



# Probability density correlation for PDM-Hamiltonians and superstatistical PDM-partition functions

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**Abstract** Schrödinger equation with position-dependent mass (PDM) allows the identification of quantum wave functions in a complex environment. Following the progress of this investigation field, in this work, we consider the non-Hermitian kinetic operators associated with the PDM Schrödinger equation. We provide a simplified picture for PDM quantum systems that admit exact solutions in confining potentials. First, we investigate the solutions for a sinusoidal and an exponential PDM distributions in an infinite potential well. Next, we consider the solutions for a PDM harmonic oscillator potential associated with a power-law PDM distribution. The results presented in this work offer a way to approach new classes of solutions for PDM quantum systems in confining potential (bound states). Complementarily, we interpret the quantum partition function of the canonical ensemble of a PDM system in the context of the superstatistics, which, in turn, allows us to express the inhomogeneity of the PDM in terms of beta distribution  $f(\beta)$ , Dirac delta distributions for  $f(\beta)$ , and effective temperatures. Our results are, hereby, reported for the sinusoidal and the exponential PDM distributions.

## 1 Introduction

Quantum mechanics is one of the most vibrant theories of physics that emerged in the 20<sup>th</sup> century. Hereby, the Schrödinger equation is the cornerstone for describing non-relativistic quantum systems involving particles within the atomic scale. The dynamical information of such particles is assumed to be indulged in the wave function, whose squared modulus gives the probability distribution of the particle. Applications of the Schrödinger equation for inhomogeneous semiconductors have manifested typical features, like abrupt discontinuities of the wave function at the heterojunctions as well as extensions of the Wannier–Slater theorem [1], that can be well modeled by a variable mass (i.e., position-dependent mass (PDM)) [2]. In the PDM framework, it is assumed that a PDM Hamiltonian with such

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inhomogeneity would result an *ordering ambiguity problem* as an attempt to compatibilize Galilean invariance along with Hermiticity of the PDM kinetic energy operator [2–4].

Furthermore, non-Hermitian extensions of the PDM Hamiltonians with real spectra have also been studied to allow comprehensive description of non-Hermitian PDM quantum systems (c.f., e.g., [5]). Such systems are natural generalizations of the so-called non-Hermitian  $\mathcal{PT}$ -symmetric quantum mechanics by Bender and Boettcher [6]. From such non-Hermitian PDM Hamiltonians, remarkable features have been obtained for a large amount of systems by means of generalized Schrödinger equations. Typical examples include semiconductors [2], Bohmian quantum theory [7], bound states [8], many-body theory [9], supersymmetrical quantum mechanics [10, 11], classical field theory [12], non-Hermitian potentials [13], etc. [14–16]. Therefore, these applications reveal that PDM quantum systems provide a way to address systems whose non-trivial potentials lead to effective mass distributions. To elucidate this idea, we can consider that the maximum value of a static potential can be reinterpreted as a minimum value in the mass distribution function. Following this insight, a non-trivial static potential generates an effective mass. Consequently, this strengthens the phenomenological point of view, which allows us to investigate the probability distributions of quantum systems with complicated potentials. Nevertheless, solving PDM quantum systems may be a challenge, which involves choosing a PDM Schrödinger equation representation, potential, and/or mass distribution.

Within this scenario, there are distinct forms to describe the PDM Schrödinger equation, all of which arise from the general form of the PDM kinetic energy operator

$$\hat{K}_R = \frac{1}{4} \left\{ [m(\hat{x})]^{-\eta} \hat{p}[m(\hat{x})]^{-1+\eta+\nu} \hat{p}[m(\hat{x})]^{-\nu} + [m(\hat{x})]^{-\nu} \hat{p}[m(\hat{x})]^{-1+\eta+\nu} \hat{p}[m(\hat{x})]^{-\eta} \right\}, \quad (1)$$

proposed by von Roos [2]. Equation (1) represents the mathematical formulation of a parametric ordering ambiguity problem, which characterizes a large range of Hermitian systems with position-dependent mass. Some of the most prominent kinetic energy operators available in the literature can be obtained in a straightforward manner: Ben Daniel and Duke (BDD) [17] ( $\eta = \nu = 0$ ), Gora and Williams (GW) [18] ( $\eta = 1, \nu = 0$ ), Zhu and Kroemer (ZK) [19] ( $\eta = \nu = \frac{1}{2}$ ), and Li and Kuhn (LK) [20] ( $\eta = 0, \nu = \frac{1}{2}$ ). However, Morrow et al. [21] have shown that only the parametric ordering constraint  $\eta = \nu$  satisfies the continuity conditions of the wave function at the boundaries of the abrupt heterojunctions between crystals. Under such parametric settings, Mustafa and Mazharimousavi [22] have shown that  $\eta = \nu = \frac{1}{4}$  allows the mapping of a PDM quantum Hamiltonian into the usual constant mass Hamiltonian by means of a point canonical transformation. They have suggested the PDM kinetic energy operator

$$\hat{K}_{MM} = \frac{1}{2} [m(\hat{x})]^{-\frac{1}{4}} \hat{p}[m(\hat{x})]^{-\frac{1}{2}} \hat{p}[m(\hat{x})]^{-\frac{1}{4}}, \quad (2)$$

which obviously adheres to the continuity conditions mentioned above. Very recently, however, Mustafa and Algaadhi [23] have constructed the PDM momentum operator

$$\hat{p}(x) = \hat{p} + \frac{i}{4} \frac{m'(x)}{m(x)} \implies \hat{K}_{MM} = \frac{1}{2} \left( \frac{\hat{p}(x)}{\sqrt{m(x)}} \right)^2 = \frac{1}{2} \hat{\Pi}(x)^2,$$

and consequently fixing the ambiguity parameters at  $\eta = \nu = \frac{1}{4}$  (i.e., MM parametric ordering). Moreover, in a straightforward manner, one may show that

$$[x, \hat{p}(x)] = i$$

which is in exact analogy with the textbook commutation relation  $[x, \hat{p}] = i$  for constant mass settings. Yet, we may map Mustafa–Mazharimousavi’s PDM kinetic energy operator into alternative forms using an auxiliary field that allows a variable change, i.e.,  $\Phi \xrightarrow{T} \Psi$ , and yields a transformation ( $T$ ) of the wave functions (see [24,25] for more details). Among the feasibly admissible alternative forms is the non-Hermitian kinetic operator

$$\hat{K} = \frac{1}{2} \hat{p}[m(\hat{x})]^{-\frac{1}{2}} \hat{p}[m(\hat{x})]^{-\frac{1}{2}}, \tag{3}$$

which is, in fact, an immediate consequence of the wave function transformation mentioned above. That is, if the transformation  $\Phi_{MM} \xrightarrow{T} [m(\hat{x})]^{-\frac{1}{4}} \Psi$  is used in the corresponding PDM Schrödinger (2), it would lead to the PDM kinetic energy operator of (3). In addition, their adjoint form was recently reported for PDM quantum systems in [26,27]. In the current proposal, we shall show that this representation allows us to find generalized classes of solutions in confining potentials framework. To the best of our knowledge, the methodology as well as the results of the current study has never been reported elsewhere.

