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Stability and consistent interactions in higher derivative matter field theories

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Abstract We present the constructions of consistent interactions among the Abelian gauge field and matter fields in the higher derivative systems. Based on the order reduction technique, we are able to gauge the higher derivative models in the BV formalism. Inserting the equations of motion of the auxiliary fields into the antighost number zero part in the deformed master action, we will recover the resulting theory in the Lagrangian density form with extra higher derivative interaction terms. Furthermore, we investigate the problems of stabilities both in the free and coupling higher derivative dynamics using a series of additional bounded integrals of motion. In this way, we show that the 00-component of the energy–momentum tensors could be positive definite and therefore the higher derivative systems are all stable before and after the deformation procedures.

1 Introduction

It is well known that most theories in physics are described by second-order ordinary or partial differential equations while in fact, theories with higher-order Lagrangians have been explored along the evolution of physics. Originally, the initial interest in such theories due to the emergence of the powerful techniques of dealing with ultraviolet divergences [1,2] and this idea has been shown to be quite successful in the study of general relativity; for instance, adding higher derivative terms may improve the renormalizability of the gravity or even asymptotically free [3,4]. The method to construct Hamiltonian formulation for such higher derivative systems was firstly proposed by Ostrogradsky [5]. In particular, the Hamiltonian obtained in such a way contains term linear in momenta which indicates that the energy of the system is unbounded from below and generically reveals instability in classical mechanics. At the quantum level, the presence of the linear terms naturally gives rise to the negative norm states, commonly known as ghost states which imply negative probabilities when the quantization procedure is considered and therefore the theory is not unitary. In view of these fatal defects and for the purpose of curing the Ostrogradsky instability, various attempts are put forward to modify the quantization scheme in higher derivative field theories. One of the most influential idea is to formulate these unstable theories as a $\mathcal{PT}$-symmetric quantum mechanics involving non-Hermitian Hamiltonians [6–9], that is to say, the original system is symmetric under combined parity reflection and time reversal. In this manner, the

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higher derivative model determines its own Hilbert space and associated positive-definite inner product which is distinct from the usual one to eliminate the ghost states. Moreover, the indefinite metric Dirac–Pauli quantization [10] has been applied into the Lee–Wick theories [11,12] in order to circumvent the Ostrogradsky ghost issues. Such instabilities also can be suppressed by complexification of the higher derivative field theories [13], where the physical meaning of the complex theories is not yet clear. Recently, Kaparulin, Lyakhovich and Sharapov consider a wide class of higher derivative systems in which the operators of the equations of motion have the factorable structure [14–19]. For the models of this type, they construct a positive integral of motion explicitly that is different from the canonical energy, while the latter is unbounded in general due to the existence of higher derivative term. In addition, utilizing the concept of a Lagrange anchor [14,20,21], they demonstrate that it is possible to connect these bounded conserved quantities of motion with the time-translation invariance which can be used to ensure the classical stability of the higher derivative dynamics. All of these quite different approaches predict a positive spectrum of the Hamiltonian and therefore, following the general spirits of quantization, it is not difficult to illustrate that the higher derivative theories are the fully acceptable quantum-mechanical theories that exhibit unitary time evolutions.

On the other hand, since the healthy theories with higher derivative terms have extra degrees of freedom that are ghost like, these theories are necessarily constrained systems. Furthermore, there may exist gauge symmetry in the higher derivative theories analogous to the usual systems which is an essential component for interesting and appealing theories in modern theoretical physics. There is no doubt that the most suitable and powerful tool to deal with the constrained systems equipped with gauge symmetry is the BRST formulation developed by Becchi et al. [22–24]. In this way, one of the core issues is how to construct consistent interactions in general gauge theories within the framework of the local BRST-cohomology [25,26]. As expounded in [27,28], a consistent deformation of the classical action and the corresponding gauge invariance induce a consistent deformation of the master action which meanwhile is preserved by the master equation. Generally speaking, there are two methods to explore these deformations, the Hamiltonian BRST formalism and the Lagrangian BV formalism. In the former one, we start with a free theory with BRST charge and BRST-invariant Hamiltonian [29]; then we require that the nilpotency of the BRST charge and the commutativity between the Hamiltonian and BRST charge are preserved after the deformations [30,31]. Through solving a set of recursive equations coming from the perturbative expansion order by order and with the analysis of the local BRST-cohomology [32–34], it makes possible for us to derive the consistent deformations in general reducible or irreducible gauge systems. Adding all these deformed quantities up, we can identify the first-class Hamiltonian with the interacting gauge theory and the deformed gauge transformations are close on-shell in such resulting system.

