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Solving procedure for the motion of infinitesimal mass in BiER4BP

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Abstract In this paper, we present a new ansatz for solving equations of motion for the *trapped orbits* of the infinitesimal mass m, which is moving near the primary M_3 in case of *bi-elliptic* restricted problem of *four* bodies (BiER4BP), where three primaries M_1, M_2, M_3 are rotating around their common center of mass on *elliptic* orbits with hierarchical configuration $M_3 \ll M_2 \ll M_1$. A new type of the solving procedure is implemented here to obtain the coordinates $\vec{r} = \{x, y, z\}$ of the infinitesimal mass m with its orbit located near the primary M_3. Meanwhile, the system of equations of motion has been successfully explored with respect to the existence of analytical or semi-analytical (approximated) way for presentation of the solution. We obtain as follows: (1) the solution for coordinate x is described by the key nonlinear ordinary differential equation of *fourth* order at simplifying assumptions, (2) solution for coordinate y is given by the proper analytical expression, depending on coordinate x and true anomaly f, (3) the expression for coordinate z is given by the equation of *Riccati*-type—it means that coordinate z should be *quasi-periodically* oscillating close to the fixed plane $\{x, y, 0\}$.

1 Introduction, BiER4BP (bi-elliptic restricted problem of four bodies)

In the restricted three-body problem (R3BP), the equations of motion describe the dynamics of an infinitesimal mass m under the action of gravitational forces effected by two celestial bodies of giant masses M_1 and M_2 ($M_2 < M_1$), which are rotating around their common center of mass on Keplerian trajectories. The small mass m is supposed to be moving (as first approximation) inside of *restricted* region of space near the planet of mass M_2 [1] (but outside the Roche-lobe's region [2] for this planet). It is worth noting that there is a large number of previous and recent fundamental works concerning analytical development with respect to the R3BP equations which should be mentioned accordingly [1–7]. We should especially emphasize the theory of orbits, which was developed in profound work [3] by Szebehely for

the case of the circular restricted problem of three bodies (CR3BP) (primaries are rotating around their common center of mass on *circular* orbits) as well as the case of the elliptic restricted problem of three bodies [4] (ER3BP, primaries are rotating around barycenter on *elliptic* orbits).

As the next step of the developing the topic under investigation, we should also mention the case of the circle restricted problem of *four* bodies (CR4BP) [5] (where particular solutions were obtained at the simplifying assumption concerning the circle orbits in case of CR4BP), and also the case of *bi-elliptic* restricted problem of *four* bodies (BiER4BP) [6, 7].

According to approach, suggested in [7], we consider in the current research the couple, consisting of two primaries of low masses $\{M_2, M_3\}$ which are rotating around their common center of mass on *elliptic* orbits (with their common center of mass $(M_2 + M_3)$ which is simultaneously rotating on *elliptic* orbit around the central giant planet or the Sun M_1 on the same plane with the couple of two primaries $\{M_2, M_3\} \ll M_1$); additionally we mean that infinitesimal mass m is not far from the primary with mass M_3, and $M_3 \ll M_2$.

So, the motions of the primaries are coplanar (but those of infinitesimal mass is not). We assume also that distance a_2 between $\{M_2, M_3\}$ much less than distance a_1 between barycenter of two primaries $\{M_2, M_3\}$ and the central giant planet or the Sun M_1

$$a_2 \ll a_1 \tag{*}$$

As for the natural restriction (of physical nature), we also mean that infinitesimal mass m is supposed to be moving outside the *double* Roche's limit [2] for the primary with mass M_3 (as first approximation, not less than 7–10 R_3 where R_3 is the radius of the primary with mass M_3).

2 Equations of motion of BiER4BP

A non-uniformly rotating coordinate system is defined to describe the motion of the fourth particle in the elliptic *planar* restricted four-body problem [7]:

- the origin O is the center of mass of all the primaries;
- z axis is perpendicular to the plane of the mutual rotation of all the primaries and directed as the angular momentum of primaries' orbital motion;
- x axis is pointing from barycenter of three primaries $\{M_1, M_2, M_3\}$ to barycenter of primaries $\{M_2, M_3\}$ (in [7] it was formulated as follows: "x axis points from primary M_1 to barycenter of primaries $\{M_2, M_3\}$"; both assumptions coincide if $\{M_2, M_3\} \ll M_1$);
- and y axis forms a right triad with x and z axes.

The dynamical equations of motion of the fourth particle in the non-uniformly rotating coordinate system can be written as in [5, 7] (we should note that Eq. (3) in [5] was obtained without assumption of finite bodies or primaries moving in circles with respect one to another with constant uniform rotation; so, all the derivation and solving procedure up to the Eq. (3) and Eq. (3) themselves coincide completely to those referred to Eq. (1) in work [7])

$$\ddot{X} - 2\dot{f}\dot{Y} - (\dot{f})^2 X - \ddot{f}Y = \frac{\partial U}{\partial X},$$
$$\ddot{Y} + 2\dot{f}\dot{X} - (\dot{f})^2 Y + \ddot{f}X = \frac{\partial U}{\partial Y},$$
$$\ddot{Z} = \frac{\partial U}{\partial Z}, \tag{1}$$

where U is the potential function and f denotes the true anomaly tracing the orbit of the barycenter of primaries $\{M_2, M_3\}$ around the main barycenter of three primaries $\{M_1, M_2, M_3\}$; let us note that for the reason that we have chosen an approximation $\{M_2, M_3\} \ll M_1$, the barycenter of three primaries $\{M_1, M_2, M_3\}$ is to be very close to the center of mass of central giant body M_1.