In this paper, we provide a thorough analysis of the PDM Schrödinger equation associated with kinetic operator shown in Eq. (3). In so doing, we organize our analysis in the following order. In Sect. 2, we contextualize the Schrödinger equation in the PDM framework and show that our approach satisfies the conservation equation for probability density. Moreover, we use a change of variable ( $x \rightarrow y(x)$ ) that allows us to rewrite the PDM Schrödinger equation in a simplified picture and find new classes of exact solutions. In Sect. 3, we solve for the one-dimensional infinite potential well by considering an arbitrary PDM,  $m(x)$ . Therefore, we present explicit solutions for two types: a sinusoidal and an exponential PDM distributions. In Sect. 4, we solve for a PDM harmonic oscillator like potential manifested by a power-law-type PDM distribution. Next, in Sect. 5 we study the quantum partition functions of the canonical ensembles of PDM systems in terms of associated superstatistical partition functions that accounts for the heterogeneity of the PDM expressed by means of local inverse temperatures  $\beta$ . Finally, in Sect. 6 we draw our conclusions and outline some perspectives.

## 2 Theoretical formulation and connections

Let us begin our analysis with the PDM kinetic operator of (3) in the corresponding PDM-Hamiltonian operator

$$\begin{aligned} \hat{H}(\hat{x}, \hat{p}) &= \frac{1}{2} \hat{p}[m(\hat{x})]^{-\frac{1}{2}} \hat{p}[m(\hat{x})]^{-\frac{1}{2}} + V(\hat{x}) \\ &= \frac{1}{2m_0} \hat{P}_\zeta^2 + V(\hat{x}), \end{aligned} \tag{4}$$

with  $\hat{P}_\zeta = \hat{p} \zeta(\hat{x})$  and  $\zeta(\hat{x}) = \sqrt{m_0/m(\hat{x})}$ . Consequently, the PDM Schrödinger equation reads

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{\Psi}{\sqrt{m(x)}} \right) + V(x)\Psi, \tag{5}$$

where  $\Psi = \Psi(x, t)$  is the quantum wave function. Multiplying, from the left, Eq. (5) by  $\Psi^*$ , and the conjugate of Eq. (5) by  $\Psi$ , we obtain in a straightforward subtraction procedure the following result

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \{ \Psi^* \Psi \} &= -\Psi^* \frac{\hbar^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{\Psi}{\sqrt{m(x)}} \right) \\
 &+ \Psi \frac{\hbar^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \frac{\Psi^*}{\sqrt{m(x)}} \right).
 \end{aligned}
 \tag{6}$$

Moreover, multiplying (from the left) both sides of (6) by  $\sqrt{m_0/m(x)}$ , we obtain (under PDM settings) the continuity equation

$$\frac{\partial}{\partial t} \rho(x, t) = -\frac{\partial}{\partial x} \mathcal{J}(x, t).
 \tag{7}$$

The PDM probability density function is given by

$$\rho(x, t) = \sqrt{\frac{m_0}{m(x)}} \Psi^*(x, t) \Psi(x, t),
 \tag{8}$$

and the PDM probability current is

$$\mathcal{J} = \sqrt{m_0} \frac{\hbar}{2i} \left( \frac{\Psi^*}{m(x)} \frac{\partial}{\partial x} \frac{\Psi}{\sqrt{m(x)}} - \frac{\Psi}{m(x)} \frac{\partial}{\partial x} \frac{\Psi^*}{\sqrt{m(x)}} \right).
 \tag{9}$$

Hence, Eq. (5) preserves the probability (i.e.,  $\int_{\Omega} \rho(x) dx = 1$ ;  $\Omega \subset \mathbb{R}$ ), and thus, the conservation law is satisfied. It is worth to note that the positivity of the PDM probability density  $\rho(x, t)$  (8) is guaranteed by the factor  $\sqrt{m_0/m(x)} > 0$ .

In order to analyze Eq. (5) in a simplified picture, we consider the following change of variable

$$y(x) = \int^x \sqrt{\frac{m(x')}{m_0}} dx,
 \tag{10}$$

studied in classical and quantum contexts [26,28–30] and also in statistical physics [31]. Obviously,  $y(x) \rightarrow x$  as  $m(x) \rightarrow m_0$  and the transformed system, along with its boundary conditions, returns back to the traditional constant mass Schrödinger settings. By means of the variable  $y$ , we can express (5) with a modified wave function  $\tilde{\Psi}(y, t)$ , as

$$i\hbar \frac{\partial}{\partial t} \tilde{\Psi}(y, t) = -\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial y^2} \tilde{\Psi}(y, t) + V(y) \tilde{\Psi}(y, t),
 \tag{11}$$

where  $V(y) = V(y(x))$  is the effective PDM potential and  $\tilde{\Psi}(y, t) = \Psi \sqrt{m_0/m(x)}$ . The mapping, through (10), between the PDM system in (5) and the constant mass system in (11) is clear, therefore. Yet, the two systems admit isospectrality, which is an immediate consequence of the substitutions  $\tilde{\Psi}(y, t) = \tilde{\psi}(y) \phi(t)$  in (11) and  $\Psi(x, t) = \psi(x) \phi(t)$  in (5), with  $\phi(t) = e^{-iEt/\hbar}$ . In the most simplistic language, in short, all well-known available solutions (eigenvalues and eigenfunctions) of the standard constant mass Schrödinger equation can be implemented for solving

$$E \tilde{\psi}(y) = -\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial y^2} \tilde{\psi}(y) + V(y) \tilde{\psi}(y),
 \tag{12}$$

corresponding to (11), regardless of the PDM distribution function. Posteriorly, we expose and report some relevant examples by considering confining potentials and rich classes of PDM distributions.

