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Review

Solution of radiative transfer equation in finite medium involving Fresnel reflection

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Abstract Radiative transfer problem in an absorbing, scattering and homogeneous finite medium with Rayleigh scattering subjected to externally incident radiation is considered. The solution is evaluated when the case of Fresnel and diffuse reflections is present at the boundary. Galerkin approximation is introduced to calculate some functionals arising in radiative transfer problems such as reflection and transmission coefficients. Numerical results are reported and compared with the available published data which show very good agreement.

1 Introduction

Rayleigh scattering occurs when the incident wavelength is greater than the size of the scattering particles. Lord Rayleigh explained why the sky is blue and sunset is red by using this scattering function. This scattering function can be used in different areas in physics, for example, astrophysics, oceanography and tissue optics.

The first work on the transfer of the polarized radiation has carried out by Gans [1], Chandrasekhar [2], and Sobolev [3]. Pomraning [4, 5] has delivered a variational expression for the albedo for a half-space in the case of isotropic phase function, Rayleigh phase function over polarization and Rayleigh scattering property accounting for polarization.

Radiative transfer equation for Rayleigh scattering was solved for different media using different methods. Bicer and Kaskas [6] solved this equation in infinite medium using Green's function. Degheidy and Abdel-Krim [7] represent the effect of Fresnel and diffuse reflectivities on light transport in half space. Abdel Krim et al. [8] presented a solution of the radiative transfer equation in a plane parallel medium involving Rayleigh scattering phase function with specular and diffuse reflecting boundaries.

A vast amount of work is available in the literature on the solution of radiative transfer equation with Rayleigh scattering and diffuse reflection in finite medium (Bicer and Kaskas [6], Abdel-Krim et al. [8], etc.). But it appears that little work has been reported of the problem of radiation transfer involving Fresnel reflection (Sieweter [9], Degheidy and Abdel-Krim [7] and Williams [10]).

In this work, we solve the time-independent, monoenergetic, radiative transfer equation for Rayleigh scattering at unpolarized case in finite medium for Fresnel case using Galerkin technique.

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2 Formulation of the problem

The unpolarized radiative transfer equation for an absorbing, gray, anisotropic scattering, plane parallel medium of optical thickness (b) can be written as [11]

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{\omega}{2} \int_{-1}^{1} f\left(\mu, \acute{\mu}\right) \psi\left(x, \acute{\mu}\right) \mathrm{d}\acute{\mu} \tag{1}$$

where $-1 < \mu < 1, \quad 0 < x < b$.

With the boundary conditions.

$$\psi(0,\mu) = F_1(\mu) + \rho(\mu, n)\psi(0, -\mu) + \rho^d J^- \tag{2}$$

and

$$\psi(b, -\mu) = F_2(\mu) \tag{3}$$

where

$$J^- = \int_0^1 \mu \psi(0, -\mu) \mathrm{d}\mu \tag{4}$$

Here, $F_1(\mu)$ and $F_2(\mu)$ are the externally incident fluxes on the left and right boundaries, respectively, x is the spatial variable, μ is the direction cosine of the radiation intensity $\Psi(x, \mu)$, ω is the single scattering albedo, and $f(\mu, \mu'$ is the scattering function which can be written for the unpolarized Rayleigh scattering case in the following form [12]

$$f\left(\mu, \acute{\mu}\right) = \frac{3}{8}\left\{\left(3 - \mu^2\right) + \left(3\mu^2 - 1\right)\acute{\mu}\,2\right\} \tag{5}$$

$\rho(\mu, n)$ and ρ^d are the Fresnel and diffuse reflectivities of the boundary, where (n) is the refractive index of the medium. The Fresnel equation is given as

$$\rho(\mu, n) = \left\{\begin{array}{ll} \frac{1}{2}\left[\left(\frac{\mu - n\mu_o}{\mu + n\mu_o}\right)^2 + \left(\frac{\mu_o - n\mu}{\mu_o + n\mu}\right)^2\right], & \mu \geq \mu_c \\ 1, & 0 < \mu < \mu_c \end{array}\right\} \tag{6}$$