It is interesting to note that in work [6] the dynamical equations of motion of the fourth particle (see Eqs. (7)–(8) in [6]) were also considered in the non-uniformly rotating coordinate system, but centered at the barycenter of primaries $\{M_2, M_3\}$ and depending on two types of true anomalies refererred to different processes of tracing the orbits (one of the true anomalies is associated with tracing an elliptic motion of the barycenter of primaries $\{M_2, M_3\}$ around the main barycenter of three primaries $\{M_1, M_2, M_3\}$, another is referred to the rotating of primaries M_2, M_3 around their common center of mass $\{M_2, M_3\}$). Nevertheless, we will follow in the current research by the approach, suggested in work [7] (and applied here for investigations of the equations of motion of massless particle, moving under the action of forces considered in the plane variant of BiER4BP).

The expression [7] for U is given by

$$U = G \cdot \sum_{i=1}^{3} \left(\frac{M_i}{R_i} \right), \tag{2}$$
$$R_i = \sqrt{(X - X_i)^2 + (Y - Y_i)^2 + (Z - Z_i)^2},$$

where G is the Gaussian constant of gravitation; R_i are distances of the infinitesimal mass m from each of the primaries M_i (i=1, 2, 3), respectively ($\{X_i, Y_i, Z_i\}$ are the coordinates of primaries M_i).

Unlike the CR4BP [5], the positions of the primaries are not fixed in the rotating frame (in case of the elliptic restricted problem)—as they move along elliptical orbits, their relative distance ρ is not constant during a time

$$\rho = \frac{a_1 \cdot (1 - e_1^2)}{1 + e_1 \cdot \cos f} \tag{3}$$

where e_1 is the eccentricity of *elliptic* orbit of the rotating barycenter of primaries $\{M_2, M_3\}$ around their common center of mass with primary M_1. By setting the scaling of mass, distance and time in such a way as in [7] that

$$\begin{cases} [M] = M_1 + M_2 + M_3 = M \\ [L] = \rho \\ [T] = [\rho^3/(G \cdot M)]^{\frac{1}{2}} = \sqrt{1 + e_1 \cdot \cos f}/\dot{f} \end{cases} \tag{4}$$

we introduce by the transformation of the previous, non-uniformly rotating coordinate system $\{X, Y, Z\}$ [used in (1)], the *pulsating* coordinate system $\{x, y, z\}$ with new scaling; so, we have with the help of (4)

$$\begin{cases} X = \rho \cdot x \\ Y = \rho \cdot y \\ Z = \rho \cdot z \end{cases} \tag{5}$$

Then, Eq. (1) become

$$x'' - 2y' = \frac{1}{1 + e_1 \cdot \cos f} \frac{\partial \Omega}{\partial x},$$
$$y'' + 2x' = \frac{1}{1 + e_1 \cdot \cos f} \frac{\partial \Omega}{\partial y},$$

$$z'' = \frac{1}{1 + e_1 \cdot \cos f} \frac{\partial \Omega}{\partial z} - z, \tag{6}$$

where symbol $'$ indicates (d/df) in (6), Ω is the scalar function

$$\Omega = \frac{1}{2}(x^2 + y^2 + z^2) + U_r$$
$$U_r = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{\mu_3}{r_3}, \tag{7}$$

and $\mu i = (M_i/M)$ $(i = 1, 2, 3)$, r_i is the dimensionless distance between the infinitesimal mass m and ith primary, respectively [7] (in [7] m_1 should be associated with M_3 in the denotations here, vice versa m_3 instead of M_1).

It is important to clarify appropriately the expressions for r_i in (7) which should be used in (6) for the further calculations [taking into account expressions (2) and transformation of coordinates (4)–(5)]. According to [7], they are given by

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$
$$\begin{aligned}
x_1 &= -(\mu_3 + \mu_2), & y_1 &= 0, & z_1 &= 0, \\
x_2 &= \mu_1 + \frac{\mu_3}{\mu_3 + \mu_2} r \cos\theta, & y_2 &= \frac{\mu_3}{\mu_3 + \mu_2} r \sin\theta, & z_2 &= 0, \\
x_3 &= \mu_1 - \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta, & y_3 &= -\frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta, & z_3 &= 0,
\end{aligned} \tag{8}$$

where r denotes the distance between primaries M_2, M_3 whereas θ denotes the angle between radius-vector from barycenter of primaries $\{M_2, M_3\}$ to the primary M_3 and the Ox axis. The expression for r can be written as [taking into account (3)]:

$$r = \left(\frac{1}{\rho}\right) \frac{a_2 \cdot (1 - e_2^2)}{1 + e_2 \cdot \cos f_2} \tag{9}$$

where a_2 is the semi-major axis of binary system $\{M_2, M_3\}$, $M_3 \ll M_2$; e_2 is the eccentricity of *elliptic* orbit of the rotating primary M_3 around the barycenter of primaries $\{M_2, M_3\}$. The parameter θ is determined as in work [7] by

$$\theta = \theta_0 + f_2 - f \Rightarrow f_2 = (\theta - \theta_0) + f \tag{10}$$

where θ_0 is the initial value of θ. Meanwhile, from (9)–(10) we obtain

$$r = \left(\frac{1 + e_1 \cdot \cos f}{a_1 \cdot (1 - e_1^2)}\right) \frac{a_2 \cdot (1 - e_2^2)}{(1 + e_2 \cdot \cos((\theta - \theta_0) + f))} \tag{11}$$