## 2.1 Alternative PDM Hamiltonians and PDM probability density correlations

Here, we discuss the connections of the PDM Schrödinger equation (5) with other parametric orderings. We start with Mustafa–Mazharimousavi’s ordering and consider a transformation of the form  $\Phi(x, t) = \sqrt[4]{m_0/m(x)}\Psi(x, t)$  in (5), and we obtain

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{\sqrt[4]{m(x)}} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt[4]{m(x)}} \frac{\partial \Phi}{\partial x} \sqrt[4]{m(x)} \right) + V(x)\Phi, \quad (13)$$

whose PDM-Hamiltonian operator is

$$\begin{aligned} \hat{H}_1 &= \frac{1}{2} [m(\hat{x})]^{-\frac{1}{4}} \hat{p} [m(\hat{x})]^{-\frac{1}{2}} \hat{p} [m(\hat{x})]^{-\frac{1}{4}} + V(\hat{x}). \\ &= \frac{1}{2m_0} \hat{\Pi}_\zeta^2 + V(\hat{x}), \end{aligned} \quad (14)$$

in which

$$\hat{\Pi}_\zeta = \frac{1}{2} (\hat{P}_\zeta^\dagger + \hat{P}_\zeta) = \sqrt{\zeta(\hat{x})} \hat{p} \sqrt{\zeta(\hat{x})}$$

is the so-called PDM pseudo-momentum operator (or the PDM Noether momentum operator) [22]. Unavoidably at this point, one has to mention that Bagchi et al. [32] have used a similar operator in their shape invariance study of PDM systems. However, our operator  $\hat{\Pi}_\zeta$  should not be confused with that of Bagchi et al.’s [32]. The operator  $\hat{\pi} = \sqrt{f(\alpha; x)} \hat{p} \sqrt{f(\alpha; x)}$  of Bagchi et al. is used to replace the standard momentum operator  $\hat{p} = -i\partial/\partial x$ , to serve for their supersymmetric treatment, and is associated with an effective potential  $V_{\text{eff}}(\mathbf{b}; x)$  (see the effective potential in Eq. (2.7) in Ref. [32]) which is still ambiguity parameters dependent, whereas our approach is based on the PDM momentum operator (discussed above and in comprehensive details in [23]) and our interaction potential in (14) is only PDM-deformed potential and is not given in terms of the ambiguity parameters (c.f., e.g., [33, 34] and related references cited therein). Strictly speaking, our effective potential would be found from Bagchi et al.’s [35] effective potential (3) and (4) for  $\alpha = -1/4$ ,  $\beta = -1/2$ . It should be, therefore, clear that our approach is completely different than that of Bagchi et al. [32].

Next, we can obtain another adjoint representation of Eq. (5) through the transformation  $\Lambda(x, t) = \sqrt[4]{m_0/m(x)}\Phi(x, t) = \sqrt{m_0/m(x)}\Psi(x, t)$ . This transforms (13) into

$$i\hbar \frac{\partial \Lambda}{\partial t} = -\frac{\hbar^2}{2} \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{m(x)}} \frac{\partial \Lambda}{\partial x} \right) + V(x)\Lambda, \quad (15)$$

which corresponds to yet another alternative PDM-Hamiltonian operator

$$\begin{aligned} \hat{H}_2(\hat{x}, \hat{p}) &= \frac{1}{2} [m(\hat{x})]^{-\frac{1}{2}} \hat{p} [m(\hat{x})]^{-\frac{1}{2}} \hat{p} + V(\hat{x}) \\ &= \frac{1}{2m_0} (\hat{P}_\zeta^\dagger)^2 + V(\hat{x}). \end{aligned} \quad (16)$$

Here,  $\hat{P}_\zeta^\dagger$  is the adjoint of the operator  $\hat{P}_\zeta$  used in (4), and the kinetic operator in (16) is the adjoint form of that in Eq. (3), i.e.,  $\hat{K}^\dagger = \frac{1}{2m_0} (\hat{P}_\zeta^\dagger)^2$ . Very recently, the two alternative PDM Hamiltonians of (14) and (16) were investigated from the PDM harmonic oscillators’ point of view in Ref. [26]. Moreover, it should be noted that the linear combination of the PDM Hamiltonian (4) and its adjoint yields  $\hat{H}_{\text{LK}} = \frac{1}{2} (\hat{H} + \hat{H}^\dagger)$ , which is in fact the PDM Hamiltonian of the Li–Kuhn’s ordering [20] (Fig. 1).

We may now rewrite the PDM probability density function of (8) in a correlation that indulges the connections/mappings among the alternative PDM Schrödinger equations (4), (14), and (16) and report that

$$\rho(x, t) = \overbrace{\zeta(x)|\Psi(x, t)|^2}^{\text{for (5)}} = \overbrace{|\Phi(x, t)|^2}^{\text{for (13)}} = \overbrace{\frac{1}{\zeta(x)}|\Lambda(x, t)|^2}^{\text{for (15)}}, \tag{17}$$

with  $\zeta(\hat{x}) = \sqrt{m_0/m(\hat{x})} > 0$ . Obviously, one would expect that the probability density will be affected by the PDM settings as a natural mathematical consequence of the PDM wave functions involved in this correlation (17). The first and the third terms on the R.H.S. of (17) manifestly suggest this effect. This is yet documented in Figs. 2, 3, 4, 5, 6 and 7. Moreover, from the probability density point of view, this result suggests that all such alternative representations are equivalent regardless of their non-Hermiticity. On the other hand, the third term on the R.H.S. of (17) secures/guarantees the positive definiteness of our PDM probability density.

In the following sections, we use some examples to illustrate the simplicity and applicability of the our methodical proposal. We use the PDM Hamiltonians of (4) and (14) along with the transformed Schrödinger equation (12) and show how to extract, with ease, some well-known exact solutions for (12) to reflect on the solutions of the PDM systems of (4) and (14).

### 3 PDM particles in an infinite potential well

In this section, we consider, within the above new perspective, PDM particles in an infinite potential well defined by

$$V(x) = \begin{cases} 0, & \text{for } x \in [0, L] \\ \infty, & \text{otherwise.} \end{cases} \tag{18}$$

This potential manifestly dictates the boundary conditions  $\psi(0) = \psi(L) = 0$  on the wave functions. The standard constant mass  $m(x) = m_0$  problem suggests a sinusoidal-type solution

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \implies E_n = \frac{n^2 \pi^2 \hbar^2}{2m_0 L^2}; \quad n = 1, 2, 3, \dots$$

Nonetheless, in order to illustrate the influence of PDM settings, we consider two examples. The first of which is an isochronic sinusoidal PDM function, as shown in Fig. 1a,

$$m(x) = \frac{m_0}{1+a} \left[ 1 + 2a \sin^2\left(k\pi \frac{x}{L}\right) \right]; \quad a > -\frac{1}{2}, \tag{19}$$

where the dimensionless parameter  $k$  is to be determined by the boundary conditions, and  $a \rightarrow 0$  recovers the usual constant mass settings, i.e.,  $m(x) = m_0$ . The second one, on the other hand, is an exponential PDM function, as shown in Fig. (1c),

$$m(x) = \frac{\alpha m_0}{(1 - e^{-\alpha})} e^{-\alpha \frac{x}{L}}, \tag{20}$$

that recovers the constant mass  $m(x) = m_0$  when  $\alpha \rightarrow 0$ . Both cases are illustrated in Fig. 1 and satisfy the following relation

$$\frac{1}{L} \int_0^L m(x) dx = m_0. \tag{21}$$

This relation shows that the distribution of the mass fluctuates around its mean value  $m_0$ , and it also allows us to analyze how the deformation of the mass affects the probability density  $\rho(x, t)$ .

