Eur. Phys. J. Plus (2021) 136:656
https://doi.org/10.1140/epjp/s13360-021-01469-x

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Check for updates

# Lie group analysis and analytic solutions for a (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko equation in fluid mechanics and plasma physics

**Fei-Yan Liu, Yi-Tian Gao[a], Xin Yu[b], Liu-Qing Li, Cui-Cui Ding, Dong Wang**

Ministry-of-Education Key Laboratory of Fluid Mechanics and National Laboratory for Computational Fluid Dynamics, Beijing University of Aeronautics and Astronautics, Beijing 100191, China

Received: 1 December 2020 / Accepted: 20 April 2021
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2021

**Abstract** Under investigation in this paper is a (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko equation in fluid mechanics and plasma physics. We obtain the Lie point symmetry generators, Lie symmetry group and symmetry reductions via the Lie group method. Hyperbolic-function, power-series, trigonometric-function, soliton and rational solutions are derived via the power-series expansion, polynomial expansion and $\left(\frac{G'}{G}\right)$ expansion method.

## 1 Introduction

Nonlinear evolution equations (NLEEs) have been applied in fluid mechanics, plasma physics and nonlinear optics [1–37]. Researchers have proposed certain methods for solving the NLEEs, such as the Darboux transformation, Bäcklund transformation, inverse scattering transformation, Lie group, consistent Riccati expansion, power-series expansion, polynomial expansion, $\left(\frac{G'}{G}\right)$ expansion and Hirota bilinear methods [38–53]. The Lie group method has been used to look for the group invariance and certain reductions to the NLEEs [38,54–57]. Based on the reduced equations, certain solutions for the NLEEs have been constructed [17,18,58,59].

A (2+1)-dimensional generalized Bogoyavlensky–Konopelchenko (gBK) equation in fluid mechanics and plasma physics has been constructed as [60]

$$\begin{aligned} &u_{xt} + k_1 u_{xxxx} + k_2 u_{xxxy} + \frac{2k_1k_3}{k_2} u_x u_{xx} + k_3 u_x u_{xy} + k_3 u_{xx} u_y \\ &+ \gamma_1 u_{xx} + \gamma_2 u_{xy} + \gamma_3 u_{yy} = 0, \end{aligned} \tag{1}$$

where $u = u(x, y, t)$ is a real function, $x$ and $y$ are the scaled space variables, $t$ is the scaled time variable, $k_1$, $k_2$, $k_3$, $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the real constants and the subscripts denote the

[a] e-mail: gaoyt163@163.com (corresponding author)
[b] e-mail: yuxin@buaa.edu.cn (corresponding author)

Published online: 14 June 2021

partial derivatives. Lump-type and lump solutions for Eq. (1) have been derived via the Hirota bilinear method [60].

Special cases for Eq. (1) in fluid mechanics and plasma physics have been seen:

- When $k_3 = 3$, $k_2 = 1$ and $k_1 = \gamma_1 = \gamma_2 = \gamma_3 = 0$, by virtue of a dimensional reduction $u_y$=$u_x$ and a potential function transformation $h(x,t) = u_x(x,t)$, Eq. (1) has been reduced to the Korteweg-de Vries equation,

$$h_t + h_{xxx} + 6hh_x = 0, \tag{2}$$

  for certain shallow water waves, stratified internal waves in a fluid or ion-acoustic waves in a plasma [61–66].
- When $k_2 = 1$, $k_3 = 4$ and $k_1 = \gamma_1 = \gamma_2 = \gamma_3 = 0$, Eq. (1) has been reduced to the Bogoyavlenskii's breaking soliton equation,

$$u_{xt} + u_{xxxy} + 4u_{xx}u_y + 4u_x u_{xy} = 0, \tag{3}$$

  for the interaction of a Riemann wave propagating along the $y$ axis and a long wave propagating along the $x$ axis in fluid mechanics [67].
- When $k_2 = 1$, $k_3 = 3$ and $k_1 = \gamma_1 = 0$, Eq. (1) has been reduced to the generalized Calogero–Bogoyavlenskii–Schiff equation [68],

$$u_{xt} + u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} + \gamma_2 u_{xy} + \gamma_3 u_{yy} = 0. \tag{4}$$

However, to our knowledge, the Lie point symmetry generators, Lie symmetry group and symmetry reductions for Eq. (1) have not been discussed. Hyperbolic-function, power-series, trigonometric-function, soliton and rational solutions for Eq. (1) have not yet been investigated via the Lie group method. In Sect. 2, the Lie point symmetry generators and Lie symmetry group for Eq. (1) will be derived. In Sect. 3, symmetry reductions as well as hyperbolic-function, power-series, trigonometric-function, soliton and rational solutions for Eq. (1) will be obtained through the Lie point symmetry generators. In Sect. 4, the conclusions will be given.

## 2 Lie group analysis for Eq. (1)

### 2.1 Lie point symmetry generators for Eq. (1)

According to the Lie group method [54], we consider a one-parameter Lie group of the infinitesimal transformations acting on the independent and dependent variables as

$$\begin{aligned} \tilde{x} &= x + \epsilon\xi(x, y, t, u) + O(\epsilon^2), \\ \tilde{y} &= y + \epsilon\eta(x, y, t, u) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\tau(x, y, t, u) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\phi(x, y, t, u) + O(\epsilon^2), \end{aligned} \tag{5}$$

where $\tilde{x}$, $\tilde{y}$, $\tilde{t}$, $\tilde{u}$, $\xi$, $\eta$, $\tau$ and $\phi$ are the real functions of $x$, $y$, $t$ and $u$, $\epsilon$ is a parameter of the infinitesimal transformation and $O(\epsilon^2)$ is the infinitesimal of the same order of $\epsilon^2$.