where

$$\mu_o^2 = 1 - n^2 + n^2\mu^2 \tag{7}$$

and

$$\mu_c = \sqrt{1 - \frac{1}{n^2}} \tag{8}$$

to solve Eq. (1), first we multiply it by $e^{x/\mu}$ and integrate the resulting equation over x from (0) to (x), we will obtain the expression for the forward radiation intensity ($\psi(x, \mu)$); second, replacing every (μ) in Eq. (1) by ($-\mu$) and multiplying it by $e^{-x/\mu}$ and then integrating the resulting equation over x from (x) to (b), we will obtain the expression for the backward radiation intensity ($\psi(x, -\mu)$). The radiation intensities will have the following forms

$$\psi(x,\mu)=\psi(0,\mu)e^{\frac{-x}{\mu}}+\frac{3\omega}{16\mu}\int_0^x e^{\frac{-\left(x-x'\right)}{\mu}}\left[\left(3-\mu^2\right)\phi\left(x'\right)+\left(3\mu^2-1\right)P\left(x'\right)\right]\mathrm{d}x'$$

$$\mu>0 \tag{9}$$

and

$$\psi(x,-\mu)=\psi(b,-\mu)e^{\frac{-(b-x)}{\mu}}+\frac{3\omega}{16\mu}\int_x^b e^{\frac{\left(x'-x\right)}{\mu}}\left[\left(3-\mu^2\right)\phi\left(x'\right)+\left(3\mu^2-1\right)P\left(x'\right)\right]\mathrm{d}x',$$

$$\mu>0 \tag{10}$$

where $\phi(x)$ and $P(x)$ are the total flux and the total pressure of the incident radiation, which are defined as

$$\phi(x)=\int_{-1}^{1}\psi(x,\mu)\mathrm{d}\mu \tag{11}$$

$$P(x)=\int_{-1}^{1}\mu^2\psi(x,\mu)\mathrm{d}\mu \tag{12}$$

Introducing Eqs. (9) and (10) into Eqs. (11) and (12), respectively, we have

$$\phi(x)=\int_0^1\psi(0,\mu)e^{\frac{-x}{\mu}}\mathrm{d}\mu+\int_0^1\psi(b,-\mu)e^{\frac{-(b-x)}{\mu}}\mathrm{d}\mu$$
$$+\frac{3\omega}{16}\int_0^b[(3E_1(|x-y|)-E_3(|x-y|))\phi(y)+(3E_3(|x-y|)-E_1(|x-y|))P(y)]\mathrm{d}y \tag{13}$$

and

$$P(x)=\int_0^1\mu^2\left\{\psi(0,\mu)e^{\frac{-x}{\mu}}+\psi(b,-\mu)e^{\frac{-(b-x)}{\mu}}\right\}\mathrm{d}\mu$$
$$+\frac{3\omega}{16}\int_0^b[(3E_3(|x-y|)-_5(|x-y|))\phi(y)+(3E_5(|x-y|)-E_3(|x-y|))P(y)]\mathrm{d}y \tag{14}$$

where $E_n(z)$ is the exponential integral function which is defined as

$$E_n(z)=\int_0^1 e^{-\frac{z}{\mu}}\mu^{n-2}\mathrm{d}\mu \tag{15}$$

3 Method of solution

In order to solve for $\phi(x)$ and $P(x)$, we will use a set of trial functions in the form

$$\phi(x) = \sum_{i=0}^{N} A_i x^i \tag{16}$$

and

$$P(x) = \sum_{i=0}^{N} B_i x^i \tag{17}$$

where A_i and B_i are the unknown expansion coefficients.

Substituting Eqs. (16) and (17) into Eqs. (13) and (14) and using the boundary conditions in Eqs. (2) and (3), then multiplying each one of the resulting equations by X^j; ($j = 0, 1, 2, \ldots., N$) and integrating over $x \in [0, b]$, we obtain a system of equations in the following form

$$\begin{aligned}&\sum_{i=1}^{N} A_i \left\{ \frac{b^{i+j+1}}{i+j+1} - \frac{3\omega}{16}\left\{ Y_1(i, j, -1) + \rho^d \text{€}_2(i, j, 2, 4) + U(i, j, 1, 3) \right\}\right\} \\ &- \frac{3\omega}{16} \sum_{i=1}^{N} B_i \left\{ Y_2(i, j, 1) + \rho^d \text{€}_2(i, j, 4, 2) + U(i, j, 3, 1) \right\} \\ &= T_1(j, 0) + T_2(j, 0) + T_3(j, 0) + \rho^d T_4(j, 2)\end{aligned} \tag{18}$$