### 3.1 An isochronic Sinusoidal PDM in infinite potential well

Here, we consider the PDM in Eq. (19) and substitute it in Eq. (10) to obtain the new coordinate

$$y(x) = \frac{L}{\pi k \sqrt{a+1}} \mathcal{E} \left( \sin \left( \frac{k\pi x}{L} \right); \sqrt{-2a} \right), \tag{22}$$

where  $\mathcal{E}$  is the incomplete elliptic integral of the second kind. By employing this change of variable in (11), we elucidate the dynamics of the problem at hand. Therefore, we must rewrite the boundary conditions in the new variable  $y(x)$  as

$$\tilde{\psi}(0) = 0 \quad \text{and} \quad \tilde{\psi}(y(L)) = 0. \tag{23}$$

At this point, moreover, one should notice that as  $a \rightarrow 0$ , not only the PDM collapses into  $m(x) = m_0$  but also  $y(L) = L$  (by virtue of (10) and as a natural consequence of returning back to the usual boundary conditions of the constant mass problem of (18)). Hence, in a straightforward manner, one can show that the condition  $y(L) = L$  manifestly introduces the restriction that  $k = 1/2$ . Considering these boundary conditions, the solution of (11) results in

$$\tilde{\psi}_n(y) = A_n \sin \left( \frac{n\pi}{y(L)} y \right), \tag{24}$$

and the energy spectra

$$E_{n,a} = \frac{n^2 \pi^4 \hbar^2}{8m_0 L^2} \frac{(a+1)}{\mathcal{E}(\sqrt{-2a})^2}, \tag{25}$$

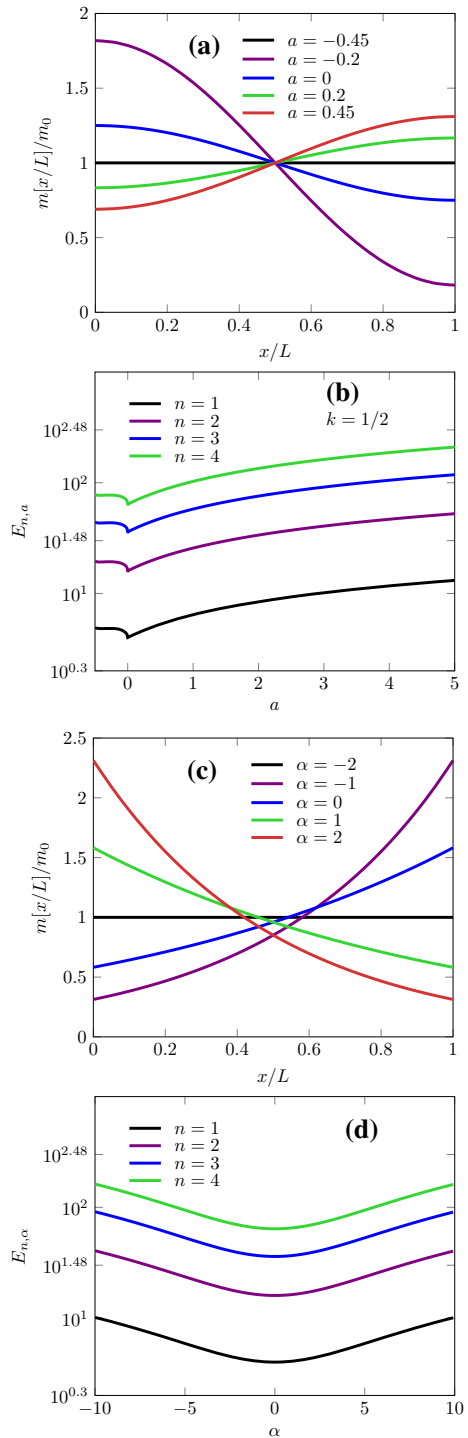
where  $\mathcal{E}(\sqrt{-2a})$  is the complete elliptic integral. Obviously, this result recovers the energy levels of the standard constant mass at the limit  $a \rightarrow 0$ . That is,

$$E_n = \lim_{a \rightarrow 0} E_{n,a} = \frac{n^2 \pi^2 \hbar^2}{2m_0 L^2}.$$

The effect of the PDM (19) on the spectra is shown in Fig. 1b where the energy levels are infected by the PDM parameter  $a$  and abandon their regular constant textbook values for constant mass setting. Now, we may use the correlation between the wave functions,  $\tilde{\psi}_n(y(x)) = \sqrt{m_0/m(x)} \psi_n(x)$ , and report the wave functions of the PDM system (19) as

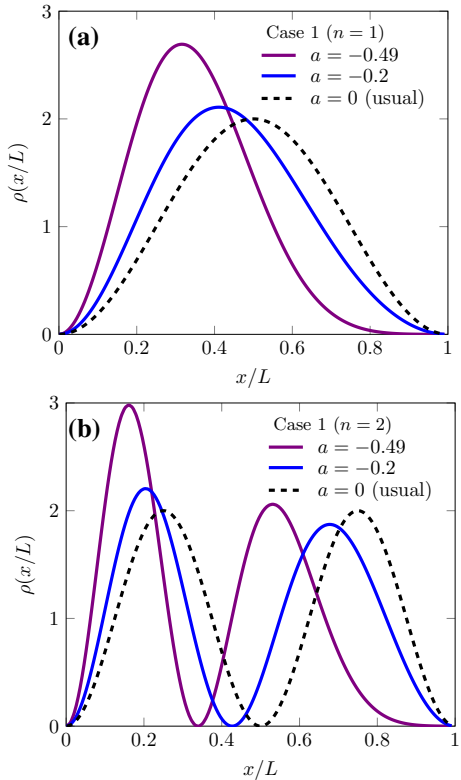
$$\psi_n(x) = \tilde{A}_n(x) \sin \left( n\pi \frac{\mathcal{E} \left( \sin \left( \frac{\pi x}{2L} \right); \sqrt{-2a} \right)}{\mathcal{E}(\sqrt{-2a})} \right) \tag{26}$$

**Fig. 1** PDM functions versus scaled position, i.e.,  $x/L$ . **a** Considers the sinusoidal mass distribution of (19) for  $k = 1/2$  and different  $a$  values. **b** Shows the effect of (19) on the energy spectrum. **c** Considers the exponential mass distribution of (20) for different  $\alpha$  values. **d** Shows the effect of (20) on the energy spectrum. For energies shown in **b, d**, we consider  $(\pi \hbar)/(2^{3/2} L \sqrt{m_0}) = 1$  in Eqs. (25) and (30)





