\partial \mathcal{L}}{\partial u_{xz}} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxz}} - \mathcal{D}_{xxx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) \\
 &\quad + W_{xx} \left( \frac{\partial \mathcal{L}}{\partial u_{xxx}} + \mathcal{D}_{zx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_{xy} \left( \frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) + W_{xz} \left( \frac{\partial \mathcal{L}}{\partial u_{xxz}} + \mathcal{D}_{xx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) \\
 &\quad + W_{xxx} \left( -\mathcal{D}_z \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_{xxz} \left( -\mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_{xxxz} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}}, \\
 C^y &= \xi^y \mathcal{L} + W \left( \frac{\partial \mathcal{L}}{\partial u_y} + \mathcal{D}_{xx} \frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) + W_x \left( -\mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) + W_{xx} \left( \frac{\partial \mathcal{L}}{\partial u_{xxy}} \right), \\
 C^z &= \xi^z \mathcal{L} + W \left( \frac{\partial \mathcal{L}}{\partial u_z} + \mathcal{D}_{xx} \frac{\partial \mathcal{L}}{\partial u_{xxz}} + \mathcal{D}_{xxx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_x \left( -\mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxz}} - \mathcal{D}_{xxx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) \\
 &\quad + W_{xx} \left( \mathcal{D}_{xx} \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_{xxx} \left( -\mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right) + W_{xxxz} \left( \frac{\partial \mathcal{L}}{\partial u_{xxxxz}} \right), \\
 C^t &= \tau \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_t},
 \end{aligned} \tag{167}$$

with  $W = \eta - \xi_x u_x - \xi_y u_y - \tau u_t$ .

In this article, we consider conservation laws related with two infinitesimal symmetries  $\mathcal{V}_3$  and  $\mathcal{V}_{10}$ . They are given as follows: For the symmetry  $\mathcal{V}_3 = \frac{\partial}{\partial t}$ , we have  $W = -u_t$ . Now, using Eqs. (162) in Eq. (167), the corresponding conservation laws are

$$\begin{aligned}
 C^x &= u_{yt} p_x - p u_{xyt} - u_{xzt} (20 p u_x + p_{xx}) - (10 p u_z + p_{xz}) u_{xxt} \\
 &\quad + u_{xt} (p_y + 10 u_z p_x + 20 p_z u_x + 30 p u_{xz} + p_{xxz}) \\
 &\quad - u_t (20 u_x p_{xz} + 40 p_x u_{xz} + p_{xy} + 10 u_z p_{xx} + 20 p_z u_{xx}) \\
 &\quad + 2p (3u_y + 60u_z u_x + 25u_{xxz}) + p_x u_{xxzt} \\
 &\quad + u_{zt} (20 p_x u_x + 20 p u_{xx} + p_{xxx}) + p_z u_{xxxz} - p u_{xxxzt}, \\
 C^y &= p_x u_{xt} - u_t (6 p u_x + p_{xx}) - p u_{xxt}, \\
 C^z &= -(20 p u_x + p_{xx}) u_{xzt} + u_{xt} (20 p_x u_x + 20 p u_{xx} + p_{xxx}) + p_x u_{xxxz} \\
 &\quad - u_t (20 u_x p_{xx} + 40 p_x u_{xx} + 30 p (2u_x^2 \\
 &\quad + u_{xxx}) + p_{xxxz}) - p u_{xxxzt}, \\
 C^t &= -p u_t + p (u_t + 6u_y u_x + 60u_z u_x^2 + 20u_x u_{xxz} + u_{xxy} + 10u_z u_{xxx} + u_{xxxxz}),
 \end{aligned} \tag{168}$$

For the symmetry  $\mathcal{V}_{10} = \frac{\partial}{\partial x}$ , we have  $W = -u_x$ . Now, using Eqs. (162) in Eq. (167), the corresponding conservation laws are

$$\begin{aligned}
 C^x &= -20u_x^2 p_{xz} + p_x u_{xy} - u_x (20 p_x u_{xz} + p_{xy} + 10 u_z p_{xx}) + p_y u_{xx} + 10 u_z p_x u_{xx} \\
 &\quad + u_{xx} p_{xxz} - p_{xx} u_{xxz} + p (u_t - 60 u_z u_x^2 + 50 u_{xz} u_{xx} - 50 u_x u_{xxz}) + u_{xz} p_{xxx} \\
 &\quad - p_{xz} u_{xxx} + p_x u_{xxxx} + p_z u_{xxxx}, \\
 C^y &= -u_x p_{xx} + p_x u_{xx} - p (6u_x^2 + u_{xxx}),
 \end{aligned}$$

$$\begin{aligned}
C^z &= -20u_x^2 p_{xx} u_{xx} p_{xxx} - p_{xx} u_{xxx} - u_x (20p_x u_{xx} + p_{xxx}) + p_x u_{xxxx} \\
&\quad - p(60u_x^3 - 20u_{xx}^2 + 50u_x u_{xxx} + u_{xxxx}), \\
C^t &= -p u_x,
\end{aligned}
\tag{169}$$