Let us note also that in the current research we neglect the effect of variable masses of the primaries [8]. As for the domain where the aforesaid infinitesimal mass m is supposed to be moving, let us consider the Cauchy problem in the whole space. Finally, it is worth noting that both spatial ER3BP and ER4BP are not conservative, and no complete integrals of motion are known [7, 9].

By appropriately transforming the right parts with regard to partial derivatives with respect to the proper coordinates $\{x, y, z\}$, system (6) can be represented as [taking into account (7)–(11)]

$$x'' - 2y' = \frac{1}{1 + e_1 \cdot \cos f}\left(x - \frac{\mu_1}{r_1^3}(x - x_1) - \frac{\mu_2}{r_2^3}(x - x_2) - \frac{\mu_3}{r_3^3}(x - x_3)\right),$$

$$y'' + 2x' = \frac{1}{1 + e_1 \cdot \cos f}\left(y - \frac{\mu_1}{r_1^3}(y - y_1) - \frac{\mu_2}{r_2^3}(y - y_2) - \frac{\mu_3}{r_3^3}(y - y_3)\right),$$

$$z'' = \frac{1}{1 + e_1 \cdot \cos f} \cdot \left[z - \frac{\mu_1}{r_1^3}(z - z_1) - \frac{\mu_2}{r_2^3}(z - z_2) - \frac{\mu_3}{r_3^3}(z - z_3) \right] - z, \qquad (12)$$

where

$$r_1 = \sqrt{(x + \mu_3 + \mu_2)^2 + y^2 + z^2},$$

$$r_2 = \sqrt{\left(x - \mu_1 - \frac{\mu_3}{\mu_3 + \mu_2} r \cos\theta \right)^2 + \left(y - \frac{\mu_3}{\mu_3 + \mu_2} r \sin\theta \right)^2 + z^2},$$

$$r_3 = \sqrt{\left(x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta \right)^2 + \left(y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta \right)^2 + z^2}. \qquad (13)$$

3 Solving procedure for the system of Eq. (12)

According to the aforesaid assumption, infinitesimal mass m is orbiting near the primary M_3; it means that we can assume $r_3/r_2 \ll 1$, $r_3/r_1 \ll 1$ in Eq. (12).

The aforesaid assumptions should simplify equations of system (12) accordingly, taking into account (13):

$$\begin{cases} x'' - 2y' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(x - \dfrac{\mu_3 \left(x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta \right)}{\left((x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta)^2 + (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)^2 + z^2 \right)^{\frac{3}{2}}} \right), \\[3ex] y'' + 2x' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(y - \dfrac{\mu_3 (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)}{\left((x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta)^2 + (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)^2 + z^2 \right)^{\frac{3}{2}}} \right), \\[3ex] z'' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \cdot \left(z - \dfrac{\mu_3 z}{\left((x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta)^2 + (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)^2 + z^2 \right)^{\frac{3}{2}}} \right) - z. \end{cases}$$

$$(14)$$

We should take into account at transforming of (14) that we have assumed previously for the mass-parameters of primaries $\mu_3 \ll \mu_2 \implies (\mu_3 \cdot \mu_2)/(\mu_3 + \mu_2) \cong \mu_3 \to 0$ (with additional assumption (*) which means $r \to 0$).

In this case, we obtain by the linear combining the second and third equations of (14)

$$z'' + z \cdot \left(1 - \left(\frac{y'' + 2x'}{y} \right) \right) \cong 0 \qquad (15)$$

where Eq. (15), which describes the dynamics of component z of the solution depending on the true anomaly f (for known components $\{x, y\}$), is known to be the ordinary differential equation of the *Riccati*-type [10].

Such the distinguished fact let us suggest that coordinate z belongs to the solutions $\vec{r} = \{x, y, z\}$ of system (14), associated with the class of *trapped motions* of the infinitesimal mass m (for which restriction $z \ll 1$ is valid; it means that infinitesimal mass m is moving or oscillating near the plane $\{x, y\}$, $z \to 0$).