While in the framework of Lagrangian consistent deformations, we should double the classical fields, the ghost fields and auxiliary fields by introducing a collection of antifields with opposite statistics compared to their partners [35]. The antibracket is defined between two local functionals on this extended phase space which is symmetric if both functionals are Grassmann-even and antisymmetric in all other cases. In this field-antifield formalism, the central role is the master action S_0 that encodes all of the necessary information about the original gauge theory including gauge transformations, the equations of motion and the Noether's identities. The master action S_0 is a functional of ghost number 0 which is in principle completely determined by requiring $(S_0, S_0) = 0$ and this is the so-called master equation [36–38]. On the other hand, the construction of such solution starts with the classical action as its boundary condition while the higher-order terms are added by assigning antifields an irreducible generating set of gauge transformations with gauge parameters replaced by ghosts. The generalized BRST transformations on the whole fields space can in general be read directly from this antibracket together with the master action [39]. Thus it follows from the above assumption that the deformed quantity should still fulfill the master equation and in this manner, it is not hard for us to receive and deal with the deformation equations by means of the perturbative expansion [40–44]. Combining all of the pieces together, we will get the total master action and if extracting the antighost number zero part, there is no difficulty in obtaining the desired interacting theory which coincides with the one derived in the Hamiltonian formalism as we can expect.

The outline of this paper is as follows. In Sect. 2, we simply illustrate the Ostrogradsky ghost problem in the free higher derivative massless real scalar field theory and handle the issue of stability utilizing a proper complexification. Then we quickly review the basic ingredients in the framework of BRST deformation procedure and show the cohomological derivation of the consistent interactions in this higher derivative model. In the same way, we address the problem of the stability in the resulting Lagrangian with coupling terms. The constructions of consistent deformations of the massive complex scalar field and Dirac spinor field are developed in Sects. 3 and 4; also with the help of a series of conserved quantities including the canonical energy, the explanations of the stability of higher derivative complex systems are presented. The final section of this paper is devoted to conclusion and discussion.

2 Massless real scalar field

2.1 Stability

We consider the free Maxwell electrodynamics with higher-order derivative scalar field ϕ in (1+3)-dimensional spacetime with metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and the Lagrangian is described by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + \frac{1}{m^2}\partial_\mu\phi\Box\partial^\mu\phi\right) \tag{2.1}$$

here the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and m is some constant. We suppose that the real scalar field possesses only the global symmetry and hence the above Lagrangian is invariant under the local gauge transformation for A_μ and rigid transformation for ϕ as follows

$$\Delta A_\mu = \partial_\mu\lambda, \qquad \Delta\phi = \xi \tag{2.2}$$

here the ξ is a constant. Varying the action gives us the equation of motion of the scalar field

$$\Box(\Box + m^2)\phi = 0 \tag{2.3}$$

In the classical regime, there is no straightforward transition from the Lagrangian (2.1) to the Hamiltonian formalism and Ostrogradsky generalized the usual construction of the Hamilton function to give a canonical description of the higher derivative systems. The main disadvantage of the Ostrogradsky's approach is that the Hamiltonian is necessarily unbounded from below due to the appearance of linear function of some momenta. Furthermore, this undesired phenomenon generally cannot be cured by trying to do any alternative canonical transformations. To be more specific, in Ostrogradsky's formalism, the time derivative of field is regarded as independent dynamic variable. Therefore we define

$$Q = \partial_0 \phi \tag{2.4}$$

as canonical coordinate in the extended phase space. The corresponding conjugate momenta are given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)}, \qquad \theta = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \left(1 + \frac{\Box}{m^2}\right)\partial_0 \phi, \qquad P = \frac{\partial \mathcal{L}}{\partial(\partial_0 Q)} = -\frac{\Box}{m^2}\phi \tag{2.5}$$

with the aid of these expressions, we are thus led to the following canonical Hamiltonian $\mathcal{H}_{Ostro}(A_i, \pi_i; \phi, \theta; Q, P)$

$$\begin{aligned}\mathcal{H}_{Ostro} &= \dot{A}^\mu \pi_\mu + \dot{\phi}\theta + \dot{Q}P - \mathcal{L} \\ &= \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A_0\partial^i\pi_i + \theta Q - \frac{1}{2}m^2P^2 + \partial_i P\partial^i\phi - \frac{1}{2}Q^2 - \frac{1}{2}\partial_i\phi\partial^i\phi\end{aligned} \tag{2.6}$$

it is obvious to see that the Hamiltonian (2.6) is linear in the canonical momentum θ, which implies that the energy can be lowered without any bound by increasing the momentum to large positive or negative values and hence the system is unstable.