Lie point symmetry generators for Eq. (1) are

$$V = \xi(x, y, t, u)\frac{\partial}{\partial x} + \eta(x, y, t, u)\frac{\partial}{\partial y} + \tau(x, y, t, u)\frac{\partial}{\partial t} + \phi(x, y, t, u)\frac{\partial}{\partial u} \tag{6}$$

with $\xi$, $\eta$, $\tau$ and $\phi$ satisfying the condition

$$Pr^{(4)}V(E)|_{E=0} = 0, \tag{7}$$

where

$$\begin{aligned} E &= u_{xt} + k_1 u_{xxxx} + k_2 u_{xxxy} + \frac{2k_1k_3}{k_2} u_x u_{xx} + k_3 u_x u_{xy} \\ &\quad + k_3 u_{xx} u_y + \gamma_1 u_{xx} + \gamma_2 u_{xy} + \gamma_3 u_{yy}. \end{aligned} \tag{8}$$

$Pr^{(4)}V(\cdot)$ represents the fourth prolongation of $V$, defined as [69],

$$\begin{aligned} Pr^{(4)}V(\cdot) &= \xi \frac{\partial}{\partial x}(\cdot) + \eta \frac{\partial}{\partial y}(\cdot) + \tau \frac{\partial}{\partial t}(\cdot) + \phi \frac{\partial}{\partial u}(\cdot) \\ &\quad + \phi^x \frac{\partial}{\partial u_x}(\cdot) + \phi^y \frac{\partial}{\partial u_y}(\cdot) + \phi^{xx} \frac{\partial}{\partial u_{xx}}(\cdot) + \phi^{xy} \frac{\partial}{\partial u_{xy}}(\cdot) \\ &\quad + \phi^{yy} \frac{\partial}{\partial u_{yy}}(\cdot) + \phi^{xt} \frac{\partial}{\partial u_{xt}}(\cdot) + \phi^{xxxx} \frac{\partial}{\partial u_{xxxx}}(\cdot) + \phi^{xxxy} \frac{\partial}{\partial u_{xxxy}}(\cdot), \\ \phi^x &= D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt}, \\ \phi^y &= D_y(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{yx} + \eta u_{yy} + \tau u_{yt}, \\ \phi^t &= D_t(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{tx} + \eta u_{ty} + \tau u_{tt}, \\ \phi^{xx} &= D_x^2(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{xxy} + \tau u_{xxt}, \\ \phi^{xt} &= D_x D_t(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxt} + \eta u_{xyt} + \tau u_{xtt}, \\ \phi^{xy} &= D_x D_y(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxy} + \eta u_{xyy} + \tau u_{xyt}, \\ \phi^{yy} &= D_y^2(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{yyt}, \\ \phi^{xxxx} &= D_x^4(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxxxx} + \eta u_{xxxxy} + \tau u_{xxxxt}, \\ \phi^{xxxy} &= D_x^3 D_y(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxxxy} + \eta u_{xxxyy} + \tau u_{xxxyt}, \end{aligned} \tag{9}$$

where $D_x$, $D_y$ and $D_t$ are the total derivative operators.

Expanding Expression (7) and splitting on the derivatives of $u$ lead to the following expressions:

$$\begin{aligned} &\tau_x = \tau_y = \tau_u = \xi_u = \eta_x = \eta_u = 0, \\ &\phi_{uu} = \phi_{xu} = \phi_{yu} = \xi_{xx} = \xi_{xy} = 0, \\ &\frac{2k_1k_3}{k_2}\phi_x - \xi_t + k_3\phi_y + \gamma_1\tau_t - \gamma_1\xi_x - \gamma_2\xi_y = 0, \\ &\phi_u - \xi_x + \tau_t - \eta_y = 0,\ k_1\tau_t - k_2\xi_y - 3k_1\xi_x = 0, \\ &k_3\phi_x - \eta_t + \gamma_2\tau_t - \gamma_2\eta_y = 0,\ \phi_{xy} + \phi_{tu} - \xi_{xt} = 0, \\ &\phi_{xt} + \gamma_2\phi_{xy} = 0,\ \tau_t + \xi_x - \phi_u = 0,\ \tau_t - 2\xi_x - \eta_y = 0. \end{aligned} \tag{10}$$

Solving Expressions (10), we have the following results:

$$\begin{aligned} \tau &= 2r_2 t + r_3, \\ \eta &= r_2(-2\gamma_2 t + 4y) + k_3 r_1 t + r_5, \\ \phi &= r_1 x + r_2 u + F_1^{'}(t) y + F_2(t), \\ \xi &= -r_2 x + \frac{5k_1 r_2}{k_2} y + k_3 F_1(t) + \frac{2k_3 r_1 k_1 - 5\gamma_2 r_2 k_1 + 3\gamma_1 r_2 k_2}{k_2} t + r_4, \end{aligned} \tag{11}$$

where $r_1$, $r_2$, $r_3$, $r_4$ and $r_5$ are the real constants, $F_1(t)$ and $F_2(t)$ are the real function of $t$, and $F_1^{'}(t)$ denotes derivative of $F_1(t)$ with respect to $t$. We derive the Lie point symmetry generators for Eq. (1) as follows:

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial y},\ V_3 = \frac{\partial}{\partial t},\ V_4 = x\frac{\partial}{\partial u} + k_3 t\frac{\partial}{\partial y} + \frac{2k_1k_3t}{k_2}\frac{\partial}{\partial x},\\
&V_5 = 2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + \frac{5k_1 y}{k_2}\frac{\partial}{\partial x} + \frac{(-5k_1\gamma_2 + 3\gamma_1 k_2)t}{k_2}\frac{\partial}{\partial x} + (-2\gamma_2 t + 4y)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},\\
&V^{[1]}_{[F_1(t)]} = k_3F_1(t)\frac{\partial}{\partial x} + F_1^{'}(t)y\frac{\partial}{\partial u},\ V^{[2]}_{[F_2(t)]} = F_2(t)\frac{\partial}{\partial u}.
\end{aligned} \tag{12}$$

Motivated by Ref. [55], commutation relations among Generators (12) are shown in Table 1, where the entries in row $i$ and column $j$ are represented by the commutators $[V_i, V_j]$, which are given by

$$[V_i, V_j] = V_iV_j - V_jV_i, (i, j = 1, 2, 3, 4, 5, F_1, F_2). \tag{13}$$

Motivated by Refs. [58,59], we take $F_1(t) = r_6 t$, $F_2(t) = r_7 t + r_8$ with the real constants $r_6$, $r_7$ and $r_8$.