In this case, we can neglect by all the terms of order z^2 and less in Eq. (14); the same assumption is valid for the other terms of second order and less (e.g., it concerns the factors

$\mu_3 \to 0$, $r \to 0$ as well). Futhermore, we obtain directly from assumption $r_3/r_2 \ll 1$ as shown below:

$$\sqrt{\left(x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2}r\cos\theta\right)^2 + \left(y + \frac{\mu_2}{\mu_3 + \mu_2}r\sin\theta\right)^2 + z^2}$$

$$\ll \sqrt{\left(x - \mu_1 - \frac{\mu_3}{\mu_3 + \mu_2}r\cos\theta\right)^2 + \left(y - \frac{\mu_3}{\mu_3 + \mu_2}r\sin\theta\right)^2 + z^2},$$

$$\Rightarrow 2((x - \mu_1)\cos\theta + y\sin\theta) << \left(\frac{\mu_3 - \mu_2}{\mu_2 + \mu_3}\right)r \cong -r < 0,$$

$$\Rightarrow (x - \mu_1)\cos\theta << -y\sin\theta \tag{16}$$

So, we have obtained 6 subclasses for the variety of partial solutions of (14), presented by (16):

$$\cos\theta = 0, \quad \theta = \pi(4n+1)/2, \quad n \in (0, 1, 2, \ldots) \to \sin\theta = 1 \Rightarrow y \ll -1 \tag{16.1}$$

$$\cos\theta = 0, \quad \theta = \pi(6n+1)/2, \quad n \in N \to \sin\theta = 1 \Rightarrow y \ll -1 \tag{16.2}$$

$$\sin\theta = 0, \quad \theta = 2\pi n, \quad n \in (0, 1, 2, \ldots) \to \cos\theta = 1 \Rightarrow (x - \mu_1) \ll -1 \tag{16.3}$$

$$\sin\theta = 0, \quad \theta = \pi(2n+1), \quad n \in (0, 1, \ldots) \to \cos\theta = -1 \Rightarrow (x - \mu_1) \gg -1 \tag{16.4}$$

$$\cos\theta > 0, \quad \theta \in (2\pi n, \pi(2n+1)/2) \cup (\pi(2n+3)/2, 2\pi n),$$
$$n \in (0, 1, 2, \ldots)(x - \mu_1) \ll -y\tan\theta \tag{16.5}$$

$$\cos\theta < 0, \quad \theta \in (\pi(2n+1)/2, \ \pi(2n+3)/2), \quad n \in (0, 1, 2, \ldots) \Rightarrow (x - \mu_1) \gg -y\tan\theta \tag{16.6}$$

where, expressions $(x - \mu_1)$ and $(y\tan\theta)$ should have different signs in (16.5) and, in opposite, the same signs in (16.6) for validity of assumption $r_3/r_2 \ll 1$(besides, $|x| > \mu_1$). Meanwhile, inequalities (16.5)–(16.6) are the most realistic cases from the aforementioned subclasses of solutions (16.1)–(16.6) [indeed, in (16.1)–(16.2) we have $|y| \gg 1$, whereas (16.3)–(16.4) holds inequality $|x - \mu_1| \gg 1$, both cases are not physically reasonable].

Thus, we obtain in case of $(x - \mu_1) \gg -y\tan\theta$ (16.6) from the first of Eq. (14) (taking into account also that $z^2 \to 0$, $\{\mu_2, \mu_3\} \ll 1$, $r \to 0$; $\theta \neq \pi(2n+1)/2$, $n \in (0, 1, \ldots)$):

$$x'' - 2y' \cong \frac{1}{1 + e_1 \cdot \cos f}\left(x - \frac{\mu_3(x - \mu_1)}{\left(\frac{(x-\mu_1)^2}{\tan^2\theta}\left[\tan^2\theta + \frac{\tan^2\theta \cdot y^2}{(x-\mu_1)^2}\right]\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow y' \cong \frac{x''}{2} - \frac{1}{2(1 + e_1 \cdot \cos f)}\left(x - \frac{\mu_3}{(x - \mu_1)^2}\right), \tag{17}$$

but from the second of Eq. (14) we should obtain in case of (16.6), taking into account additionally expression (17) $\{|x| > \mu_1\}$:

$$y'' + 2x' \cong \frac{1}{1 + e_1 \cdot \cos f}\left(y - \frac{\mu_3 y}{(x - \mu_1)^3\left(\frac{1}{\tan^2\theta}\left[\tan^2\theta + \frac{\tan^2\theta \cdot y^2}{(x-\mu_1)^2}\right]\right)^{\frac{3}{2}}}\right),$$

$$\Rightarrow \frac{d\left(\frac{(y''+2x')\cdot(1+e_1\cdot\cos f)}{1-\frac{\mu_3}{(x-\mu_1)^3}}\right)}{d f} \cong y' \cong \frac{x''}{2} - \frac{1}{2(1+e_1\cdot\cos f)}\left(x - \frac{\mu_3}{(x-\mu_1)^2}\right),$$

$$\left\{y' \cong \frac{x''}{2} - \frac{1}{2(1+e_1\cdot\cos f)}\left(x - \frac{\mu_3}{(x-\mu_1)^2}\right)\right\} \tag{18}$$

Equation (18) above is the nonlinear differential equation of 4-th order in regard to the coordinate x (with respect to the true anomaly f), which can obviously be solved by numerical methods only.