In order to avoid the Ostrogradsky ghosts, recently Raidal and Veermae discussed that, at least in the Pais–Uhlenbeck oscillator case, these ghosts could be reinterpreted as physical particles through a canonical way. The essential point of their work is that for the purpose of the energy spectrum of the theory be bounded, the ghost degrees of freedom should be necessarily complex [13]. Therefore after a proper complexification, the complex higher derivative system can be consistently quantized using the rules of canonical quantization and the resulting system possesses all good properties of the known quantum physics including positive definite Hamiltonian which yields a stable and unitary quantum correspondence, the standard probabilistic interpretation, and no Ostrogradsky instability. On the other hand, it is well known that any higher derivative Lagrangian could be reduced to a normal one through new dynamical variables. In the present situation, this can be achieved by means of an auxiliary field Z

$$\tilde{\mathcal{L}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu Z + \frac{1}{8}m^2\phi\phi + \frac{1}{8}m^2ZZ - \frac{1}{4}m^2\phi Z \tag{2.7}$$

in which the equations of motion become

$$\left(1 + 2\frac{\Box}{m^2}\right)\phi = Z, \qquad \left(1 + 2\frac{\Box}{m^2}\right)Z = \phi \tag{2.8}$$

after substituting these coupled equations into (2.7), it is direct to show that the above reduced Lagrangian turns out to be (2.1) and this equivalent ordinary Lagrangian now possesses new degrees of freedom. Based on this, through the following decomposition or complexification [13,45]

$$\phi = X + iY, \qquad Z = X - iY \tag{2.9}$$

we are capable of converting the Lagrangian into the form of

$$\tilde{\mathcal{L}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu X\partial^\mu X + \frac{1}{2}\partial_\mu Y\partial^\mu Y - \frac{1}{2}m^2Y^2 \tag{2.10}$$

consequently, there is no difficulty in asserting that the energy of (2.10) is positive and in this sense, the unitarity is preserved and the original higher derivative massless scalar field theory (2.1) is quite an acceptable physical system.

2.2 Consistent interactions

Now let us turn the attentions to the BV deformations of the higher derivative Lagrangian or in other words, gauging the higher derivative free scalar field theory. Firstly, noting that (2.7) is equipped with the following symmetry transformations

$$\Delta A_\mu = \partial_\mu \lambda, \qquad \Delta\phi = \xi, \qquad \Delta Z = \xi \tag{2.11}$$

from the Noether's theorem, the on-shell conserved matter current J^μ is determined from the above global transformations of scalar field ϕ, Z as

$$\partial_\mu J^\mu + \frac{\delta\tilde{\mathcal{L}}}{\delta\phi} + \frac{\delta\tilde{\mathcal{L}}}{\delta Z} = 0 \tag{2.12}$$

to be more precise, in our situation

$$\delta\tilde{\mathcal{L}}/\delta\phi = -\frac{1}{2}\Box Z + \frac{1}{4}m^2\phi - \frac{1}{4}m^2 Z, \qquad \delta\tilde{\mathcal{L}}/\delta Z = -\frac{1}{2}\Box\phi + \frac{1}{4}m^2 Z - \frac{1}{4}m^2\phi \tag{2.13}$$

through a simple calculation it follows that

$$J^\mu = \frac{1}{2}(\partial^\mu\phi + \partial^\mu Z) \tag{2.14}$$

Within the framework of standard BV formalism, we denote all these fields collectively as $\phi^A = (A_\mu, \phi, Z, \eta)$ with the ghost field η arising from the irreducible generator of the Abelian local gauge symmetry. Also we introduce antifields for all of them by $\phi^*_A = (A^{*\mu}, \phi^*, Z^*, \eta^*)$ and these fields have opposite statistics compared to the original ones. The necessary ingredients of Grassmann parities, antighost, pure ghost and ghost numbers of the whole fields are listed as follows