and

$$\begin{aligned}&- \frac{3\omega}{16} \sum_{i=1}^{N} A_i \left\{ Y_1(i, j, 1) + \rho^d \text{€}_4(i, j, 2, 4) + U(i, j, 3, 5) \right\} \\ &+ \sum_{i=1}^{N} B_i \left\{ \frac{b^{i+j+1}}{i+j+1} - \frac{3\omega}{16}\left\{ Y_2(i, j, 3) + \rho^d \text{€}_4(i, j, 4, 2) + U(i, j, 5, 3) \right\}\right\} \\ &= T_1(j, 2) + T_2(j, 2) + T_3(j, 2) + \rho^d T_4(j, 4)\end{aligned} \tag{19}$$

where

$$Y_1(i, j, p) = \int_0^b x^i \mathrm{d}x \int_0^b y^j \mathrm{d}y \int_0^1 \left(3\mu^p - \mu^{p+2}\right)\rho(\mu, n) e^{-(x+y)/\mu} \mathrm{d}\mu, \tag{20}$$

$$Y_2(i, j, p) = \int_0^b x^i \mathrm{d}x \int_0^b y^j \mathrm{d}y \int_0^1 \left(3\mu^p - \mu^{p-2}\right)\rho(\mu, n) e^{-(x+y)/\mu} \mathrm{d}\mu, \tag{21}$$

$$\text{€}_k(i, j, l, r) = \int_0^b E_k(x) x^j \mathrm{d}x \int_0^b y^i \{3E_l(y) - E_r(y)\} \mathrm{d}y, \tag{22}$$

$$U(i, j, l, r) = \int_0^b x^j \mathrm{d}x \int_0^b y^i \{3E_l(|x-y|) - E_r(|x-y|)\} \mathrm{d}y, \tag{23}$$

$$T_1(i, p) = \int_0^b x^i \mathrm{d}x \int_0^1 \mu^p F_1(\mu) e^{\frac{-x}{\mu}} \mathrm{d}\mu, \tag{24}$$

Table 1 Reflectivity (R) for selected values of thickness (b) and normal incidence of radiation ($k = 0$), $\rho^d = 0$ without Fresnel reflection and $F_2(\mu) = 0$

ω/b	1.0	*	**	2.0	*	3.0	*	**
0.1	0.02101	0.02101	0.021	0.02195	0.02196	0.02201	0.02203	0.022
0.2	0.04442	0.04442	–	0.04668	0.04669	0.04685	0.04689	–
0.3	0.07072	0.07073	0.0707	0.07490	0.07491	0.07526	0.07530	0.0753
0.4	0.10060	0.10060	–	0.10759	0.10760	0.10829	0.10833	–
0.5	0.13496	0.13496	0.1349	0.14621	0.14622	0.14752	0.14757	0.1475
0.6	0.17506	0.17506	–	0.19300	0.19301	0.19549	0.19554	–
0.7	0.22272	0.22272	0.2227	0.25164	0.25165	0.25661	0.25664	0.2566
0.8	0.28063	0.28063	–	0.32873	0.32873	0.33950	0.33952	–
0.9	0.35301	0.35301	0.3530	0.43755	0.43756	0.46455	0.46456	0.4646
0.95	0.39663	0.39663	–	0.51236	0.51236	0.55900	0.55900	–

*Bicer and Kaskas [6]
**Abdel Krim et al. [8]

Table 2 Transmission coefficient (T) for selected values of thickness (b) and normal incidence of radiation ($k = 0$), $\rho^d = 0$ without Fresnel reflection and $F_2(\mu) = 0$