Thus, we derive the Lie point symmetry generators for Eq. (1) as follows:

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial x}, \Gamma_2 = \frac{\partial}{\partial y}, \Gamma_3 = \frac{\partial}{\partial t}, \Gamma_4 = t\frac{\partial}{\partial u}, \Gamma_5 = \frac{\partial}{\partial u},\\
&\Gamma_6 = k_3t\frac{\partial}{\partial x} + y\frac{\partial}{\partial u}, \Gamma_7 = x\frac{\partial}{\partial u} + k_3t\frac{\partial}{\partial y} + \frac{2k_1k_3t}{k_2}\frac{\partial}{\partial x},\\
&\Gamma_8 = 2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + \frac{5k_1y}{k_2}\frac{\partial}{\partial x} + \frac{(-5k_1\gamma_2 + 3\gamma_1k_2)\,t}{k_2}\frac{\partial}{\partial x} + (-2\gamma_2t + 4y)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.
\end{aligned} \tag{14}$$

## 2.2 Lie symmetry group for Eq. (1)

In order to obtain the group transformation, which is generated by the infinitesimal generators $\Gamma_i$, we need to solve the following initial problems:

$$\begin{aligned}
&\frac{d\bar{x}(\epsilon)}{d\epsilon} = \xi[\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{t}(\epsilon), \bar{u}(\epsilon)],\ \bar{x}|_{\epsilon=0} = x,\\
&\frac{d\bar{y}(\epsilon)}{d\epsilon} = \eta[\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{t}(\epsilon), \bar{u}(\epsilon)],\ \bar{y}|_{\epsilon=0} = y,\\
&\frac{d\bar{t}(\epsilon)}{d\epsilon} = \tau[\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{t}(\epsilon), \bar{u}(\epsilon)],\ \bar{t}|_{\epsilon=0} = t,\\
&\frac{d\bar{u}(\epsilon)}{d\epsilon} = \phi[\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{t}(\epsilon), \bar{u}(\epsilon)],\ \bar{u}|_{\epsilon=0} = u.
\end{aligned} \tag{15}$$

**Table 1** Commutator table of the Lie algebra for Eq. (1)

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V^{[1]}_{[F_1(t)]}$ | $V^{[2]}_{[F_2(t)]}$ |
|---|---|---|---|---|---|---|---|
| $V_1$ | 0 | 0 | 0 | $V_5$ | $-V_1$ | 0 | 0 |
| $V_2$ | 0 | 0 | $V_5$ | 0 | $\frac{5k_1}{k_2}V_1+4V_2$ | $V^{[2]}_{\left[F_1'\right]}$ | 0 |
| $V_3$ | 0 | 0 | 0 | $k_3V_2+\frac{2k_1k_3}{k_2}V_1$ | $\frac{-5k_1\gamma_2+3\gamma_1k_2}{k_2}V_1$ $+2V_3-2\gamma_2V_2$ | $V^{[1]}_{[F_1'']}$ | $V^{[2]}_{[F_2']}$ |
| $V_4$ | $-V_5$ | 0 | $-k_3V_2-\frac{2k_1k_3}{k_2}V_1$ | 0 | $-V^{[2]}_{\left[\frac{-5k_1\gamma_2+3\gamma_1k_2}{k_2}t\right]}$ $+2V_4-V^{[1]}_{\left[\frac{5k_1}{k_2}t\right]}$ | $V^{[2]}_{\left[k_3tF_1'-k_3F_1\right]}$ | 0 |
| $V_5$ | $V_1$ | $-\frac{5k_1V_1}{k_2}-4V_2$ | $-\frac{(-5k_1\gamma_2+3\gamma_1k_2)V_1}{k_2}$ $-2V_3+2\gamma_2V_2$ | $V^{[2]}_{\left[\frac{-5k_1\gamma_2+3\gamma_1k_2}{k_2}t\right]}$ $-2V_4+V^{[1]}_{\left[\frac{5k_1}{k_2}t\right]}$ | 0 | $V^{[1]}_{[2F_1'+F_1]}-V^{[2]}_{\left[2\gamma_2tF_1'\right]}$ | $V^{[2]}_{[2tF_2'-F_2]}$ |
| $V^{[1]}_{[F_1]}$ | 0 | $-V^{[2]}_{\left[F_1'\right]}$ | $-V^{[1]}_{\left[F_1''\right]}$ | $-V^{[2]}_{\left[k_3tF_1'-k_3F_1\right]}$ | $-V^{[1]}_{[2F_1'+F_1]}+V^{[2]}_{[2\gamma_2tF_1']}$ | 0 | 0 |
| $V^{[2]}_{[F_2]}$ | 0 | 0 | $-V^{[2]}_{\left[F_2'\right]}$ | 0 | $-V^{[2]}_{[2tF_2'-F_2]}$ | 0 | 0 |

Then, the Lie symmetry group $g_i$'s generated by $V_i$ can be derived as

$$
\begin{aligned}
&g_1 : (x, y, t, u) \rightarrow (x + \epsilon, y, t, u),\ g_2 : (x, y, t, u) \rightarrow (x, y + \epsilon, t, u),\\
&g_3 : (x, y, t, u) \rightarrow (x, y, t + \epsilon, u),\ g_4 : (x, y, t, u) \rightarrow (x, y, t, u + t\epsilon),\\
&g_5 : (x, y, t, u) \rightarrow (x, y, t, u + \epsilon),\ g_6(x, y, t, u) \rightarrow (x + k_3 t\epsilon, y, t, u + y\epsilon),\\
&g_7 : (x, y, t, u) \rightarrow \left(x + \frac{2k_1k_3}{k_2} t\epsilon, y + k_3 t\epsilon, t, u + x\epsilon\right),\\
&g_8 : (x, y, t, u) \rightarrow \left[\frac{(y - \gamma_2 t)(e^{4\epsilon} - e^{-\epsilon}) + \gamma_1 k_2 t\left(e^{2\epsilon} - e^{-\epsilon}\right)}{k_2}\right.\\
&\qquad \left. + xe^{-\epsilon}, \gamma_2 t e^{2\epsilon} + (y - \gamma_2 t)e^{4\epsilon}, te^{2\epsilon}, ue^{\epsilon}\right].
\end{aligned} \tag{16}
$$