**Fig. 2** Probability density (8) of the wave function versus  $x/L$  for the sinusoidal mass (solid curves) and the uniform mass (dashed curves). The figures present different cases of the fundamental state and the first excited cases. The **a** and **b** correspond to the values  $(m_0, k, n) = (1, 0.5, 1)$  and  $(m_0, k, n) = (1, 0.5, 2)$ , respectively



with

$$\tilde{A}_n(x) = \frac{A_n}{\sqrt{1+a}} \left( 1 + 2a \sin^2 \left( \frac{\pi}{2L} x \right) \right)^{\frac{1}{2}}, \tag{27}$$

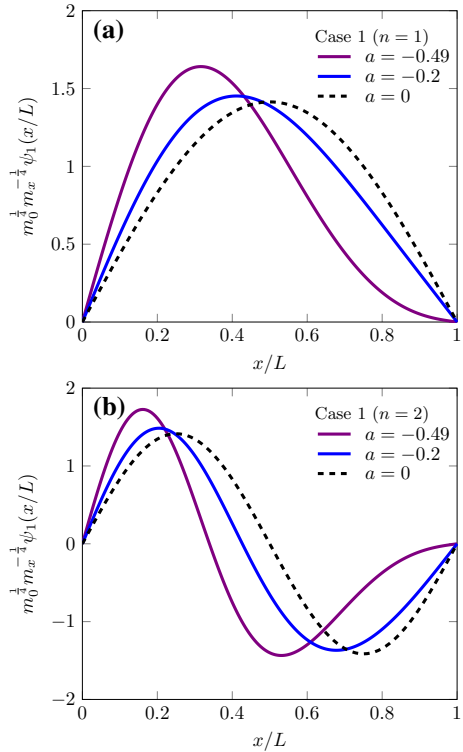
where  $A_n$  is the normalization constant. In Fig. 2a, b, we show the probability densities of the wave functions corresponding to the standard mass case and the sinusoidal PDM for some values of the quantum number  $n$  and the PDM parameter  $a$ . It can be seen that as the value of  $a$  decreases, the probability density curve is amplified and pends to the left toward  $L = 0$ . This would indicate that the state becomes more localized to the left as a consequence of PDM, which manifestly introduces a new dynamical effect on the standard quantum systems. Similar trend is observed for the wave functions as in Fig. 3a, b. Such irregular dynamics are nothing but a manifestation of the PDM deformation in the effective potential force field as it adapts to PDM settings.

### 3.2 An Exponential PDM in infinite potential well

We now consider the exponential mass distribution of (20) (e.g., Gönül et al. [28]) and use our transformation (10) to obtain

$$y(x) = \frac{2L}{\sqrt{\alpha(1 - e^{-\alpha})}} \left( 1 - e^{-\frac{\alpha x}{2L}} \right). \tag{28}$$

**Fig. 3** Wave function multiplied by a mass factor versus  $x/L$  for the sinusoidal mass (see Eq. (19), solid curves) and uniform-standard mass (dashed-black curve). The figures present different cases of fundamental state and the first excited cases. The **a** and **b** correspond to the values  $(m_0, k, n) = (1, 0.5, 1)$  and  $(m_0, k, n) = (1, 0.5, 2)$ , respectively



Implementing the boundary conditions  $\tilde{\psi}(0) = 0 = \tilde{\psi}(y(L))$  would again result in the general solution in (24). In turn, along with  $\psi_n(x) = \sqrt{m(x)/m_0} \tilde{\psi}_n(y(x))$ , this allows us to find

$$\psi_n(x) = A_n \frac{\sqrt{\alpha} e^{-\frac{\alpha}{2} \frac{x}{L}}}{(1 - e^{-\alpha})^{\frac{1}{2}}} \sin \left[ n\pi \frac{\left(1 - e^{-\frac{\alpha x}{L}}\right)}{\left(1 - e^{-\frac{\alpha}{2}}\right)} \right], \tag{29}$$

which is the wave function for the exponential PDM (20) particle moving in the infinite potential well (18). Consequently, we obtain the energy spectra for different  $\alpha$  values as

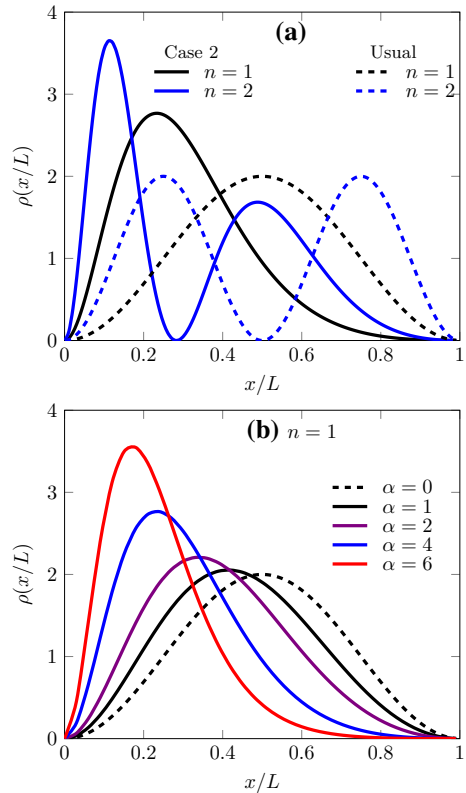
$$E_{n,\alpha} = \frac{\hbar^2 \pi^2 n^2}{8m_0 L^2} \frac{\alpha(1 - e^{-\alpha})}{\left(1 - e^{-\frac{\alpha}{2}}\right)^2}. \tag{30}$$

As before, from this result we recover the constant mass energies

$$E_n = \lim_{\alpha \rightarrow 0} E_{n,\alpha} = \frac{n^2 \pi^2 \hbar^2}{2m_0 L^2}.$$

Moreover, each value of  $\alpha$  yields an energy spectrum. Figure 1d shows the effect of the PDM (20) on the spectra and clearly indicates that the spectra are infected by the PDM parameter and once again abandon the textbook constant values for the energy levels for constant mass. Yet, again, we see that when  $\alpha \rightarrow 0$ , not only the PDM collapses into  $m(x) = m_0$  but also  $y(L) = L$  and it allows us to recover the usual boundary conditions of the constant mass problem of (18). In Fig. 4, we show the probability densities of the wave functions

**Fig. 4** Probability densities (8) of the wave functions versus  $x/L$  for the exponential PDM of (20) (solid curves) and for the constant mass (dashed curves). **a** Shows the probability density of the ground state and first excited state for  $\alpha = 4$ . **b** Shows the probability densities of the ground state for different  $\alpha$  values. In both figures, we considered  $m_0 = 1$