ω/b	1.0	*	**	2.0	*	3.0	*	**
0.1	0.23211	0.23211	0.2321	0.06610	0.06610	0.02016	0.02016	0.0202
0.2	0.24674	0.24674	–	0.07321	0.07321	0.02309	0.02309	–
0.3	0.26373	0.26373	0.2637	0.08206	0.08206	0.02692	0.02692	0.0269
0.4	0.28369	0.28369	–	0.09332	0.09332	0.03210	0.03210	–
0.5	0.30746	0.30746	0.3075	0.10803	0.10803	0.03939	0.03939	0.0394
0.6	0.33623	0.33623	–	0.12793	0.12793	0.05016	0.05015	–
0.7	0.37172	0.37172	0.3717	0.15602	0.15602	0.06714	0.06714	0.0671
0.8	0.41653	0.41653	–	0.19799	0.19799	0.09647	0.09647	–
0.9	0.47473	0.47473	0.4747	0.26586	0.26586	0.15459	0.15459	0.1546
0.95	0.51083	0.51083	–	0.31747	0.31747	0.20863	0.20862	–

*Bicer and Kaskas [6]
**Abdel Krim et al. [8]

$$T_2(i, p) = \int_0^b x^i \mathrm{d}x \int_0^1 \mu^p F_2(\mu) e^{\frac{-(b-x)}{\mu}} \mathrm{d}\mu \tag{25}$$

$$T_3(i, p) = \int_0^b x^i \mathrm{d}x \int_0^1 \mu^p F_2(\mu) \rho(\mu, n) e^{\frac{-(b+x)}{\mu}} \mathrm{d}\mu, \tag{26}$$

and

$$T_4(i, m) = \int_0^b x^i E_m(x) \mathrm{d}x \int_0^1 \mu F_2(\mu) e^{\frac{-b}{\mu}} \mathrm{d}\mu. \tag{27}$$

Table 3 Reflectivity for the medium when $b = 1.0$, $\rho^d = 0$ and selected values of k (without Fresnel reflection), $F_2(\mu) = 0$

ω/k	1	*	2	*	3	*	4	*
0.1	0.01932	0.01932	0.01847	0.01847	0.01797	0.01797	0.01764	0.01764
0.2	0.04085	0.04085	0.03906	0.03906	0.03799	0.03799	0.03730	0.03730
0.3	0.06509	0.06509	0.06222	0.06222	0.06052	0.06052	0.05940	0.05940
0.4	0.09265	0.09265	0.08857	0.08857	0.08613	0.08613	0.08452	0.08452
0.5	0.12440	0.12440	0.11894	0.11894	0.11564	0.11564	0.11346	0.11346
0.6	0.16154	0.16154	0.15445	0.15445	0.15016	0.15015	0.14729	0.14729
0.7	0.20576	0.20576	0.19677	0.19677	0.19128	0.19128	0.18761	0.18761
0.8	0.25963	0.25963	0.24835	0.24835	0.24141	0.24141	0.23674	0.23674
0.9	0.32713	0.32713	0.31303	0.31303	0.30428	0.30428	0.29836	0.29836
0.95	0.36790	0.34566	0.35212	0.32891	0.34229	0.31843	0.33562	0.31129

*Bicer and Kaskas [6]

Table 4 Transmission coefficient for the medium when $b = 1.0$, $\rho^d = 0$ and selected values of k (without Fresnel reflection), $F_2(\mu) = 0$

ω/k	1	*	2	*	3	*	4	*
0.1	0.27104	0.27104	0.29466	0.29466	0.31023	0.31023	0.32116	0.32116
0.2	0.28572	0.28572	0.30929	0.30929	0.32480	0.32480	0.33567	0.33567
0.3	0.30269	0.30269	0.32615	0.32615	0.34154	0.34154	0.35232	0.35232
0.4	0.32253	0.32253	0.34579	0.34579	0.36102	0.36102	0.37166	0.37166
0.5	0.34604	0.34604	0.36901	0.36901	0.38399	0.38399	0.39444	0.39444
0.6	0.37436	0.37436	0.39689	0.39689	0.41153	0.41153	0.42171	0.42171
0.7	0.40913	0.40913	0.43102	0.43102	0.44517	0.44517	0.45499	0.45499
0.8	0.45281	0.45281	0.47379	0.47379	0.48726	0.48726	0.49657	0.49657
0.9	0.50931	0.50931	0.52896	0.52896	0.54146	0.54146	0.55004	0.55004
0.95	0.54424	0.55625	0.56301	0.57544	0.57487	0.58756	0.58297	0.59585