4 Final presentation of the solution

Let us present the solution $\vec{r} = \{x, y, z\}$ of system (14) for the *trapped motion* of the infinitesimal mass m, which is moving near the primary M_3 [we suppose that restriction $r_3 / r_2 \ll 1$ is valid for Eq. (14)]:

- If $\cos\theta < 0$, $\theta \in (\pi(2n + 1)/2, \pi(2n + 3)/2)$, $n \in (0, 1, 2, \ldots)$, the restriction $(x - \mu_1) \gg -y\tan\theta$ should be valid for the solutions (where, expressions $(x - \mu_1)$ and $(y\tan\theta)$ should have the same signs for validity of assumption $r_3/r_2 \ll 1$, here $|x| > \mu_1$); besides, the solution for coordinate x is described by the key nonlinear ordinary differential equation of *fourth* order (18)

$$\frac{d\left(\frac{(y''+2x')\cdot(1+e_1\cdot\cos f)}{1-\frac{\mu_3}{(x-\mu_1)^3}}\right)}{d f} \cong \frac{x''}{2} - \frac{1}{2(1+e_1\cdot\cos f)}\left(x - \frac{\mu_3}{(x-\mu_1)^2}\right),$$

$$\left\{y' \cong \frac{x''}{2} - \frac{1}{2(1+e_1\cdot\cos f)}\left(x - \frac{\mu_3}{(x-\mu_1)^2}\right)\right\}$$

- But solution for coordinate y is described by the expression which stems from Eq. (17)

$$y \cong \frac{x'}{2} - \frac{1}{2}\int\left(\frac{1}{(1+e_1\cdot\cos f)}\left(x - \frac{\mu_3}{(x-\mu_1)^2}\right)\right)d f$$

- The expression for coordinate z is given in (15) by equation of *Riccati*-type:

$$z'' + z\cdot\left(1 - \left(\frac{y''+2x'}{y}\right)\right) \cong 0.$$

Meanwhile, all the coordinates $\{x, y, z\}$ should be calculated under the optimizing *natural* restriction (of physical nature) so that the infinitesimal mass m is moving outside the *double* Roche's limit [2] for the primary M_3 (not less than 7–10 R_3 where R_3 is the radius of the primary).

5 Discussion

As we can see from the derivation above, equations of motion (1) even for the case of *trapped motion* $\vec{r} = \{x, y, z\}$ [described by appropriate approximation of equations of system (14)] of the infinitesimal mass m, which is moving near the primary M_3, are proved to be very hard to solve analytically.

Nevertheless, at first step we have succeeded in obtaining from third equation of system (14) the Eq. (15) of a *Riccati*-type for the coordinate z (if the solutions for coordinates $\{x, y\}$ are already obtained)—it means that coordinate z should be *quasi-periodically* oscillating (close to the plane $\{x, y, 0\}$). Then we suggest a kind of reduction in a form (17)–(18) for the first and second equations of system (14), which describes the dynamics of coordinates $\{x, y\}$.

Meanwhile, we should note that due to the special character of the solutions of *Riccati*-type ODEs, there is the possibility for sudden *jumping* in the magnitude of the solution [11, 12] (in [12] see e.g. the solutions of *Riccati*-type for *Abel* ordinary differential equation of 1-st order). It means that any solution, which could be in principle derived from the equations of motion (1) in the form (14), are obviously not to be stable during a long the rotation of infinitesimal mass m around the primary M_3 (including solutions for the libration points [6], [13–15]).

At the second step we have obtained the *approximated* solutions (17)–(18) of Eq. (14) for coordinates $\{x, y\}$ under simplifying assumption (16.6), where $|x| > \mu_1$. Hereafter we have considered case with assumptions $z^2 \to 0, \{\mu_2, \mu_3\} \ll 1, r \to 0; \theta \neq \pi(2n+1)/2, n \in (0, 1, \ldots)$ (for the sake of simplicity). The key *nonlinear* ODE of fourth-order in regard to the coordinate $x(f)$ is obtained to be presented in (18). The appropriate solutions under simplifying assumption (16.5) can be obtained in the same manner.

Ending discussion, let us list below the obvious assumptions (or simplifications) which have been used at the formulation or the solution of the problem:

- we consider in the current research two primaries of low masses $\{M_2, M_3\}$ which are rotating on *elliptic* orbits (around barycenter) with their barycenter rotating on *elliptic* orbit around the third, giant primary M_1;
- the motions of the primaries are coplanar (but those of infinitesimal mass is not);
- we consider the hierarchical configuration of primaries $M_1 \gg M_2 \gg M_3$;
- all the coordinates $\{x, y, z\}$ should be calculated under the *natural* restriction (of physical nature) so that the infinitesimal mass m is moving outside the *double* Roche's limit [2] for the primary M_3 (not less than 7–10 R_3 where R_3 is the radius of the primary). If we obtain during the solving procedure the solutions closer to the primary M_3 than the *double* Roche's limit, calculations should be stopped immediately at this forbidden zone;
- distance a_2 between $\{M_2, M_3\}$ much less than distance a_1 between barycenter of two primaries $\{M_2, M_3\}$ and the giant primary M_1 ($a_2 \ll a_1$);
- small mass m is supposed to be moving inside the *restricted* region of space or the so-called *Hill sphere* [2] of primary M_3 with radius (the maximum distance from M_3): $r_H = a_1 \cdot \left(\frac{M_2}{3M_1}\right)^{\frac{1}{3}}$;
- moreover, infinitesimal mass m is orbiting near the primary M_3 [in this case, we can assume $r_3/r_2 \ll 1, \ r_3/r_1 \ll 1$ in Eq. (12)];
- coordinate z belongs to the class of *trapped motions* (i.e., infinitesimal mass m is assumed to be moving or oscillating near the plane $\{x, y\}, \ z \to 0$);
- in the current research we neglect the effect of variable masses of the primaries.