Similarly, one can obtain other conservation laws corresponding to each Lie point symmetry. Making use of explicit solutions of Eq. (165), nonlocal conservation laws in case of each Lie point symmetries can be obtained for (3+1)-dimensional KdV-equation.

## 6 Results and discussion

The exact analytical wave solutions are obtained with the help of symbolic computation via Lie symmetry analysis for a (3+1)-KdV equation. The dynamical behavior of obtained solutions demonstrates the different lump-type soliton solutions for adequate choice of arbitrary independent functions and free parameters. Some important physical structures are obtained, and a summary of the profiles of the solutions is as follows:

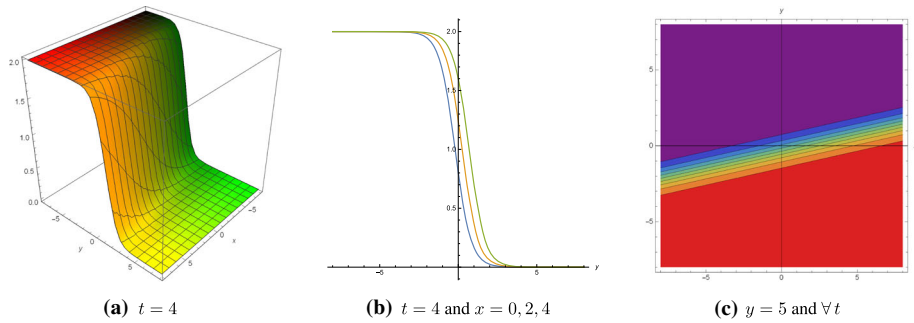
Figure 1 shows the physical structures of Lump-type solitons and multi-soliton profiles for (55) with parameters  $c_1 = 154$ ,  $c_2 = 0.003$ ,  $c_3 = 2717$ . Properties of such solitons reveal that solitons do not transform their size and shapes when they are associated with each other and complete elastic nature is described by the wave structures. In this figure, we have shown elastic behavior with parameters (a)  $a = 0.3$ ,  $b = 1$ , (b)  $a = 1$ ,  $b = 1$ , (c)  $a = 1.5$ ,  $b = 1$ , (d)  $a = 0.3$ ,  $b = 4$ , (e)  $a = 1$ ,  $b = 4$ , (f)  $a = 1.5$ ,  $b = 4$ . Moreover, annihilation has been achieved for parameters (g)  $a = 1$ ,  $b = 1$ ,  $c_1 = 1$ ,  $c_2 = 0.3$ ,  $c_3 = 1$ , (h)  $a = 1$ ,  $b = 5$ ,  $c_1 = 1$ ,  $c_2 = 0.3$ ,  $c_3 = 1$ , (i)  $a = 1$ ,  $b = 20$ ,  $c_1 = 1$ ,  $c_2 = 0.3$ ,  $c_3 = 1$ . In this figure, we have observed annihilation of multisoliton into a stationary wave profile for Eq. (55) by changing the specific arbitrary constants as  $c_1 = 1$ ,  $c_2 = 0.3$ ,  $c_3 = 1$ .

Figure 2 exhibits the physical structure of W-shaped soliton profile for (104) presents doubly soliton, traveling wave along the  $x$ -axis for  $g\left(\frac{t}{y}, \frac{3zt-5y^2}{3t}\right) = \tanh^2\left(\frac{3tz-5y^2}{3y}\right) + \tanh^2\left(\frac{2t^3}{y^3} + \frac{2(3tz-5y^2)}{t}\right) + \tanh^2\left(\frac{(3tz-5y^2)^2}{9t^2} + \frac{2t^2}{y^2} + 1\right)$ . Moreover, two-dimensional wave profiles and contour plots are demonstrated in this figure for parameters  $z = 0.9654$ ,  $t = 6$  and  $x = 1, 5, 10$ . An important observation is the amplitude, the velocity and the shape of the soliton remains constant.