On account of Lie Symmetry Group (16), if $\bar{f}(x, y, t)$ is a known solutions for Eq. (1), the corresponding solutions can be obtained as

$$
\begin{aligned}
u^{(1)} &= \bar{f}(x - \epsilon, y, t),\\
u^{(2)} &= \bar{f}(x, y - \epsilon, t),\\
u^{(3)} &= \bar{f}(x, y, t - \epsilon),\\
u^{(4)} &= t\epsilon + \bar{f}(x, y, t),\\
u^{(5)} &= \epsilon + \bar{f}(x, y, t),\\
u^{(6)} &= y\epsilon + \bar{f}(x - k_3 t\epsilon, y, t),\\
u^{(7)} &= x\epsilon + \bar{f}\left(x - \frac{2k_1k_3}{k_2} t\epsilon, y - k_3 t\epsilon, t\right),\\
u^{(8)} &= e^{\epsilon}\bar{f}\left[xe^{-\epsilon} + \frac{(y - \gamma_2 t)\left(e^{-4\epsilon} - e^{\epsilon}\right) + \gamma_1 k_2 t(e^{-2\epsilon} - e^{\epsilon})}{k_2},\right.\\
&\qquad \left.(y - \gamma_2)e^{-4\epsilon} + \gamma_2 t e^{-2\epsilon}, te^{-2\epsilon}\right].
\end{aligned} \tag{17}
$$

## 3 Symmetry reductions and analytic solutions for Eq. (1)

In this section, we use the combination of Generators (14) to derive the reduction equations and construct some analytic solutions for Eq. (1).

**Case 1**: For the Lie point symmetry $V_2 = \partial_y$, we have the following group-invariant solutions:

$$u = H(x_1, t_1), \tag{18}$$

where $x_1 = x$, $t_1 = t$ and $H$ is a function of $x_1$ and $t_1$. Substituting Expression (18) into Eq. (1) gives rise to the following reduced equation:

$$H_{x_1t_1} + k_1 H_{x_1x_1x_1x_1} + \frac{2k_1k_3}{k_2} H_{x_1} H_{x_1x_1} + \gamma_1 H_{x_1x_1} = 0. \tag{19}$$

We suppose that the solutions for Eq. (1) be as follows:

$$H(x_1, t_1) = \frac{ae^{cx_1+dt_1+k}}{b + e^{cx_1+dt_1+k}}, \tag{20}$$

where $a$, $b$, $c$, $d$ and $k$ are the real constants. Substituting Expression (20) into Eq. (19), we can obtain

$$a = \frac{6ck_2}{k_3},\ b_1 = -k_1c^3 - \gamma_1 c. \tag{21}$$

Therefore, soliton solutions for Eq. (1) can be derived as

$$u = \frac{6ck_2 e^{cx+\left(-k_1c^3-\gamma_1 c\right)t+k}}{k_3\left[b + e^{cx+(-k_1c_1-\gamma_1 c)t+k}\right]}. \tag{22}$$

**Case 2**: For the Lie point symmetry $V_3 = \partial_t$, we have the following group-invariant solutions:

$$u = P(x_2, y_2), \tag{23}$$

where $x_2 = x$ and $y_2 = y$, while $P$ is a function of $x_2$ and $y_2$. Substituting Expression (23) into Eq. (1) gives rise to the following reduced equation:

$$\begin{aligned} &k_1 P_{x_2x_2x_2x_2} + k_2 P_{x_2x_2x_2y_2} + \frac{2k_1k_3}{k_2} P_{x_2} P_{x_2x_2} + k_3 P_{x_2} P_{x_2y_2} \\ &+ k_3 P_{x_2x_2} P_{y_2} + \gamma_1 P_{x_2x_2} + \gamma_2 P_{x_2y_2} + \gamma_3 P_{y_2y_2} = 0. \end{aligned} \tag{24}$$

Applying the Lie group method on Eq. (24), we obtain

$$\xi_1 = s_1x_2 + \frac{5k_1s_1}{k_2} y_2 + s_2,\ \eta_1 = 4s_1y_2 + s_3,\ \phi_1 = s_1P - \frac{\gamma_1 s_1}{k_1} x_1 + F_3(y_2), \tag{25}$$

where $s_1$, $s_2$ and $s_3$ are the real constants, while $F_3(y_2)$ is a real function of $y_2$. Motivated by Refs. [58,59], we take $F_3(y_2) = s_4$ with a real constant $s_4$. Thus, we derive the Lie point symmetry generators for Eq. (24) as follows:

$$\begin{aligned} &\Gamma_1 = \frac{\partial}{\partial x_2},\ \Gamma_2 = \frac{\partial}{\partial y_2},\ \Gamma_3 = \frac{\partial}{\partial P}, \\ &\Gamma_4 = \left(\frac{5k_1y_2}{k_2} - x_2\right)\frac{\partial}{\partial x_2} + 4y_2\frac{\partial}{\partial y_2} + \left(P - \frac{\gamma_1 x_2}{k_1}\right)\frac{\partial}{\partial P}. \end{aligned} \tag{26}$$

For the Lie point symmetry $\Gamma_1 + \Gamma_2$, the symmetry produces the following invariants:

$$f = x_2 - y_2,\ P = \Phi(f), \tag{27}$$

where $\Phi$ is a real function of $f$. Substituting Expressions (27) into Eq. (24) gives rise to the following reduced equation:

$$(k_1 - k_2)\Phi_{ffff} + \left(\frac{2k_1k_3}{k_2} - 2k_3\right)\Phi_f\Phi_{ff} + (\gamma_1 - \gamma_2 + \gamma_3)\Phi_{ff} = 0. \tag{28}$$

We suppose that some solutions for Eq. (28) have the following form:

$$\Phi = \sum_{i=0}^{m} a_i \left(\frac{G^{'}}{G}\right)^i, \tag{29}$$

where $m$ is a positive integer, while $a_i$'s are the real constants. Here, $G$ satisfies the second-order linear ordinary differential equation, i.e.,

$$G^{''} + BG^{'} + AG = 0, \tag{30}$$

where $G^{'} = \frac{dG}{df}$ and $G^{''} = \frac{d^2G}{df^2}$, while $A$ and $B$ are the real constants. $m$ can be determined via homogeneous balance method between the highest order derivative term and the nonlinear term appearing in Eq. (28). We get $m = 1$. Substituting Eq. (29) into Eq. (28) with Constraint (30) and setting the coefficients of $(\frac{G^{'}}{G})$ equal to zero, we obtain

$$a_1 = \frac{6k_2}{k_3}, \; A = \frac{B^2k_1 - B^2k_2 + \gamma_1 - \gamma_2 + \gamma_3}{4(k_1 - k_2)}. \tag{31}$$

When $\sqrt{B^2 - 4A} > 0$, we derive some solutions for Eq. (1) as

$$u(x, y, t) = \frac{6k_2}{k_3}\left\{\sqrt{\frac{\gamma_1 - \gamma_2 + \gamma_3}{4(k_2 - k_1)}}\frac{C_1 \cosh\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_2-k_1)}}\right] + C_2 \sinh\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_2-k_1)}}\right]}{C_2 \cosh\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_2-k_1)}}\right] + C_1 \sinh\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_2-k_1)}}\right]}\right\}$$
$$+ a_0 - \frac{3Bk_2}{k_3}, \tag{32}$$

where $C_1$ and $C_2$ are the real constants.