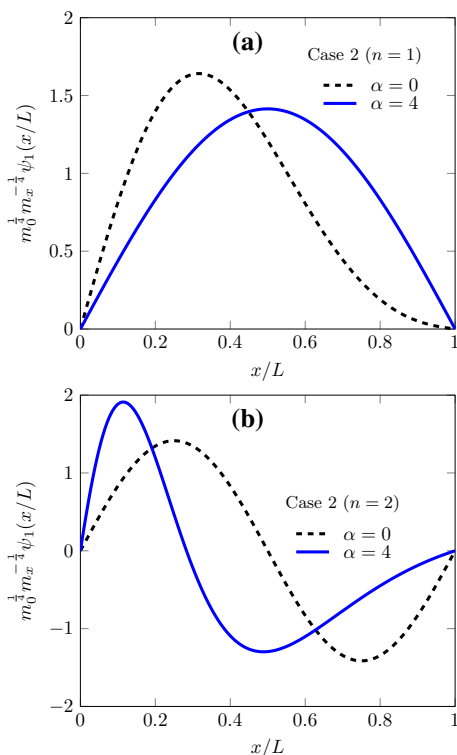


corresponding to the exponential PDM and the standard constant mass for the ground and the first excited states. Figure 5 represents the ground and first excited states wave functions for the exponential PDM. These figures clearly show similar trends as those observed for the PDM of (19). Namely, as the PDM parameter  $\alpha$  increases, the probability density curves are amplified and tend to shift to the left toward  $L = 0$  that can be considered a natural dynamical effect as the effective potential force field adapts to PDM settings.

#### 4 Probability density for a power-law PDM in a PDM-oscillator potential

The harmonic oscillator potential  $V(x) = \frac{1}{2}m_0\omega_0^2x^2$  is not only a standard textbook potential of pedagogical academic interest, but it is also immensely used as a model of a wide range of applicability and research interest in classical and quantum physics (cf., e.g., [26,36] and related references cited therein). Within the PDM settings, moreover, this potential has been addressed through some analytical and/or numerical approaches using different parametric settings for the von Roos PDM kinetic operators [37–39]. However, very recently, it has been asserted that under PDM setting, the potential energy transforms/deforms in a completely different manner than the kinetic energy (in both classical and quantum systems) (c.f., e.g., [26] and related references cited therein). That is, the oscillator potential would inherit a new PDM form so that  $V(x) = \frac{1}{2}m_0\omega_0^2(\int^x \sqrt{m(x')/m_0}dx')^2$  [26,37]. Consequently, if we recollect the new variable  $y(x)$  in (10), then the PDM oscillator potential reads

**Fig. 5** Wave functions  $\tilde{\psi}(x/L)$  (24) multiplied by a mass factor versus  $x/L$  for the ground ( $n = 1$ ) and the first ( $n = 2$ ) excited states for the exponential PDM (20) (solid curves) and for the constant mass (dashed curves), with  $m_0 = 1$  and  $\alpha = 4$



$$V(x) = \frac{1}{2}m_0\omega_0^2 \left( \int^x \sqrt{m(x')/m_0} dx' \right)^2 \iff V(y(x)) = \frac{1}{2}m_0\omega_0^2 y(x)^2. \tag{31}$$

We now consider a power-law-type PDM function in the form of

$$m(x) = m_0 \left| \frac{x}{x_0} \right|^{-\alpha}, \tag{32}$$

which when substituted in (10) implies

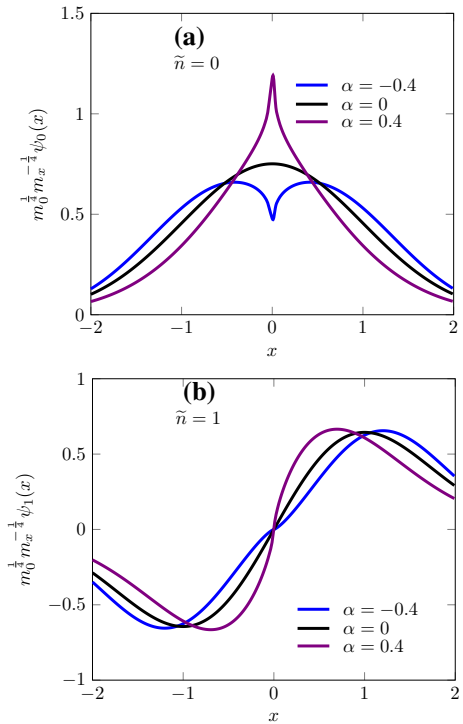
$$y(x) = \frac{x}{1 - \frac{\alpha}{2}} \left| \frac{x}{x_0} \right|^{-\frac{\alpha}{2}}; \quad \alpha \neq 2. \tag{33}$$

This would, in turn, allow us to obtain the PDM harmonic oscillator potential (31) as

$$V(x) = V(y(x)) = m_0\omega_0^2 \frac{2x^2}{(2 - \alpha)^2} \left| \frac{x}{x_0} \right|^{-\alpha}, \tag{34}$$

which recovers the usual constant mass case for  $\alpha \rightarrow 0$ . At this point, one may observe that this PDM-deformed harmonic oscillator in  $y$ -space, (31), represents the same mathematical structure of the standard one in (12). The consequence of this is that the energy spectrum is preserved due to a subtle balance between the deformation in the standard kinetic energy operator and in the potential terms of the PDM Schrödinger equation. The two systems are isospectral and share the same energy levels [26]

**Fig. 6** Wave function  $\tilde{\psi}(x)$  (35) multiplied by a mass factor versus  $x$  for the fundamental state ( $\tilde{n} = 0$ ) and the first excited state ( $\tilde{n} = 1$ ) to **a** and **b**, respectively. The power-law mass distribution (32) in Schrödinger equation is represented by blue and violet curves, and standard case, i.e., constant mass, is represented for black curve. In both situations, we considered  $\omega_0 m_0 / \hbar = 1$  and  $x_0 = 1$



$$E_{\tilde{n}} = \hbar\omega_0 \left( \tilde{n} + \frac{1}{2} \right); \quad \tilde{n} = 0, 1, 2, \dots$$

Within this context, we may assume the symmetry of the wave functions  $\tilde{\psi}(y) = \tilde{\psi}(-y)$  and their asymptotic convergence limits, i.e.,  $\lim_{y \rightarrow \pm\infty} \tilde{\psi}(y) = 0$ . Following the procedure used above, this would allow us to write the PDM wave functions in  $x$ -space as