$$\begin{array}{llll}
\epsilon(A_\mu, \phi, Z) = 0, & \epsilon(A^{*\mu}, \phi^*, Z^*) = 1, & \epsilon(\eta) = 1, & \epsilon(\eta^*) = 0, \\
\mathrm{agh}(A_\mu, \phi, Z) = 0, & \mathrm{agh}(A^{*\mu}, \phi^*, Z^*) = 1, & \mathrm{agh}(\eta) = 0, & \mathrm{agh}(\eta^*) = 2, \\
\mathrm{pgh}(A_\mu, \phi, Z) = 0, & \mathrm{pgh}(A^{*\mu}, \phi^*, Z^*) = 0, & \mathrm{pgh}(\eta) = 1, & \mathrm{pgh}(\eta^*) = 0, \\
\mathrm{gh}(A_\mu, \phi, Z) = 0, & \mathrm{gh}(A^{*\mu}, \phi^*, Z^*) = -1, & \mathrm{gh}(\eta) = 1, & \mathrm{gh}(\eta^*) = -2
\end{array} \tag{2.15}$$

in this manner, one obtains a space constituted by the local functionals of the whole fields which is naturally endowed with an odd Poisson bracket (,), called antibracket and hence this resulting field/antifield space acquires an odd phase space structure. Concretely, for arbitrary two local functionals $F(\phi^A, \phi^*_A)$, $G(\phi^A, \phi^*_A)$, the antibracket is given by [35]

$$(F, G) = \int_M \left(\frac{\delta_r F}{\delta\phi^A}\frac{\delta_l G}{\delta\phi^*_A} - \frac{\delta_r G}{\delta\phi^A}\frac{\delta_l F}{\delta\phi^*_A} \right) d^n x \tag{2.16}$$

here the summation over A is understood and the l, r superscripts on the functional derivatives denote that they are taken from the left or from the right, respectively. In such a way, the fields ϕ^A and antifields ϕ^*_B behave as coordinates and momenta and we can regard them as conjugate variables. To say more, the antibracket satisfies graded commutation, distribution and Jacobi relations as we can imagine, and in particular, the antibracket has ghost number 1.

On the other hand, in this large configuration space, the minimal solution of the master action for the Lagrangian (2.7) is constructed by [35,36]

$$S_0 = \int d^4x(\tilde{\mathcal{L}} + A^*_\mu \partial^\mu \eta) \tag{2.17}$$

as we pointed out previously, the BRST transformations on the extended phase space (ϕ^A, ϕ^*_A) determined from the master action as well as the BV-antibracket are provided by $s = (S_0, \cdot)$ which generally can be divided into two parts $s = \delta + \gamma$; here δ is the Koszul–Tate differential and γ is the exterior longitudinal derivative associated to the constraint surface and gauge orbits [35,36]. More specifically, the action of these operators on the generators of the BRST complex are given by

$$\begin{aligned}
&\delta A_\mu = 0, \quad \gamma A_\mu = \partial_\mu \eta, \quad \gamma A^{*\mu} = 0, \quad \delta A^{*\mu} = \partial_\nu F^{\mu\nu}, \\
&\delta\phi = 0, \quad \gamma\phi = 0, \quad \gamma\phi^* = 0, \quad \delta\phi^* = \frac{1}{2}\left(\Box Z - \frac{1}{2}m^2\phi + \frac{1}{2}m^2 Z\right), \\
&\delta Z = 0, \quad \gamma Z = 0, \quad \gamma Z^* = 0, \quad \delta Z^* = \frac{1}{2}\left(\Box\phi - \frac{1}{2}m^2 Z + \frac{1}{2}m^2\phi\right), \\
&\delta\eta = 0, \quad \gamma\eta = 0, \quad \gamma\eta^* = 0, \quad \delta\eta^* = -\partial_\mu A^{*\mu}
\end{aligned} \tag{2.18}$$

now if we do the deformation by the introduction of parameter g and express the deformed master action in terms of the parameter as [40–43]

$$S = S_0 + gS_1 + g^2 S_2 + \cdots\cdots \tag{2.19}$$

then it follows from the above assumption that the deformed quantity should still fulfill the master equation; thus, by expanding and comparing the power series of g order by order, it is not hard for us to obtain the following deformation equations of the master action [44]

$$\begin{aligned}
&1 : (S_0, S_0) = 0, \quad g^1 : 2(S_0, S_1) = 0, \quad g^2 : 2(S_0, S_2) + (S_1, S_1) = 0, \\
&g^3 : (S_0, S_3) + (S_1, S_2) = 0, \quad \ldots\ldots
\end{aligned} \tag{2.20}$$