*Bicer and Kaskas [6]

Solving Eqs. (18) and (19), we obtain the unknown coefficients, which are used in calculating some quantities such as hemispherical reflectivity (R) and transmissivity (T) that are defined as

$$R = \frac{\int_0^1 \mu \Psi(0, -\mu) d\mu}{\int_0^1 \mu \Psi(0, \mu) d\mu} \tag{28}$$

and

$$T = \frac{\int_0^1 \mu \Psi(b, \mu) d\mu}{\int_0^1 \mu \Psi(0, \mu) d\mu} \tag{29}$$

Table 5 Reflectivity and transmissivity of the medium for selected values of R^d and thickness (b), $k = 0$, $n = 1.3$ and $F_2(\mu) = 0$

ω/R^d	$b = 1$				$b = 3$			
	0.1		0.5		0.1		0.5	
	T	R	T	R	T	R	T	R
0.1	0.23346	0.02131	0.23346	0.02131	0.0203	0.0223	0.0203	0.0223
0.2	0.24998	0.04577	0.24998	0.04577	0.0235	0.0482	0.0235	0.0482
0.3	0.26966	0.07421	0.26966	0.07420	0.0276	0.0788	0.0276	0.0788
0.4	0.29347	0.10778	0.29347	0.10777	0.0332	0.1158	0.0332	0.1158
0.5	0.32285	0.14816	0.32285	0.14814	0.0413	0.1618	0.0413	0.1618
0.6	0.35998	0.19786	0.35996	0.19783	0.0536	0.2211	0.0536	0.2211
0.7	0.40827	0.26079	0.40825	0.26074	0.0738	0.3018	0.0738	0.3017
0.8	0.47348	0.34350	0.47345	0.34343	0.1106	0.4210	0.1106	0.4209
0.9	0.56606	0.45779	0.56601	0.45768	0.1905	0.6244	0.1905	0.6242
0.93	0.60184	0.50124	0.60178	0.50112	0.2340	0.7197	0.2339	0.7194
0.95	0.62845	0.53334	0.62838	0.53322	0.2727	0.7992	0.2726	0.7990
0.97	0.65762	0.56836	0.65755	0.56823	0.3229	0.8972	0.3228	0.8969
0.99	0.68974	0.60673	0.68966	0.60658	0.3900	1.0218	0.3898	1.0214

Table 6 Reflectivity (R) for $n = 1.3$, $R^d = 0$, $F_2(\mu) = 0$ and selected values of b and k

ω/k	$b = 1.0$			$b = 3.0$		
	1	2	3	1	2	3
0.1	0.01958	0.01872	0.01821	0.02075	0.01999	0.01954
0.2	0.04205	0.04018	0.03907	0.04490	0.04326	0.04229
0.3	0.06818	0.06512	0.06330	0.07352	0.07084	0.06925
0.4	0.09905	0.09457	0.09188	0.10818	0.10430	0.10197
0.5	0.13620	0.13000	0.12626	0.15142	0.14608	0.14286
0.6	0.18196	0.17363	0.16859	0.20745	0.20034	0.19602
0.7	0.23998	0.22896	0.22223	0.28417	0.27484	0.26912
0.8	0.31634	0.30178	0.29283	0.39841	0.38620	0.37862
0.9	0.42203	0.40261	0.39058	0.59519	0.57900	0.56878
0.93	0.46226	0.44100	0.42779	0.68807	0.67034	0.65906
0.95	0.49199	0.46937	0.45531	0.76584	0.74692	0.73484
0.97	0.52444	0.50034	0.48533	0.86187	0.84163	0.82862
0.99	0.56000	0.53429	0.51824	0.98420	0.96244	0.94836

4 Numerical results

In order to illustrate the application of the foregoing analysis, we consider anisotropic intensity of the form $F_1(\mu) = \mu^k$, $k = 0, 1, 2, 3, \ldots$ incident on the boundary.