When $\sqrt{B^2 - 4A} = 0$, we derive some solutions for Eq. (1) as

$$u(x, y, t) = a_0 + \frac{6k_2}{k_3}\left[-\frac{B}{2} + \frac{C_4}{C_3 + C_4(x - y)}\right], \tag{33}$$

where $C_3$ and $C_4$ are the real constants.

When $\sqrt{B^2 - 4A} < 0$, we derive some solutions for Eq. (1) as

$$u(x, y, t) = \frac{6k_2}{k_3}\left\{\sqrt{\frac{\gamma_1 - \gamma_2 + \gamma_3}{4(k_1 - k_2)}}\frac{C_6 \cos\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_1-k_2)}}\right] - C_5 \sin\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_1-k_2)}}\right]}{C_5 \cos\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_1-k_2)}}\right] + C_6 \sin\left[(x-y)\sqrt{\frac{\gamma_1-\gamma_2+\gamma_3}{4(k_1-k_2)}}\right]}\right\}$$
$$+ a_0 - \frac{3Bk_2}{k_3}, \tag{34}$$

where $C_5$ and $C_6$ are the real constants.

**Case 3**: For the Lie point symmetry $V_{(1)} = V_1 + V_2 + V_3 + V_5$, we have the following group-invariant solutions:

$$f_1 = x - t, h_1 = y - t, u = t + R(f_1, h_1), \tag{35}$$

where $R$ is a function of $f_1$ and $h_1$. Substituting Expressions (35) into Eq. (1) gives rise to the following reduced equation:

$$- R_{f_1h_1} - R_{f_1f_1} + k_1R_{f_1f_1f_1f_1} + k_2R_{f_1f_1f_1h_1} + \frac{2k_1k_3}{k_2}R_{f_1}R_{f_1f_1}$$
$$+ k_3R_{f_1}R_{f_1h_1} + k_3R_{f_1f_1}R_{h_1} + \gamma_1R_{f_1f_1} + \gamma_2R_{f_1h_1} + \gamma_3R_{h_1h_1} = 0. \tag{36}$$

Applying the Lie group method on Eq. (36), we obtain

$$\xi_2 = -s_5f_1 + \frac{5k_1s_5h_1}{k_2} + s_6, \eta_2 = 4s_5h_1 + s_7,$$
$$\phi_2 = s_5R + \frac{2(\gamma_2 - 1)s_5f_1}{3k_2} - \frac{s_5\left[3(\gamma_1 - 1)k_2 - k_1(\gamma_2 - 1)\right]h_1}{3k_2^2} + s_8, \tag{37}$$

where $s_5$, $s_6$, $s_7$ and $s_8$ are the real constants. Thus, we derive the Lie point symmetry generators for Eq. (36) as follows:

$$\begin{aligned}
\Upsilon_1 &= \frac{\partial}{\partial f_1},\ \Upsilon_2 = \frac{\partial}{\partial h_1},\ \Upsilon_3 = \frac{\partial}{\partial R},\\
\Upsilon_4 &= \frac{3k_2^2 R + 2k_2(\gamma_2 - 1) f_1 - h_1[3(\gamma_1 - 1)k_2 - k_1(\gamma_2 - 1)]}{3k_2^2}\frac{\partial}{\partial R}\\
&\quad + \frac{5k_1 h_1 - k_2 f_1}{k_2}\frac{\partial}{\partial f_1} + 4h_1\frac{\partial}{\partial h_1}.
\end{aligned} \tag{38}$$

For the Lie point symmetry $n_1\Upsilon_1 + n_2\Upsilon_2$, the symmetry produces the following invariants:

$$z = n_2 f_1 - n_1 h_1,\ R = Q(z), \tag{39}$$

where $n_1$ and $n_2$ are the real constants, and $Q$ is a real function of $z$. Substituting Expressions (39) into Eq. (36) gives rise to the following reduced equation:

$$\begin{aligned}
&n_2^2\left(\frac{2k_1k_3n_2 - 2n_1k_2k_3}{k_2}\right) Q_{zz}Q_z + n_2^3(k_1n_2 - n_1k_2)Q_{zzzz}\\
&\quad + \left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right) Q_{zz} = 0.
\end{aligned} \tag{40}$$

Seeking the solutions for Eq. (40) in a power series of the form

$$Q = \sum_{q=0}^{\infty} c_q z^q, \tag{41}$$

and substituting Expression (41) into Eq. (40), we obtain

$$\begin{aligned}
&24c_4n_2^3(n_2k_1 - n_1k_2) + n_2^3(n_2k_1 - n_1k_2)\sum_{q=1}^{\infty}(q+4)(q+3)(q+2)(q+1)c_{q+4}z^q\\
&+n_2^2\left(\frac{2k_1k_3n_2 - 2k_3k_2n_1}{k_2}\right)\sum_{q=1}^{\infty}\sum_{k=0}^{q}(k+1)(q-k+2)(q-k+1)c_{k+1}c_{q-k+2}z^q\\
&+2n_2^2\left(\frac{2k_1k_3n_2 - 2k_3n_1k_2}{k_2}\right)c_1c_2 + 2\left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right)c_2\\
&+\left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right)\sum_{q=1}^{\infty}(q+2)(q+1)c_{q+2}z^q = 0,
\end{aligned} \tag{42}$$

where $c_q$'s are the real constants. From Expression (42), equating the coefficients of each order of $z$, we can calculate $c_q$ for the case of $q = 0$, so that

$$c_4 = \frac{n_2^2(2k_1k_3n_2 - 2k_3k_2n_1)c_1c_2 + (-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2)k_2c_2}{12k_2n_2^3(k_2n_1 - k_1n_2)}. \tag{43}$$