Let us solve these deformation master equations and in the beginning, we concentrate on the first-order deformation term in (2.20) which in general takes the form of $S_1 = \int d^4x\omega_1$ and satisfies the functional equation [42–44]

$$0 = sS_1 = \int d^4x s\omega_1 \tag{2.21}$$

here ω_1 is a local functional and we see that the first-order deformation of the master action is s-cocycle modulo the total derivative d at ghost number zero. Using the decomposition of $s = \delta + \gamma$, the above s-exact equation is equivalent to

$$s\omega_1 = \delta\omega_1 + \gamma\omega_1 = \partial_\mu k_1^\mu \tag{2.22}$$

here k_1^μ is a local current functional and to find out the solution of (2.22), let us expand the ω_1 according to the antighost number [42,44]

$$\omega_1 = \omega_1^{(0)} + \omega_1^{(1)} + \cdots + \omega_1^{(I)} \tag{2.23}$$

here the antighost number of $\omega_1^{(i)}$ is i and from (2.15), (2.18), we realize that the Koszul-Tate differential δ lowers the antighost number while the exterior longitudinal derivative γ keeps the antighost number. In this way, comparing the antighost number on both sides of (2.22), the equality thus leads to a series of recursive equations

$$\gamma\omega_1^{(I)} = \partial_\mu k_1^{\mu(I)}, \quad \delta\omega_1^{(I)} + \gamma\omega_1^{(I-1)} = \partial_\mu k_1^{\mu(I-1)}, \quad \delta\omega_1^{(i+1)} + \gamma\omega_1^{(i)} = \partial_\mu k_1^{\mu(i)} \tag{2.24}$$

for $i = 0, \ldots, I-2$ and the terms in (2.23) are determined successively from these equations. As explained in [34,42], the highest antighost number I term should be strictly satisfied $\gamma\omega_1^{(I)} = 0$ and the general form is given by

$$\omega_1^{(I)} = b_I(\eta)^I \tag{2.25}$$

due to the fermonic nature of η, we can assume that the first-order deformation ω_1 is truncated at the $\omega_1^{(1)}$, or in other words we have

$$\omega = \omega_1^{(0)} + \omega_1^{(1)} \tag{2.26}$$

and the (2.24) turns out to be

$$\gamma\omega_1^{(1)} = 0, \qquad \delta\omega_1^{(1)} + \gamma\omega_1^{(0)} = \partial_\mu k_1^{\mu(0)} \tag{2.27}$$

with $\omega_1^{(1)} = b_1\eta$, here b_1 belongs to $H_1(\delta|d)$ [32–34,42] that can be solved as

$$\omega_1^{(1)} = \phi^*\eta + Z^*\eta \tag{2.28}$$

which in addition gives rise to

$$\delta\omega_1^{(1)} = -\frac{\delta\tilde{\mathcal{L}}}{\delta\phi}\eta - \frac{\delta\tilde{\mathcal{L}}}{\delta Z}\eta = \frac{1}{2}(\Box\phi + \Box Z)\eta = \partial_\mu J^\mu\eta \tag{2.29}$$

then we pick up the solution

$$\omega_1^{(0)} = J^\mu A_\mu \tag{2.30}$$

and it is simple to check

$$\delta\omega_1^{(1)} + \gamma\omega_1^{(0)} = \partial_\mu(J^\mu\eta) \tag{2.31}$$

therefore we obtain

$$S_1 = \int d^4x((\phi^* + Z^*)\eta + \frac{1}{2}(\partial^\mu\phi + \partial^\mu Z)A_\mu) \tag{2.32}$$

next making using of the canonical relations

$$\begin{aligned}(\phi(x), \phi^*(y)) &= (\phi^*(y), \phi(x)) = -\delta^4(x-y),\\ (Z(x), Z^*(y)) &= (Z^*(y), Z(x)) = -\delta^4(x-y)\end{aligned} \tag{2.33}$$

and applying the partial integration, we achieve

$$(S_1, S_1) = 2\int d^4x\eta\partial^\mu A_\mu = -s\left(\int d^4x A^\mu A_\mu\right) \tag{2.34}$$

that resulting in

$$S_2 = \frac{1}{2}\int d^4x A^\mu A_\mu \tag{2.35}$$

it is easy to verify $(S_1, S_2) = 0$ which of course produces $S_3 = 0$, and moreover the other higher-order deformations can be chosen $S_i = 0$ for $i \geq 4$ that the deformation equations in