The last but not least, we should note that we can see from the aforementioned list of simplifications that the real conditions for orbiting of infinitesimal mass m near the primary M_3 in BiER4BP are definitely very far from such the ideal formulation of the *bi-elliptic* restricted problem of *four* bodies (albeit the mathematical formulation relies heavily on simplifications based on such assumptions).
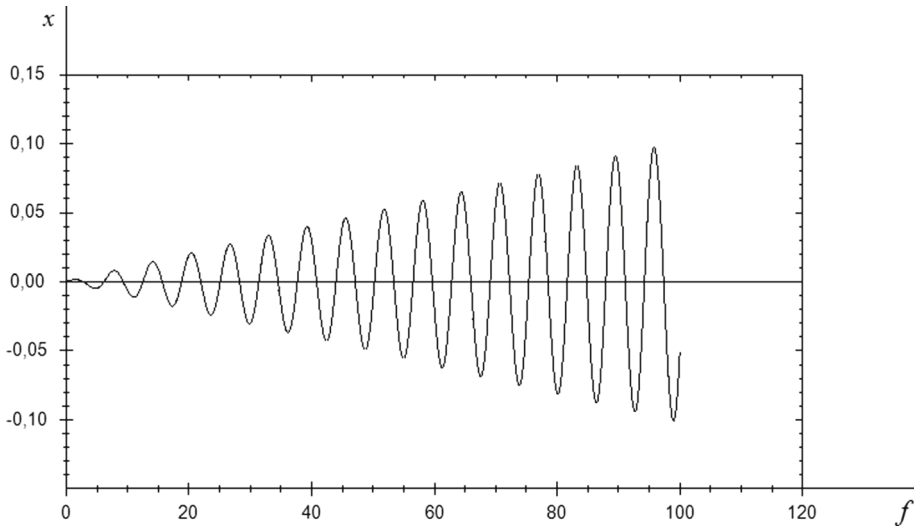
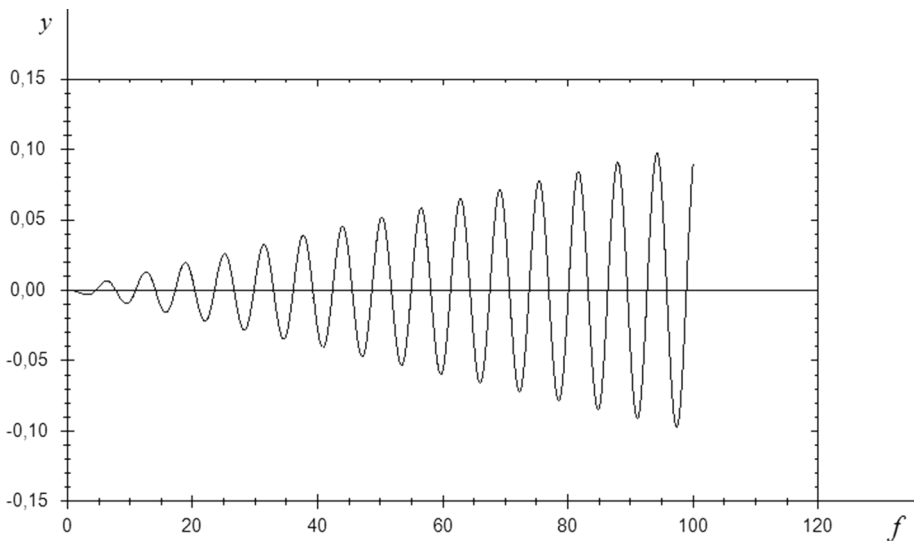**Fig. 1** Results of numerical calculations of the coordinate x

Fig. 2 Results of numerical calculations of the coordinate y

We provide the numerical calculation of the approximated solutions of first and second equations of system (14) under appropriate simplifying assumptions (see "Appendix A1"). We should note that we have used for calculating the data the Runge–Kutta fourth-order method with step 0.0001 starting from initial values. All the results of numerical calculations we schematically imagine at Figs. 1, 2 and 3.
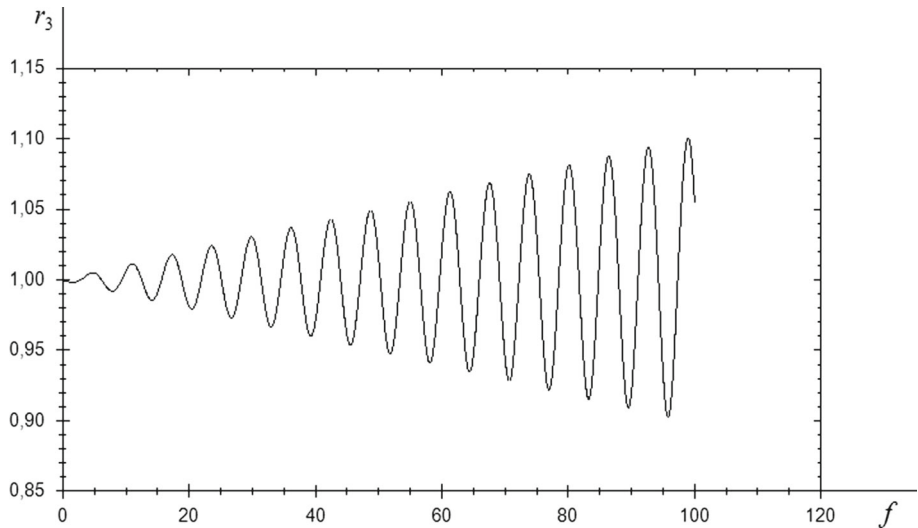
Fig. 3 Results of numerical calculations of r_3