For $n \geq 1$, we obtain

$$c_{q+4} = \frac{1}{(q+4)(q+3)(q+2)(q+4)(k_2n_1 - k_1n_2)} \Big[\left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right)(q+2)(q+1)c_{q+2} + n_2^2\left(\frac{2k_1k_3n_2 - 2k_3k_2n_1}{k_2}\right)\sum_{k=0}^{q}(k+1)(q-k+2)(q-k+1)c_{k+1}c_{q-k+2}\Big]. \tag{44}$$

Then, we derive the power series solutions for Eq. (1) as

$$\begin{aligned} u(x,y,t) &= c_0 + c_1[n_2(x-t) - n_1(y-t)] + c_2[n_2(x-t) - n_1(y-t)]^2 \\ &+ c_3[n_2(x-t) - n_1(y-t)]^3 + \frac{[n_2(x-t) - n_1(y-t)]^4}{12k_2n_2^3(k_2n_1 - k_1n_2)} \\ &\left[n_2^2(2k_1k_3n_2 - 2k_3k_2n_1)c_1c_2 + \left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right)k_2c_2\right] \\ &+ \sum_{q=1}^{\infty}\frac{[n_2(x-t) - n_1(y-t)]^{q+4}}{(q+4)(q+3)(q+2)(q+4)(k_2n_1 - k_1n_2)} \\ &\left[n_2^2\left(\frac{2k_1k_3n_2 - 2k_3k_2n_1}{k_2}\right)\sum_{k=0}^{q}(k+1)(q-k+2)(q-k+1)c_{k+1}c_{q-k+2}\right. \\ &\left. + \left(-n_2^2\gamma_1 + n_1n_2\gamma_2 + n_1^2\gamma_3 - n_1n_2 + n_2^2\right)(q+2)(q+1)c_{q+2}\right] + t. \end{aligned} \tag{45}$$

**Case 4**: For the Lie point symmetry $V_{(2)} = V_1 + V_2 + V_3$, we have the following group-invariant solutions:

$$f_2 = x - t, h_2 = y - t, u = S(f_2, h_2). \tag{46}$$

where $S$ is a function of $f_2$ and $h_2$. Substituting Expressions (46) into Eq. (1) gives rise to the following reduced equation:

$$\begin{aligned} &- S_{f_2h_2} - S_{f_2f_2} + k_1S_{f_2f_2f_2f_2} + k_2S_{f_2f_2f_2h_2} + \frac{2k_1k_3}{k_2}S_{f_2}R_{f_2f_2} \\ &+ k_3S_{f_2}S_{f_2h_2} + k_3S_{f_2f_2}S_{h_2} + \gamma_1S_{f_2f_2} + \gamma_2S_{f_2h_2} + \gamma_3S_{h_2h_2} = 0. \end{aligned} \tag{47}$$

Applying the Lie group method on Eq. (47), we obtain

$$\begin{aligned} \xi_2 &= -s_9f_2 + \frac{5k_1s_9h_2}{k_2} + s_{10}, \eta_2 = 4s_9h_2 + s_{11}, \\ \phi_2 &= s_9S + \frac{2(\gamma_2 - 1)s_9f_2}{3k_2} - \frac{s_9[3(\gamma_1 - 1)k_2 - k_1(\gamma_2 - 1)]h_2}{3k_2^2} + s_{12}, \end{aligned} \tag{48}$$

where $s_9$, $s_{10}$, $s_{11}$ and $s_{12}$ are the real constants. Thus, we derive the Lie point symmetry generators for Eq. (46) as follows:

$$\begin{aligned} \Theta_1 &= \frac{\partial}{\partial f_2}, \ \Theta_2 = \frac{\partial}{\partial h_2}, \ \Theta_3 = \frac{\partial}{\partial S}, \\ \Theta_4 &= \frac{3k_2^2S + 2k_2(\gamma_2 - 1)f_2 - h_2[3(\gamma_1 - 1)k_2 - k_1(\gamma_2 - 1)]}{3k_2^2}\frac{\partial}{\partial S} \\ &+ \frac{5k_1h_2 - k_2f_2}{k_2}\frac{\partial}{\partial f_2} + 4h_2\frac{\partial}{\partial h_2}. \end{aligned} \tag{49}$$

For the Lie point symmetry $n_3\Theta_1 + \Theta_2$, the symmetry produces the following invariants:

$$z_1 = f_2 - n_3 h_2,\ L = S(z_1), \tag{50}$$

where $n_3$ is a real constant and $L$ is a real function of $z_1$. Substituting Expressions (50) into Eq. (47) gives rise to the following reduced equation:

$$\left(\frac{2k_1k_3 - 2n_3k_2k_3}{k_2}\right) L_{z_1z_1}L_{z_1} + (k_1 - n_3k_2)L_{z_1z_1z_1z_1} \\ +(\gamma_1 - n_3\gamma_2 + n_3^2\gamma_3 + n_3 - 1)L_{z_1z_1} = 0. \tag{51}$$

We suppose that the solutions for Eq. (51) have the following form:

$$L = b_0 + \sum_{j=1}^{M} b_j W(z_1)^j + \sum_{j=1}^{M} d_j W(z_1)^{-j}, \tag{52}$$

where $M$ is a positive integer, $b_0$, $b_j$'s and $d_j$'s are the real constants, $W$ satisfies

$$\frac{dW}{dz_1} = W^2 + p_1 z_1 + p_2, \tag{53}$$

while $p_1$ and $p_2$ are the real constants. $M$ can be determined via homogeneous balance method between the highest order derivative term and the nonlinear term appearing in Eq. (51). We get $M = 1$. Substituting Expression (52) into Eq. (51) with Constraint (53) and setting the coefficients of $W(z_1)$ equal to zero, we obtain the following results:

Case 4.1:

$$b_1 = p_1 = 0,\ d_1 = \frac{k_2(\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1)}{2k_3(k_1 - n_3k_2)},\ p_2 = \frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}.$$

Hereby, the solutions for Eq. (1) are obtained as

$$u = b_0 + \frac{k_2(\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1)\cot\left\{[c + x - t - n_3(y - t)]\sqrt{\frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}}\right\}}{2k_3(k_1 - n_3k_2)\sqrt{\frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}}}, \tag{54}$$

where $c$ is a real constant.