(2.20) are satisfied automatically. In final, the total solution $S = S_0 + S_1 + S_2$ is described by

$$S = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu Z + \frac{1}{8}m^2\phi\phi + \frac{1}{8}m^2 ZZ - \frac{1}{4}m^2\phi Z + A^*_\mu\partial^\mu\eta \right.$$
$$\left. + g((\phi^* + Z^*)\eta + \frac{1}{2}(\partial^\mu\phi + \partial^\mu Z)A_\mu) + \frac{1}{2}g^2 A^\mu A_\mu \right) \tag{2.36}$$

Based on the above result, the explicit formula of the antighost number zero part in the deformed master action admits the form of

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi + gA_\mu)(\partial^\mu Z + gA^\mu) + \frac{1}{8}m^2\phi\phi + \frac{1}{8}m^2 ZZ - \frac{1}{4}m^2\phi Z \tag{2.37}$$

in this expression, it is not difficult to derive the equations of motion for real scalar field ϕ and auxiliary field Z

$$\left(1 + 2\frac{\Box}{m^2}\right)\phi - \frac{2g}{m^2}\partial^\mu A_\mu = Z, \qquad \left(1 + 2\frac{\Box}{m^2}\right)Z - \frac{2g}{m^2}\partial^\mu A_\mu = \phi \tag{2.38}$$

by substituting the algebraic solution of Z back into (2.37) and modulo the total derivative terms, we finally arrive at the following equivalent form

$$\tilde{\mathcal{L}}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi - gA_\mu)(\partial^\mu\phi - gA^\mu) + \frac{1}{2m^2}(\partial_\mu\phi - gA_\mu)\Box(\partial^\mu\phi - gA^\mu) \tag{2.39}$$

It fairly obvious to see that such Lagrangian is invariant under the following transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda, \qquad \phi \to \phi' = \phi + g\lambda \tag{2.40}$$

for arbitrary real function λ. By a comparison of (2.2) with (2.40), we then conclude that after the deformation procedure, the system containing consistent interactions modifies the symmetry transformation but preserves the number of the gauge symmetry. Indeed, it is exactly the Stueckelberg-like couplings [27,42] between Abelian gauge and real scalar fields and the deformed model can be understood in the sense of gaugings of the relevant global shift symmetry from the original free theory in the derivative terms of the matter field.

To this end, we proceed to deal with the Ostrogradsky instability of the resulting system and by the same token, making using of the choice (2.9), one simply obtains

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu X + gA_\mu)(\partial^\mu X + gA^\mu) + \frac{1}{2}\partial_\mu Y\partial^\mu Y - \frac{1}{2}m^2 Y^2 \tag{2.41}$$

this demonstrates that the coupling system is stable.

3 Massive complex scalar field

3.1 Stability

We consider the following Lagrangian density between the Abelian gauge field and a complex massive scalar field $(\varphi, \bar{\varphi})$ with higher derivative term

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \partial_\mu\varphi\partial^\mu\bar{\varphi} - \frac{1}{m^2}\Box\varphi\Box\bar{\varphi} - M^2\varphi\bar{\varphi} \tag{3.1}$$

here m, M are some real constants. Firstly let us find out whether the spectrum is still unbounded from below and as was also the case in the previous example, due to the presence of higher derivatives, the phase space should involve extra canonical coordinates and we introduce the following two independent new dynamic variables

$$Q = \partial_0 \varphi, \qquad \bar{Q} = \partial_0 \bar{\varphi} \tag{3.2}$$

together with the canonical conjugate momenta

$$\begin{aligned} \theta &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \left(1 + \frac{\Box}{m^2}\right)\partial_0 \bar{\varphi}, \qquad P = \frac{\partial \mathcal{L}}{\partial(\partial_0 Q)} = -\frac{\Box}{m^2}\bar{\varphi}, \\ \bar{\theta} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\varphi})} = \left(1 + \frac{\Box}{m^2}\right)\partial_0 \varphi, \qquad \bar{P} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{Q})} = -\frac{\Box}{m^2}\varphi \end{aligned} \tag{3.3}$$