6 Conclusion

In this paper, we present a new ansatz for solving equations of motion for the *trapped orbits* of the infinitesimal mass m, which is moving near the primary M_3 in case of *bi-elliptic* restricted problem of *four* bodies (BiER4BP), where three primaries M_1, M_2, M_3 are rotating around their common center of mass on *elliptic* orbits with hierarchical configuration $M_3 \ll M_2 \ll M_1$.

A new type of the solving procedure is implemented here to obtain the coordinates $\vec{r} = \{x, y, z\}$ of the infinitesimal mass m with its orbit located near the primary M_3. Meanwhile, the system of equations of motion has been successfully explored with respect to the existence of analytical or semi-analytical (approximated) way for presentation of the solution.

We obtain as follows: (1) the solution for coordinate x is described by the key nonlinear ordinary differential equation of *fourth* order at simplifying assumptions, (2) solution for coordinate y is given by the proper analytical expression, depending on coordinate x and true anomaly f, (3) the expression for coordinate z is given by the equation of *Riccati*-type—it means that coordinate z should be *quasi-periodically* oscillating close to the fixed plane $\{x, y, 0\}$.

We have pointed out the optimizing procedure for the solutions $\vec{r} = \{x, y, z\}$ (the distance of the infinitesimal mass m from the primary M_3 should exceed the level of minimal distances outside the *double* Roche-limit for the chosen primary).

The suggested approach can be used in future researches for optimizing the maneuvers of spacecraft (in space) which is moving near the smallest body M_3 belonging to the triplet of massive objects which are rotating around their common center of mass on *elliptic* orbits with hierarchical configuration $M_3 \ll M_2 \ll M_1$.

Also, some remarkable articles should be cited, which concern the problem under consideration, [16–19].

Dr. Nikolay Emelyanov for schematical graphical plot, presented at Fig. 4 (which was used previously in our working discussion regarding schematical solutions in R2BP).

Author contribution In this research, Dr. Sergey Ershkov is responsible for the general ansatz and the solving procedure, simple algebra manipulations, calculations, results of the article in Sects. 1–4 and also is responsible for the search of approximated solutions. Dr. Dmytro Leshchenko is responsible for theoretical investigations as well as for the deep survey in literature on the problem under consideration. Dr. Alla Rachinskaya is responsible for approximated solving the *nonlinear* ordinary differential equations by means of advanced numerical methods (see "Appendix A1") as well as is responsible for numerical data of calculations and graphical plots of numerical solutions. All the authors agreed with results and conclusions of each other in Sects. 1–6.

Compliance with ethical standards

Conflict of interest Authors declare that there is no conflict of interests regarding publication of article.

Appendix A1 (results of numerical calculations)

Let us provide the numerical calculation of the approximated solutions of first and second equations of system (14) (we consider approximation $z^2 \to 0$)

$$
\begin{cases}
x'' - 2y' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(x - \dfrac{\mu_3 \left(x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta \right)}{\left((x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta)^2 + (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)^2 + z^2 \right)^{\frac{3}{2}}} \right), \\[4mm]
y'' + 2x' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(y - \dfrac{\mu_3 \left(y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta \right)}{\left((x - \mu_1 + \frac{\mu_2}{\mu_3 + \mu_2} r \cos\theta)^2 + (y + \frac{\mu_2}{\mu_3 + \mu_2} r \sin\theta)^2 + z^2 \right)^{\frac{3}{2}}} \right),
\end{cases}
$$

which under appropriate simplifying assumptions such as hierarchical configuration $M_3 \ll M_2 \ll M_1$ and (*) [it means that $r \to 0$ in expression (11)] can be reduced as below

$$
\begin{cases}
x'' - 2y' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(x - \dfrac{\mu_3 (x - \mu_1)}{\left((x - \mu_1)^2 + y^2 \right)^{\frac{3}{2}}} \right), \\[4mm]
y'' + 2x' \cong \dfrac{1}{1 + e_1 \cdot \cos f} \left(y - \dfrac{\mu_3 y}{\left((x - \mu_1)^2 + y^2 \right)^{\frac{3}{2}}} \right).
\end{cases}
$$

We should note that we have used for calculating the data the Runge–Kutta fourth-order method with step 0.0001 starting from initial values. We have chosen for our calculations (for modeling the triple system "Sun-Earth-Moon" $\{M_1, M_2, M_3\}$) as follows:

$$ e_1 = 0.0167, \quad \mu_2 \cong (1/332946), \quad \mu_3 \cong (1/328901) - \mu_2, \quad \mu_1 = 1 - (\mu_2 + \mu_3). $$

As for the initial data, we have chosen as follows: (1) $x_0 = 0.001$, $(x')_0 = 0$; (2) $y_0 = 0.0001$, $(y')_0 = 0$; (3) $z_0 = 0$, $(z')_0 = 0$. All the results of numerical calculations we schematically imagine at Figs. 1, 2 and 3.