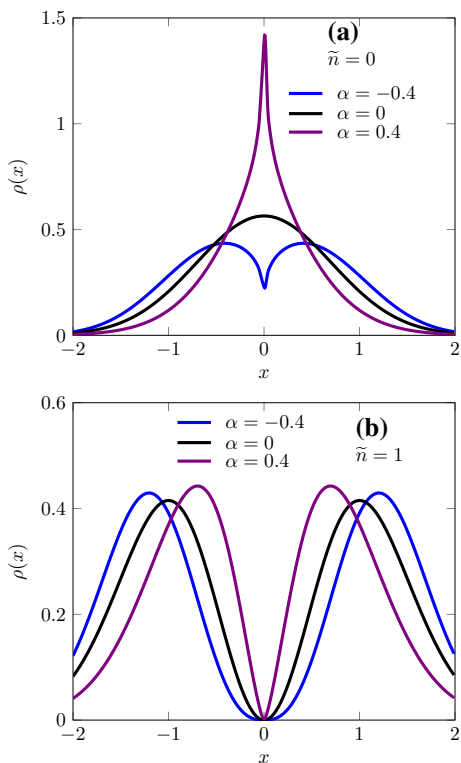
$$\psi_{\tilde{n}}(x) = \frac{1}{\sqrt{2^n \tilde{n}!}} \left( \frac{m_0 \omega_0}{\hbar \pi} \right)^{\frac{1}{4}} \left| \frac{x}{x_0} \right|^{-\frac{\alpha}{2}} \exp \left( -\frac{m_0 \omega_0}{2\hbar} y(x)^2 \right) H_{\tilde{n}} \left( \sqrt{\frac{m_0 \omega_0}{\hbar}} y(x) \right), \quad (35)$$

where  $H_{\tilde{n}}$  are the Hermite polynomials and  $y(x)$  is defined in (33). For  $\alpha = 0$  (i.e.,  $m(x) \rightarrow m_0$ ), we recover the standard solution for quantum oscillators. In order to illustrate the differences of the mass variation as a function of the position, in Fig. 6 we show the behavior of the solution (35) for the first two quantum states, in Fig. 7 we show the same situation from point of view of the probability density distribution.

### 5 PDM quantum partition function via superstatistics

In this section, we calculate the partition functions for quantum systems with PDM. Our motivation is to interpret the heterogeneity of the PDMs (19) and (20) in the infinite potential well in terms of a continuous superposition of the canonical ensembles of inverse temperature

**Fig. 7** Probability distribution function versus  $x$  for the fundamental state ( $\tilde{n} = 0$ ) and the first excited state ( $\tilde{n} = 1$ ) to **a** and **b**, respectively. The power-law mass distribution (32) in Schrödinger equation is represented by blue and violet curves, and standard case, i.e., constant mass, is represented for black curve. In both situations, we considered  $\omega_0 m_0 / \hbar = 1$  and  $x_0 = 1$



$\beta$ . We shall, therefore, appeal to the superstatistical partition function [40,41],

$$\Gamma = \sum_n B(E_n), \tag{36}$$

where

$$B(E_n) = \int_0^\infty d\beta f(\beta) \exp(-\beta E_n) \tag{37}$$

is the superstatistical beta and  $f(\beta)$  is its distribution. The standard partition function is recovered from (36) and (37) for  $f(\beta) = \delta(\beta - \beta_0)$ , where  $\beta_0$  is constant. Next, we consider two quantum systems: The first is for a PDM  $m(x)$  described by  $\tilde{H}$  and the second for a constant mass  $m_0$  described by  $H$ , with  $\tilde{E}_n$  and  $E_n$  their respective energies. By assuming that  $\tilde{H}$  is a PDM deformation of  $H$ , then it results in natural to consider that  $\tilde{H}$  is a non-degenerate if  $H$  is a non-degenerate one. This situation fits very well with the one-dimensional motion and so is the case we consider here.

With the aim of linking PDM systems with superstatistics, we can represent the partition function of  $\tilde{H}$  by means of the superstatistical one (36) and the energies  $E_n$  for a suitable  $f(\beta)$ . In other words, we impose the condition

$$\exp(-\beta_0 \tilde{E}_n) = \int_0^\infty d\beta f(\beta) \exp(-\beta E_n) \tag{38}$$

that represents the arising of the spectrum of a PDM system, as the results of the associated superstatistics applied over a constant mass system and given by  $f(\beta)$ . Moreover, we see

that (38) allows us to connect between the position-dependent mass  $m(x)$  and  $f(\beta)$ , which has never been reported before (to the best of our knowledge). One of the advantages of (38) is that both the PDM Hamiltonian  $\tilde{H}$  and the standard Hamiltonian  $H$  are arbitrary. Under such PDM-superstatistical settings, we consider the sinusoidal PDM of (19) and exponential PDM of (20) in the infinite potential well (18).

5.1 PDM-superstatistics for a sinusoidal and an exponential PDMs in an infinite potential well

Here, we elaborate on the PDM-superstatistical results for the sinusoidal (19) and the exponential (20) PDM functions, in the infinite potential well, with their corresponding energies spectra reported above in (25) and (30), respectively. Then, equation (38) along with the relationships  $\tilde{E}_n = E_{n,a}$ ,  $\tilde{E}_n = E_{n,\alpha}$  (given, respectively, by (25) and (30)) and  $E_n = n^2\pi^2\hbar^2/2m_0L^2$  leads to

$$\exp\left(-\beta_0 \frac{n^2\pi^4\hbar^2}{8m_0L^2} \frac{(a+1)}{\mathcal{E}(\sqrt{-2a})^2}\right) = \int_0^\infty d\beta f_{\sin}(\beta) \exp\left(-\beta \frac{\hbar^2\pi^2n^2}{2m_0L^2}\right), \tag{39}$$

and

$$\exp\left(-\beta_0 \frac{\hbar^2\pi^2n^2}{8m_0L^2} \frac{\alpha(1-e^{-\alpha})}{\left(1-e^{-\frac{\alpha}{2}}\right)^2}\right) = \int_0^\infty d\beta f_{\exp}(\beta) \exp\left(-\beta \frac{\hbar^2\pi^2n^2}{2m_0L^2}\right). \tag{40}$$

Here,  $f_{\sin}(\beta)$  and  $f_{\exp}(\beta)$  are the distributions to be determined for the sinusoidal PDM (19) and the exponential PDM (20), respectively. From (39) and (40), we see that  $f_{\sin}(\beta)$  and  $f_{\exp}(\beta)$  are Dirac delta distributions. More precisely, using the property

$$g(B\beta_0) = \int_0^\infty d\beta \delta(\beta - B\beta_0/A)g(A\beta) \tag{41}$$

with  $g(\beta) = \exp(-\beta)$ ,

$$A = \hbar^2\pi^2n^2/(2m_0L^2),$$

$$B_{\sin} = \frac{n^2\pi^4\hbar^2}{8m_0L^2} \frac{(a+1)}{\mathcal{E}(\sqrt{-2a})^2},$$

and

$$B_{\exp} = \frac{\hbar^2\pi^2n^2}{8m_0L^2} \frac{\alpha(1-e^{-\alpha})}{\left(1-e^{-\frac{\alpha}{2}}\right)^2},$$