Figure 3 shows the physical structure of M-shaped soliton profile for (104) shows doubly soliton; 2D plot exhibits solitary waves for  $g\left(\frac{t}{y}, \frac{3zt-5y^2}{3t}\right) = \operatorname{sech}^2\left(\frac{3tz-5y^2}{3y}\right) + \operatorname{sech}^2\left(\frac{2t^3}{y^3} + \frac{2(3tz-5y^2)}{t}\right) + \operatorname{sech}^2\left(\frac{(3tz-5y^2)^2}{9t^2} + \frac{2t^2}{y^2} + 1\right)$ . In this figure, three-dimensional plot is sketched using  $z = 0.9654$ ,  $t = 6$ , two-dimensional plot shows M-wave propagation for  $x = 3.5$ ,  $z = 0.9654$  and a contour plot is also exhibited for  $z = 0.9654$ ,  $t = 6$ . Such solitary waves type precisely M-shaped and W-shaped solitons described the propagation of ultrashort pulses in optical fibers [39].

Figure 4 represents V-type physical structure of single soliton profile for (108) with  $g\left(\frac{z}{y}, \frac{3zt-5y^2}{3z}\right) = \tanh\left[\frac{3tz-5y^2}{3y}\right]^2$  shows perspective view of the real part of the dark-bright soliton solution and the wave propagation pattern of the wave along the  $y$ -axis. It includes three-dimensional plots for  $z = 1$ ,  $t = 15$ , wave propagation using two-dimensional plot for  $x = 2$ ,  $z = 1$ ,  $t = 15$  and contour plot for  $z = 1$ ,  $t = 15$ .





**Fig. 6** The physical structure of kink wave soliton profile (111) for  $c_1 = 0.4873$ ,  $c_2 = 0.3374$  and  $c_3 = 0.3984$  shows strip single soliton in 3D plot; 2D plot with three values of  $x$  and corresponding contour shape

Figure 5 exhibits the physical structure of single soliton profile for (108) with  $g\left(\frac{z}{y}, \frac{3zt-5y^2}{3z}\right) = \text{sech}\left[\frac{3tz-5y^2}{3y}\right]^2$  and shows perspective view of the real part of the bright-dark soliton solution, the wave propagation pattern of the wave along the  $y$ -axis. It includes three-dimensional plot for  $z = 1, t = 14$ , wave propagation using two-dimensional plot for  $x = 2, z = 1, t = 14$  and contour plot for  $z = 1, t = 14$ . Such types of solutions are new soliton solutions obtained in this work that can help us with understanding the propagation of solitons through a nonlinear medium.

Figure 6 shows the physical structure of Kink wave soliton profile (111) for  $c_1 = 0.4873$ ,  $c_2 = 0.3374$  and  $c_3 = 0.3984$  which describes strip single soliton in 3D plot; 2D plot with three values of  $x$  and corresponding contour shape [40]. Using the values of certain free parameters, we can control the solitons propagation direction and speed and reduce the interactions between them as well.

It is interesting to notify that the established solutions in this article have not been reported in the literature. Furthermore, the wide diversity of features and physical parameters of these generated soliton solutions are illustrated with the assistance of 3D plots, considering the suitable choice of involved arbitrary independent functions and other constants. Such type of investigation is highly recommended in the areas of progressive research and development.

### 7 Conclusion

Applications of the Lie group-theoretic method are well defined for constructing Lie point symmetries and group-invariant solutions of three-dimensional KdV-type equation. The geometric vector fields spanned by ten basic Lie point symmetries are obtained with the help of computerized symbolic computation *Maple*. An optimal system of ten symmetry subalgebras is established to classify all the symmetry reductions. By using the optimal system, Eq. (2) is converted into numerous NPDEs with less order. In this work, symbolic computation is used for numerical simulation of various solutions and different types of solutions are derived and interpreted via three-dimensional and two-dimensional graphs through *Mathematica 11.3*. The obtained solutions are given by equations (44), (55), (70), (75), (80), (83), (96), (104), (108), (111), (100), (114), (124), (132), (140) which are entirely different compared to the works [33,34]. Using the Ibragimov approach, we constructed nonlocal conservation laws for some Lie point symmetries. The obtained conservation laws can be used in the construc-

tion of new numerical schemes and stability analysis of solutions so obtained. Some exact analytic solutions in the shapes of kink waves, traveling waves, single solitons, doubly solitons, curved-shaped multi-solitons and explicit WeierstrassZeta are constructed by Lie group of transformation method. Moreover, this work reveals that some traveling waves which propagate are solitary wave types precisely *M*-shaped, *W*-shaped solitons and dark-bright solitons.