Meanwhile, the numerical approximation for the dynamics of the infinitesimal mass m in this case (see Figs. 1, 2, 3) means that test particle is moving near barycenter of the system
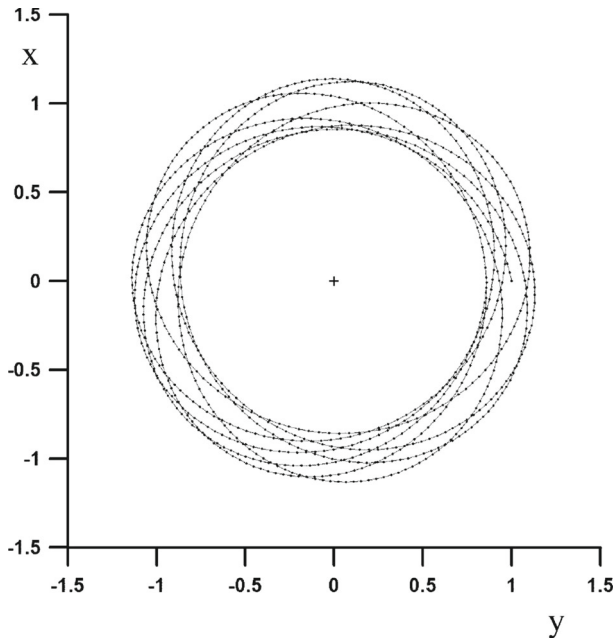
Fig. 4 Schematically imagined trajectories of motion of infinitesimal particle near the Moon M_3 for the Sun–Earth–Moon system $\{M_1, M_2, M_3\}$ in BiER4BP

"Sun-Earth-Moon", but the dynamics for the components of the solution is unstable. As for the example of the admissible ranges of the restrictions for the distance the infinitesimal mass m from Moon M_3 for the Sun-Earth-Moon system $\{M_1, M_2, M_3\}$ we provide it as follows:

- the *double* Roche's limit [2] for the Moon M_3 (not less than 7–10 R_3 where $R_3 = 1731$ km is the radius of the primary; 1 AU $= 1.496 \cdot 10^8$ km) is at maximum circa $(10 \cdot 1731)/(1.496 \cdot 10^8) \cong 1.16 \cdot 10^{-4}$;

- the *Hill sphere* of the Earth M_2 is circa 0.101 AU, according to the formula $r_H = a_1 \cdot \left(\frac{M_2}{3M_1}\right)^{\frac{1}{3}}$ in [2] (it seems to be obvious fact that the *Hill sphere* of the Earth M_2 includes the *Hill sphere* of the Moon M_3). Meanwhile, it should mean that test particle reaches the admissible range of the restriction circa at value of true anomaly $f \cong 100$ (Fig. 3) or circa 16 full turns of the orbit around the Moon M_3 starting from initial point.

Finally, we have schematically imagined trajectories of motion of infinitesimal particle near the Moon M_3 for the Sun-Earth-Moon system $\{M_1, M_2, M_3\}$ in BiER4BP.

Let us additionally discuss and comment Figs. 1, 2, 3 and 4 (with inclusion of a more detailed analysis of these graphical plots) with the main aim to clarify what they are displaying and how they support the formulation presented:

- Figure 1 presents the results of numerical calculations of the coordinate x. Namely, we can see from Fig. 1 that coordinate x is oscillating in a narrow range $[-0.1; 0.1]$ in the *pulsating* coordinate system $\{x, y, z\}$ (with new scaling) during a long period of changing the true anomaly f, up to the value $f = 100$ (which corresponds circa to the 16 full turns of orbit's for the rotating barycenter of the primaries $\{M_2, M_3\}$ around their common center of mass with primary M_1);

- Figure 2 presents the results of numerical calculations of the coordinate y. The same as displayed above on Fig. 1, we can see that coordinate y is oscillating in a narrow range [−0.1; 0.1] in the *pulsating* coordinate system $\{x, y, z\}$, also during a long period of changing the true anomaly up to value $f = 100$;
- Figure 3 presents the results of numerical calculations of r_3 (besides, it displays the fact that infinitesimal mass m is indeed orbiting near the primary M_3), where r_3 is oscillating in a narrow range [0.9; 1.1] in the *pulsating* coordinate system $\{x, y, z\}$ (with new scaling), during a long period of changing the true anomaly up to value $f = 100$ as well;
- Figure 4 schematically displays the possible trajectories of motion of infinitesimal particle near the Moon M_3 for the Sun-Earth-Moon system $\{M_1, M_2, M_3\}$ in the *pulsating* coordinate system $\{x, y, z\}$ in BiER4BP.

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