Case 4.2:

$$d_1 = p_1 = 0,\ b_1 = -\frac{6k_2}{k_3},\ p_2 = \frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}.$$

Hereby, the solutions for Eq. (1) are obtained as

$$u = b_0 - \frac{6k_2\sqrt{\frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}}\tan\left\{[c + x - t - n_3(y - t)]\sqrt{\frac{\gamma_3 n_3^2 - \gamma_2 n + \gamma_1 + n_3 - 1}{4(k_1 - n_3k_2)}}\right\}}{k_3}. \tag{55}$$

Case 4.3:

$$p_1 = 0,\ \ p_2 = \frac{\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1}{16(k_1 - n_3 k_2)},$$
$$b_1 = -\frac{6k_2}{k_3},\ d_1 = \frac{3k_2(\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1)}{8k_3(k_1 - n_3 k_2)}.$$

Hereby, the solutions for Eq. (1) are obtained as

$$u = b_0 - \frac{6k_2\sqrt{\frac{\gamma_3 n_3^2-\gamma_2 n_3+\gamma_1+n_3-1}{16(k_1-n_3k_2)}}\tan\left\{[c+x-t-n_3(y-t)]\sqrt{\frac{\gamma_3 n_3^2-\gamma_2 n_3+\gamma_1+n_3-1}{16(k_1-n_3k_2)}}\right\}}{k_3} + \frac{3k_2(\gamma_3 n_3^2 - \gamma_2 n_3 + \gamma_1 + n_3 - 1)\cot\left\{[c+x-t-n_3(y-t)]\sqrt{\frac{\gamma_3 n_3^2-\gamma_2 n_3+\gamma_1+n_3-1}{16(k_1-n_3k_2)}}\right\}}{8k_3(k_1-n_3k_2)\sqrt{\frac{\gamma_3 n_3^2-\gamma_2 n_3+\gamma_1+n_3-1}{16(k_1-n_3k_2)}}}. \tag{56}$$

## 4 Conclusions

In this paper, a (2+1)-dimensional gBK equation in fluid mechanics and plasma physics, i.e., Eq. (1), has been investigated. Lie Point Symmetry Generators (14) and Lie Symmetry Group (16) for Eq. (1) have been derived via the Lie group method. Symmetry Reductions (19), (28), (40) and (51) for Eq. (1) have been obtained from Cases 1-4. Soliton Solutions (22), Hyperbolic-Function Solutions (32), Rational Solutions (33), Power-Series Solutions (45) as well as Trigonometric-Function Solutions (34) and (54)–(56) for Eq. (1) have been derived.

**Acknowledgements** The authors express their sincere thanks to the members of their discussion group for their valuable suggestions. This work has been supported by the National Natural Science Foundation of China under Grant No. 11772017, and by the Fundamental Research Funds for the Central Universities.

## References

1. V.B. Matveev, M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991)
2. M. Boiti, F. Pempinelli, G. Soliani, *Nonlinear Evolution Equations and Dynamical Systems* (Springer, Berlin, 1980)
3. M. Mirzazadeh, M. Ekici, A. Sonmezoglu, S. Ortakaya, M. Eslami, A. Biswas, Eur. Phys. J. Plus **131**, 166 (2016)
4. X.Y. Gao, Y.J. Guo, W.R. Shan, Appl. Math. Lett. **104**, 106170 (2020)
5. H.Y. Tian, B. Tian, Y.Q. Yuan, C.R. Zhang, *Phys. Scr.* **96**, 045213 (2021)
6. Y. Shen, B. Tian, S.H. Liu, D.Y. Yang, *Phys. Scr.* **96**, 075212 (2021)
7. D.Y. Yang, B. Tian, Q.X. Qu, C.R. Zhang, S.S. Chen, C.C. Wei, Chaos Solitons Fract. (2021). https://doi.org/10.1016/j.chaos.2020.110487
8. K. Hosseini, S. Salahshour, M. Mirzazadeh, A. Ahmadian, D. Baleanu, A. Khoshrang, Eur. Phys. J. Plus **136**, 206 (2021)
9. Y.Q. Chen, B. Tian, Q.X. Qu, H. Li, X.H. Zhao, H.Y. Tian, M. Wang, Int. J. Mod. Phys. B **34**, 2050226 (2020)
10. S.S. Chen, B. Tian, J. Chai, X.Y. Wu, Z. Du, Wave. Random Complex **30**, 389 (2020)
11. Y. Shen, B. Tian, C.R. Zhang, H.Y. Tian, S.H. Liu, Mod. Phys. Lett. B **35**, 2150261 (2021)
12. O.V. Kaptsov, D.O. Kaptsov, Eur. Phys. J. Plus **135**, 723 (2020)
13. M. Wang, B. Tian, C.C. Hu, S.H. Liu, *Appl. Math. Lett.* **119**, 106936 (2021)