**Declaration**

**Conflict of interest** None.

**8 Appendix I**

$$A = \begin{pmatrix} 1 & A_{12} & -\epsilon_3 & \frac{1}{4}e^{\epsilon_5/2}\epsilon_4 & 0 & A_{16} & A_{17} & A_{18} & \frac{\epsilon_9}{4} & A_{110} \\ 0 & e^{\frac{\epsilon_5-\epsilon_1}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{32} & e^{\epsilon_1} & 0 & 0 & \frac{1}{20}(-3)e^{\epsilon_1+\epsilon_5}\epsilon_4^2 & A_{37} & 0 & 0 & A_{310} \\ 0 & A_{42} & 0 & e^{\frac{\epsilon_5-\epsilon_1}{2}} & 0 & -e^{\frac{\epsilon_5-\epsilon_1}{2}}\epsilon_7 & A_{38} & A_{48} & 0 & A_{410} \\ 0 & A_{52} & 0 & -\frac{1}{2}e^{\epsilon_5/2}\epsilon_2 & 1 & \frac{1}{2}e^{\epsilon_5/2}\epsilon_2\epsilon_7 - \epsilon_6 & -\frac{\epsilon_7}{2} & A_{58} & \frac{3\epsilon_9}{4} & A_{510} \\ 0 & \frac{e^{\epsilon_5}\epsilon_9}{10} & 0 & 0 & 0 & e^{\epsilon_5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}e^{\frac{\epsilon_1+\epsilon_5}{2}}\epsilon_8 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_1+\epsilon_5}{2}} & 0 & 0 & A_{710} \\ 0 & A_{82} & 0 & 0 & 0 & 0 & 0 & A_{88} & 0 & A_{810} \\ 0 & A_{92} & 0 & 0 & 0 & 0 & 0 & \frac{1}{10}(-3)e^{-\frac{\epsilon_1}{4}-\frac{\epsilon_5}{4}}\epsilon_4 & A_{98} & A_{910} \\ 0 & \frac{1}{20}e^{\frac{\epsilon_1+\epsilon_5}{4}}\epsilon_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1010} \end{pmatrix} \tag{170}$$

where  $A_{ij}$  are given below

$$A_{12} = \frac{18e^{\epsilon_5}\epsilon_3\epsilon_9\epsilon_4^2 + 15e^{\epsilon_5/2}(\epsilon_{10} - 4\epsilon_3\epsilon_8 - 2\epsilon_7\epsilon_9)\epsilon_4 + 300e^{\epsilon_5/4}\epsilon_2 - 100\epsilon_7\epsilon_8}{1200},$$

$$A_{52} = \frac{1}{120}(-30e^{\epsilon_5/4}\epsilon_2 - 3e^{\epsilon_5/2}(\epsilon_{10} - 2\epsilon_7\epsilon_9)\epsilon_2 - 2(5\epsilon_7\epsilon_8 + 6\epsilon_6\epsilon_9)),$$