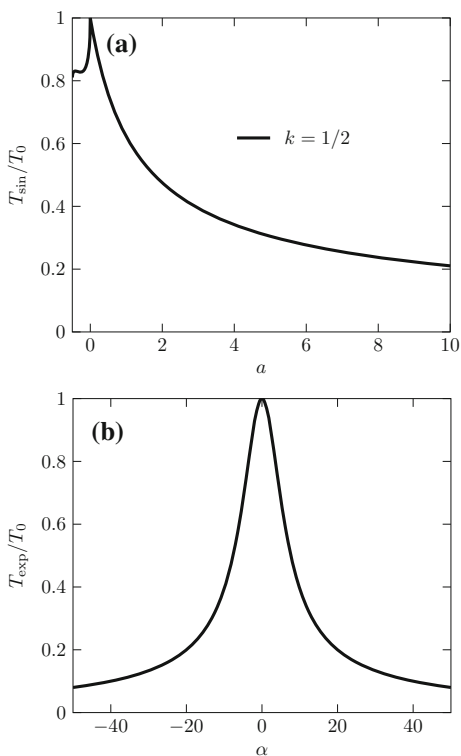
we, respectively, obtain

$$f_{\sin}(\beta) = \delta\left(\beta - \frac{\pi^2(a+1)}{4\mathcal{E}(\sqrt{-2a})^2}\beta_0\right), \tag{42}$$

and

$$f_{\exp}(\beta) = \delta\left(\beta - \frac{\alpha(1-e^{-\alpha})}{4\left(1-e^{-\frac{\alpha}{2}}\right)^2}\beta_0\right). \tag{43}$$

**Fig. 8** Shows the characteristic temperature ratios **a** for the isochronic sinusoidal PDM function, (44), and **b** for the exponential PDM function, (45)



In turn, the relationship  $\beta_0 = 1/(k_B T_0)$  allows us to identify the characteristic temperatures in (42) and (43), respectively, as

$$\frac{T_{\text{sin}}}{T_0} = \frac{4\mathcal{E}(\sqrt{-2a})^2}{\pi^2(a + 1)}, \tag{44}$$

and

$$\frac{T_{\text{exp}}}{T_0} = \frac{4\left(1 - e^{-\frac{\alpha}{2}}\right)^2}{\alpha(1 - e^{-\alpha})} \tag{45}$$

with the suffixes standing for the sinusoidal and the exponential masses, respectively. Moreover, in a straightforward manner, one can show that  $\lim_{a \rightarrow 0} T_{\text{sin}} = T_0$  and  $\lim_{\alpha \rightarrow 0} T_{\text{exp}} = T_0$ , where  $T_0$  represents the temperature for the constant mass case  $m_0$ . Such limits tendencies are also obvious in Fig. 8. Yet, we may very well observe from Fig. 8 that a PDM characteristic temperature inherits a maximum possible value  $T_0$ . Moreover, the PDM parametric effects are observed to lower such characteristic temperatures as the mass parameters increase or decrease far from their zero limits. Equations (44) and (45) express that the effect of a sinusoidal mass or an exponential one can be thermodynamically mimicked by the standard infinite potential well but with an effective temperature  $T_{\text{sin}}$  or  $T_{\text{exp}}$  whose magnitudes depend on the parameters of the corresponding mass distributions. It is worth to note that other mass distributions could give place to beta distributions different from the Dirac delta one.



## 6 Conclusions

In a broad context, we have provided a thorough analysis of non-Hermitian PDM kinetic energy operator ( $\hat{K}$  in Eq. (3)) in the Schrödinger equation framework. We have shown that our proposal satisfies the continuity equation and can be mapped into a standard Schrödinger equation in which a PDM becomes implicitly indulged in a new scaled spatial variable (i.e., transforming Eq. (5) into Eq. (11)). We have shown how this simplified picture of PDM systems connects with other orderings including a Hermitian one. Our approach easily allows to obtain solutions for PDM quantum systems in confining potentials, such as infinite potential well and the quantum harmonic oscillator.

Moreover, we have observed that PDM particles in infinite potential well have disrupted the traditional (i.e., constant mass) energy spectra which only explicitly depends on the principle quantum number into some energy spectra which is of explicit dependence on the principle quantum number as well as the PDM parameters (as documented in Fig. 1b, d, whereas for the PDM-oscillator potential (a nonzero potential unlike the infinite potential well), we have retrieved the exact traditional constant mass energy spectrum. Such results would basically emphasize that a deformation in the coordinate system may very well introduce position-dependent effective mass PDEM [25,26,32,33,42] or in short PDM.

The PDM Schrödinger equation investigated in this work presents two major remarkable features. One of them is that the orderings investigated here allow a connection with that of Mustafa–Mazharimousavi [22], and presumably with others like [25,26], thus showing an equivalence from the probability density point of view. The other feature is also an advantage, such that the solutions can be determined for general classes regardless of the explicit form of the position-dependent mass distributions.

Regarding superstatistics, we were able to characterize the quantum partition functions corresponding to a sinusoidal (19) and an exponential (20) PDM distributions in terms of the superstatistical partition function of the one-dimensional infinite potential well with constant mass. From this characterization, we have obtained Dirac delta distributions for  $f(\beta)$  along with the effective temperatures corresponding to each mass distribution. For the examples studied, we have seen that the effect of a PDM in the one-dimensional infinite potential well implies the standard partition function of the canonical ensemble (for a constant mass particle) but with an effective temperature given in terms of the parameters of the variable mass.

This work opens new perspectives to approach quantum processes with position-dependent mass as well as to study the thermodynamics of their corresponding superstatistical canonical ensemble. For example, once the partition functions are obtained through (36) and (37), one would be able to study some other thermodynamical properties like the Helmholtz free energy, the mean energy, the entropy, the specific heat, etc. [43]. Yet, non-Hermitian  $PT$ -symmetric PDM-Hamiltonians (e.g., [44,45]) are feasible applications. It has been shown that the  $PT$ -symmetry of the non-Hermitian Hamiltonians is a sufficient and necessary condition for the existence of the unitary time evolution [45].

Finally, our proposal offers a simple and practical method to calculate analytical solutions for PDM Schrödinger equation discussed in (5). As an example of implementation in future researches, we mention that the formalism presented could be applied to study unbounded potentials and Gaussian packet of wave function evolving in time.

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