14. X.Y. Gao, Y.J. Guo, W.R. Shan, Chaos Solitons Fract. **142**, 110367 (2021)
15. M. Wang, B. Tian, Y. Sun, Z. Zhang, Comput. Math. Appl. **79**, 576 (2020)
16. Y.Q. Chen, B. Tian, Q.X. Qu, H. Li, X.H. Zhao, H.Y. Tian, M. Wang, Mod. Phys. Lett. B **34**, 2050287 (2020)
17. X.X. Du, B. Tian, Y.Q. Yuan, Z. Du, Ann. Phys. (Berlin) **531**, 1900198 (2019)
18. X.X. Du, B. Tian, Q.X. Qu, Y.Q. Yuan, X.H. Zhao, Chaos Solitons Fract. **134**, 109709 (2020)
19. B. Ghanbari, C.K. Kuo, Eur. Phys. J. Plus **134**, 334 (2019)
20. M. Wang, B. Tian, Y. Sun, H.M. Yin, Z. Zhang, Chin. J. Phys. **60**, 440 (2019)
21. D.Y. Yang, B. Tian, Q.X. Qu, H. Li, X.H. Zhao, S.S. Chen, C.C. Wei, Phys. Scr. **96**, 045210 (2021)
22. X. Zhao, B. Tian, Q.X. Qu, Y.Q. Yuan, X.X. Du, M.X. Chu, Mod. Phys. Lett. B **34**, 2050282 (2020)
23. V.F. Morales-Delgado, J.F. Gomez-Aguilar, S. Kumar, M.A. Taneco-Hernandez, Eur. Phys. J. Plus **133**, 200 (2018)
24. X.Y. Gao, Y.J. Guo, W.R. Shan, Chaos Solitons Fract. **138**, 109950 (2020)
25. X.Y. Gao, Y.J. Guo, W.R. Shan, Chin. J. Phys. **70**, 264 (2021)
26. S.S. Chen, B. Tian, Y. Sun, C.R. Zhang, Ann. Phys. (Berlin) **531**, 1900011 (2019)
27. X.Y. Gao, Y.J. Guo, W.R. Shan, Y.Q. Yuan, C.R. Zhang , S.S. Chen, Appl. Math. Lett. **111**, 106627 (2021)
28. X.Y. Gao, Y.J. Guo, W.R. Shan, Acta Mech. **231**, 4415 (2020)
29. C.R. Zhang, B. Tian, Y. Sun, H. M. Yin, EPL **127**, 40003 (2019)
30. H.Y. Tian, B. Tian, C.R. Zhang, S.S. Chen, Int. J. Comput. Math. (2021). https://doi.org/10.1080/00207160.2021.1900568
31. M. Wang, B. Tian, S.H. Liu, W.R. Shan, Y. Jiang, Eur. Phys. J. Plus **136**, 635 (2021)
32. X. Zhao, B. Tian, H.Y. Tian, D.Y. Yang, Nonlinear Dyn. **103**, 1785 (2021)
33. S.H. Liu, B. Tian, Q.X. Qu, H. Li, X. H. Zhao, X. X. Du, S. S. Chen, Int. J. Comput. Math. **98**, 1130 (2021)
34. S.H. Liu, B. Tian, Q.X. Qu, C.R. Zhang, C.C. Hu, M. Wang, Mod. Phys. Lett. B **34**, 2050243 (2020)
35. Y. Shen, B. Tian, Appl. Math. Lett. (2021). https://doi.org/10.1016/j.aml.2021.107301
36. C.R. Zhang, B. Tian, Q.X. Qu, L. Liu, H.Y. Tian, Z. Angew, Math. Phys. **71**, 18 (2020)
37. Y. Shen, B. Tian, S.H. Liu, Phys. Lett. A **405**, 127429 (2021)
38. P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 2000)
39. H. Gao, R.X. Zhao, Appl. Math. Comput. **215**, 2781 (2009)
40. C.C. Hu, B. Tian, H.M. Yin, C.R. Zhang, Z. Zhang, Comput. Math. Appl. **78**, 166 (2019)
41. F. Chand, A.K. Malik, Int. J. Nonlinear Sci. **14**, 416 (2012)
42. M. Alquran, A. Qawasmeh, J. Appl. Anal. Comput. **4**, 221 (2014)
43. W.A. Huang, Chaos Solitons Fract. **29**, 365 (2006)
44. R. Hirota, Phys. Rev. Lett. **27**, 1192 (1971)
45. L.L. Feng, S.F. Tian, T.T. Zhang, Bull. Malays. Math. Sci. Soc. **43**, 141 (2020)
46. X.E. Zhang, Y. Chen, Appl. Math. Lett. **98**, 306 (2019)
47. X.B. Wang, B. Han, J. Math. Anal. Appl. **487**, 123968 (2019)
48. L.Q. Li, Y.T. Gao, L. Hu, T.T. Jia, C.C. Ding, Y.J. Feng, Nonlinear Dyn. **100**, 2729 (2020)
49. X.Y. Gao, Y.J. Guo, W.R. Shan, Phys. Lett. A **384**, 126788 (2020)
50. M. Wang, B. Tian, Q.X. Qu, X.X. Du, C.R. Zhang, Z. Zhang, Eur. Phys. J. Plus **134**, 578 (2019)
51. C. Chen, Y.L. Jiang, Commun. Theor. Phys. **68**, 295 (2017)
52. S.Y. Lou, X.B. Hu, J. Math. Phys. **38**, 6401 (1997)
53. P.G. Estévez, G.A. Hernáez, (2000). Darboux transformations for a Bogoyavlenskii equation in (2+ 1) dimensions, in *Proceedings of the Workshop on Nonlinearity, Integrability and All That–Twenty Years After NEEDS'79: Lecce, Italy, 1-10 July 1999*, (World Sci., 2000), p. 117
54. G.W. Bluman, S. Kumei, *Symmetries and Differential Equations* (Springer, Berlin, 1989)
55. P. Albares, J.M. Conde, P.G. Estévez, Appl. Math. Comput. **355**, 585 (2019)
56. P.G. Estévez, D. Lejarreta, C. Sardón, Nonlinear Dyn. **87**, 13 (2017)
57. B. Gao, Y. Zhang, Phys. A **525**, 1058 (2019)
58. H.C. Hu, Y.H. Lu, Mod. Phys. Lett. B **34**, 2050106 (2020)
59. M. Kumar, D.V. Tanwar, R. Kumar, Comput. Math. Appl. **75**, 218 (2018)
60. Q. Li, T. Chaolu, Y.H. Wang, Comput. Math. Appl. **77**, 2077 (2019)
61. D.J. Korteweg, G. de Vries, Philos. Mag. **39**, 422 (1895)
62. N.J. Zabusky, M.D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965)
63. C.S. Gardner, J.M. Greene, M.D. Krustal, R.M. Miura, Phys. Rev. Lett. **19**, 1095 (1967)
64. M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge Univ. Press, New York, 1991)
65. A. Jeffrey, T. Kakutani, SIAM Rev. **14**, 582 (1972)
66. R.M. Miura, SIAM Rev. **18**, 412 (1976)

67. T.C. Xia, S.Q. Xiong, Comput. Math. Appl. **60**, 919 (2010)
68. S.T. Chen, W.X. Ma, Comput. Math. Appl. **76**, 1680 (2018)
69. W. Rui, Y. Zhang, F. Yang, Nonlinear Anal. -Real **43**, 467 (2018)