$$\begin{aligned}
 A_{16} &= \frac{3}{20} e^{\epsilon_5} \epsilon_3 \epsilon_4^2 - \frac{1}{4} e^{\epsilon_5/2} \epsilon_4 \epsilon_7, & A_{32} &= \frac{1}{20} e^{\epsilon_1 + \frac{\epsilon_5}{2}} \epsilon_4 \epsilon_8 - \frac{3}{200} e^{\epsilon_1 + \epsilon_5} \epsilon_4^2 \epsilon_9, \\
 A_{17} &= \frac{1}{10} \left( -3e^{\epsilon_5/2} \epsilon_3 \epsilon_4 - 5\epsilon_7 \right), & A_{37} &= \frac{3}{10} e^{\epsilon_1 + \frac{\epsilon_5}{2}} \epsilon_4, \\
 A_{18} &= \frac{3}{40} \left( 10\epsilon_8 + e^{\epsilon_5/2} \epsilon_4 \epsilon_9 \right), & A_{38} &= \frac{-3}{10} e^{\frac{\epsilon_5 - \epsilon_1}{2}} (\epsilon_3 + \epsilon_4), \\
 A_{42} &= \frac{1}{20} e^{\frac{\epsilon_5 - \epsilon_1}{2}} (\epsilon_{10} - (\epsilon_3 + \epsilon_4)\epsilon_8 - 2\epsilon_7 \epsilon_9), & A_{48} &= \frac{3}{10} e^{\frac{\epsilon_5 - \epsilon_1}{2}} \epsilon_9, \\
 A_{58} &= \frac{1}{20} \left( 5\epsilon_8 - 3e^{\epsilon_5/2} \epsilon_2 \epsilon_9 \right), & A_{88} &= e^{-\frac{3\epsilon_1}{4} - \frac{\epsilon_5}{4}}, \\
 A_{82} &= -\frac{1}{60} e^{-\frac{3\epsilon_1}{4} - \frac{\epsilon_5}{4}} \left( 3e^{\epsilon_5/2} \epsilon_3 \epsilon_4 + 10\epsilon_7 \right), & A_{92} &= \frac{1}{20} e^{-\frac{\epsilon_1}{4} - \frac{3\epsilon_5}{4}} \left( e^{\epsilon_5/2} \epsilon_4 \epsilon_7 - 2\epsilon_6 \right), \\
 A_{98} &= e^{-\frac{\epsilon_1}{4} - \frac{3\epsilon_5}{4}}, & A_{510} &= \frac{1}{4} (\epsilon_{10} - 2\epsilon_7 \epsilon_9), \\
 A_{110} &= \frac{1}{20} \left( -6e^{\epsilon_5/2} \epsilon_3 \epsilon_4 \epsilon_9 - 5(\epsilon_{10} + 4\epsilon_3 \epsilon_8 + 2\epsilon_7 \epsilon_9) \right), & A_{710} &= e^{\frac{\epsilon_1 + \epsilon_5}{2}} \epsilon_9, \\
 A_{310} &= e^{\epsilon_1} \epsilon_8 + \frac{3}{10} e^{\epsilon_1 + \frac{\epsilon_5}{2}} \epsilon_4 \epsilon_9, & A_{810} &= -e^{-\frac{3\epsilon_1}{4} - \frac{\epsilon_5}{4}} \epsilon_3, \\
 A_{410} &= \frac{-3}{10} e^{\frac{\epsilon_5 - \epsilon_1}{2}} (\epsilon_3 + \epsilon_4) \epsilon_9, & A_{910} &= -e^{-\frac{\epsilon_1}{4} - \frac{3\epsilon_5}{4}} \epsilon_7, \\
 & & A_{1010} &= e^{\frac{\epsilon_1 - \epsilon_5}{4}}.
 \end{aligned}$$

**References**

1. G.B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974)
2. P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 1991)
3. G.W. Bluman, S. Kumei, *Symmetries and Differential Equations* (Springer, Berlin, 1989)
4. L.V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982)
5. P.J. Olver, P. Rosenau, Group-invariant solutions of differential equations. *SIAM J. Appl. Math.* **47**, 263–278 (1987)
6. N.H. Ibragimov, A new conservation theorem. *J. Math. Anal. Appl.* **333**, 311–328 (2007)
7. S.V. Meleshko, Group classification of the equations of two dimensional motions of a gas. *J. Appl. Maths. Mech.* **58**, 629–635 (1994)
8. N.H. Ibragimov, M. Torrisi, A. Valenti, Preliminary group classification of equations  $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$ . *J. Math. Phys.* **32**, 2988 (1991)
9. Y. Zhang, W.X. Ma, Rational solutions to a KdV-like equation. *Appl. Math. Comp.* **256**, 252–256 (2015)
10. F. Batool, G. Akram, New solitary wave solutions of the time-fractional Cahn-Allen equation via the improved  $\left(\frac{G'}{G}\right)$ -expansion method. *Eur. Phys. J. Plus* **133**, 171 (2018)
11. D.J. Korteweg, G. Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary wave. *Philos. Mag.* **39**, 422–443 (1895)
12. A.J. Khattak, A comparative study of numerical solutions of a class of KdV equation. *App. Math. Comput.* **199**, 425–434 (2008)
13. S.A. Khuri, Soliton and periodic solutions for higher order wave equations of KdV type. *Chaos Solitons Fractals* **26**, 25–32 (2005)
14. N.J. Zabusky, M.D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15**, 240–243 (1965)
15. J.M. Miles, Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11–43 (1980)
16. O. Darrigol, *Worlds of Flow* (Oxford University Press, Oxford, 2005)
17. R. Hirota, *The Direct Method in Soliton Theory*, 155 (Cambridge University Press, Cambridge, 2004)
18. M. Wang, Y. Zhou, Z. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Phys. Lett. A* **216**, 67–75 (1996)
19. A. Biswas, D. Milovic, Bright and dark solitons of the generalized nonlinear Schrödinger’s equation. *Commun. Nonlinear Sci. Numer. Simul.* **15**, 1473–1484 (2010)
20. N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons Fractals* **24**, 1217–1231 (2005)