Tables 1 and 2 show reflectivity (R) and transmissivity (T) for selected values of thickness ($b = 1, 2, 3$), $k = 0$ and $\rho^d = 0$, without Fresnel reflection and $F_2(\mu) = 0$. The results are

Table 7 Transmissivity (T) for $n = 1.3$, $R^d = 0$ and selected values of b and k, $F_2(\mu) = 0$

ω/k	$b = 1.0$			$b = 3.0$		
	1	2	3	1	2	3
0.1	0.27231	0.29590	0.31147	0.02573	0.02972	0.03275
0.2	0.28878	0.31226	0.32772	0.02920	0.03341	0.03661
0.3	0.30824	0.33152	0.34682	0.03375	0.03822	0.04159
0.4	0.33164	0.35457	0.36962	0.03992	0.04469	0.04828
0.5	0.36031	0.38271	0.39737	0.04865	0.05379	0.05763
0.6	0.39628	0.41788	0.43198	0.06173	0.06733	0.07148
0.7	0.44277	0.46316	0.47642	0.08289	0.08907	0.09360
0.8	0.50515	0.52371	0.53570	0.12103	0.12794	0.13293
0.9	0.59318	0.60884	0.61888	0.20263	0.21043	0.21593
0.93	0.62708	0.64156	0.65082	0.24668	0.25482	0.26045
0.95	0.65225	0.66584	0.67450	0.28580	0.29410	0.29980
0.97	0.67982	0.69242	0.70041	0.33640	0.34485	0.35060
0.99	0.71015	0.72163	0.72888	0.40391	0.41246	0.41819

Table 8 Reflectivity and transmissivity for $n = 1.5$, $b = 1$, $R^d = 0$ and different values of k

ω	$k = 1$		$k = 2$		$k = 3$	
	R	T	R	T	R	T
0.1	0.01963	0.27457	0.01876	0.29765	0.01825	0.31298
0.2	0.04227	0.29349	0.04038	0.31594	0.03927	0.33093
0.3	0.06875	0.31569	0.06567	0.33738	0.06383	0.35195
0.4	0.10026	0.34219	0.09573	0.36297	0.09302	0.37702
0.5	0.13852	0.37452	0.13223	0.39418	0.12843	0.40756
0.6	0.18615	0.41504	0.17766	0.43327	0.17250	0.44577
0.7	0.24737	0.46751	0.23606	0.48387	0.22914	0.49517
0.8	0.32943	0.53847	0.31435	0.55225	0.30507	0.56185
0.9	0.44586	0.64014	0.42549	0.65017	0.41285	0.65724
0.93	0.49108	0.68099	0.46867	0.68843	0.45473	0.69449
0.95	0.52484	0.71082	0.50091	0.71706	0.48600	0.72235
0.97	0.56200	0.74374	0.53640	0.74864	0.52043	0.75308
0.99	0.60312	0.78024	0.57569	0.78366	0.55854	0.78714

compared with that given by Bicer and Kaskas [6] and also with that given by Abdel-Krim et al. [8] which show good agreement.

Tables 3 and 4 show reflectivity and transmissivity for selected values of (k), $b = 1.0$ and $\rho^d = 0$, without Fresnel reflection and $F_2(\mu) = 0$. The results are compared with that given by Bicer and Kaskas [6].

Table 5 shows the reflectivity and transmissivity of the medium for selected values of R^d and thickness (b), $k = 0$, $n = 1.3$ and $F_2(\mu) = 0$.

Tables 6 and 7 represent the reflectivity (R) and transmissivity (T), respectively, for $n = 1.3$, $R^d = 0$, $F_2(\mu) = 0$ and selected values of b and k.

In Table 8 we represent the reflectivity (R) and transmissivity (T) for $n = 1.5$, $R^d = 0$, $F_2(\mu) = 0$, $b = 1$ and $k = 1, 2, 3$.

5 Conclusions

The time-independent, monoenergetic radiation transfer equation in plane-parallel medium of anisotropic scattering, and diffuse and Fresnel reflecting boundary, was solved using Galerkin method. The calculations were considered for media of Rayleigh scattering phase function. The reflection and transmission coefficients were calculated for three thicknesses; $b = 1$, 2 and 3; and external radiation intensity of the form $F_1(\mu) = \mu^k$, $k = 0, 1, 2, 3$ and 4. Some of results were compared with that given by Bicer and Kaskas [6] and also with that given by Abdel-Krim et al. [8] which showed good agreement.

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