21. M. Lakshmanan, P. Kaliappan, Lie transformations, nonlinear evolution equations, and Painlevé forms. *J. Math. Phys.* **24**, 795–806 (1983)
22. M. Kumar, R. Kumar, A. Kumar, On similarity solutions of Zabolotskaya-Khokhlov equation *Comput. Math. Appl.* **68**, 454–463 (2018)
23. M. Xu, M. Jia, Exact solutions, symmetry reductions, painlevé test and Bäcklund transformations of a coupled KdV equation. *Commun. Theor. Phys.* **68**, 417–444 (2017)
24. S. Kumar, D. Kumar, Solitary wave solutions of (3+1)-dimensional extended Zakharov-Kuznetsov equation by Lie symmetry approach *Comput. Math. Appl.* **77**(8), 2096–2113 (2019)
25. S. Kumar, D. Kumar, A.M. Wazwaz, Group invariant solutions of (3+1)-dimensional generalized B-type Kadomtsev Petviashvili equation using optimal system of Lie subalgebra. *Phys. Scr.* **94** (2019)
26. D. Kumar, S. Kumar, Some new periodic solitary wave solutions of (3+1)-dimensional generalized shallow water wave equation by Lie symmetry approach. *Comput. Math. Appl.* **78**, 857–877 (2019)
27. S. Kumar, A. Kumar, H. Kharbanda, Lie symmetry analysis and generalized invariant solutions of (2+1)-dimensional dispersive long wave (DLW) equations. *Phys. Scr.* **95** (2020)
28. D.V. Tanwar, A.M. Wazwaz, Lie symmetries and dynamics of exact solutions of dissipative Zabolotskaya-Khokhlov equation in nonlinear acoustics. *Eur. Phys. J. Plus* **135**, 520 (2020)
29. S.Y. Lou, Dromion like structures in a (3 + 1) dimensional KdV-type equation. *J. Phys. A Math. Gen.* **29**, 5989–6001 (1996)
30. A.M. Wazwaz, Soliton solutions for two (3+1)-dimensional non-integrable KdV-type equations. *Math. Comp. Mod.* **55**, 1845–1848 (2012)
31. W. Hereman, A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations. *Math. Comput. Simul.* **43**, 13–27 (1997)
32. Ö. Ünsal, Complexiton solutions for (3+1)-dimensional KdV-type equation. *Comp. Math. Appl.* **75**(7), 2466–2472 (2018)
33. N. Liu, Y. Liu, Homoclinic breather wave, rogue wave and interaction solutions for a (3+1)-dimensional KdV-type equation. *Phys. Scr.* **94** (2018)
34. J.J. Mao, S.F. Tian, T.T. Zhang, Rogue waves, homoclinic breather waves and soliton waves for a (3+1)-dimensional nonintegrable KdV-type equation. *Int. J. Numer. Method H* **29**, 763–772 (2019)
35. A.M. Wazwaz, New solutions for two integrable cases of a generalized fifth-order nonlinear equation. *Mod. Phys. Lett. B* **29**, 1550065 (2015)
36. X. Hu, Y. Li, Y. Chen, A direct algorithm of one dimensional optimal system for the group invariant solutions. *J. Math. Phys.* **56** (2015)
37. P. Satapathy, T.R. Sekhar, Optimal system, invariant solutions and evolution of weak discontinuity for isentropic drift flux model. *Appl. Math. Comp.* **334**, 107–116 (2018)
38. N.H. Ibragimov, Nonlinear self-adjointness and conservation laws. *J. Phys. A Math. Theor.* **44** (2011)
39. G. Dieu-donne, C.G.L. Tiofack, A. Seadawy et al., Propagation of W-shaped, M-shaped and other exotic optical solitons in the perturbed Fokas-Lenells equation. *Eur. Phys. J. Plus* **135**, 371 (2020)
40. S. Kumar, D. Kumar, A. Kumar, Lie symmetry analysis for obtaining the abundant exact solutions, optimal system and dynamics of solitons for a higher-dimensional Fokas equation. *Chaos Soliton Fractals* **142** (2021)