after a standard Legendre transformation, the canonical Hamiltonian $\mathcal{H}_{Ostro}$ can be brought in the form of

$$\begin{aligned} \mathcal{H}_{Ostro} &= \dot{A}^{\mu}\pi_{\mu} + \dot{\varphi}\theta + \dot{\bar{\varphi}}\bar{\theta} + \dot{Q}P + \dot{\bar{Q}}\bar{P} - \mathcal{L} \\ &= \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A_0\partial^i\pi_i + \theta Q + \bar{\theta}\bar{Q} + m^2 P\bar{P} \\ &\quad + \partial_i P\partial^i\varphi + \partial_i\bar{P}\partial^i\bar{\varphi} - Q\bar{Q} - \partial_i\varphi\partial^i\bar{\varphi} + M^2\varphi\bar{\varphi} \end{aligned} \tag{3.4}$$

it is evident to see that the Hamiltonian is linear in terms of the momenta $\theta, \bar{\theta}$ which is worse and terrible.

Compared to the real scalar field theory, in the present complex case, it is suitable to apply a more general scheme to investigate the problem of the stability of the higher derivative system by using the concept of Lagrangian anchors. More specifically, this new method mainly provides an extension of the usual Noether theorem to produce a class of conserved quantities associated with a given symmetry [14,17–19]. Following this way, the Lagrangian anchor connects conserved quantities to symmetries for any system of field equations whether they are Lagrangian or not. Especially, for the time translation invariance, this approach will result in different conserved quantities which could be identified with a Hamiltonian and more importantly, some of them would recover the stability of higher derivative dynamics. For a given system, the Lagrangian anchor is not necessarily unique and if the theory possesses multiple Lagrangian anchors, the same symmetry can be connected to different conserved quantities. Under this framework, it allows us to add consistent interactions into field equations of motion by means of the proper deformation scheme for any given Lagrange anchors. In addition, if the anchor connects the symmetry with the bounded quantity, the system remains stable upon inclusion of interactions which can be seen clearly in [19,46].

In our situation, at first, the equations of motion for the complex fields $\varphi, \bar{\varphi}$ are given by

$$\Box^2\varphi + m^2\Box\varphi + m^2M^2\varphi = 0, \qquad \Box^2\bar{\varphi} + m^2\Box\bar{\varphi} + m^2M^2\bar{\varphi} = 0 \tag{3.5}$$

noting that these expressions have the factorable structure with the operators $\mathcal{P}$ and $\mathcal{Q}$

$$\mathcal{P} = \frac{\Box + m_1^2}{m_1^2 - m_2^2}, \qquad \mathcal{Q} = \frac{\Box + m_2^2}{m_2^2 - m_1^2} \tag{3.6}$$

where

$$m_1^2 = \frac{m^2 + \sqrt{m^4 - 4m^2M^2}}{2}, \qquad m_2^2 = \frac{m^2 - \sqrt{m^4 - 4m^2M^2}}{2} \tag{3.7}$$

with the following properties

$$\mathcal{P}+\mathcal{Q}=1, \qquad [\mathcal{P},\mathcal{Q}]=0, \qquad \mathcal{P}\mathcal{Q}\varphi=0, \quad \mathcal{P}\mathcal{Q}\bar{\varphi}=0 \tag{3.8}$$

here we suppose $m > 2M$ and in this assumption, m_1^2, m_2^2 are different real numbers. Subsequently, with the help of the above factorization we introduce

$$\xi=\mathcal{Q}\varphi, \qquad \eta=\mathcal{P}\varphi, \qquad \bar{\xi}=\mathcal{Q}\bar{\varphi}, \qquad \bar{\eta}=\mathcal{P}\bar{\varphi} \tag{3.9}$$

in terms of these new dynamical fields, we construct an alternative action functional as follows

$$S_1[\xi(x),\bar{\xi}(x);\eta(x),\bar{\eta}(x)]=\int(\bar{\xi}\mathcal{P}\xi+\bar{\eta}\mathcal{Q}\eta)d^4x \tag{3.10}$$

together with the equations of motion

$$\mathcal{P}\xi=0, \qquad \mathcal{Q}\eta=0, \qquad \mathcal{P}\bar{\xi}=0, \qquad \mathcal{Q}\bar{\eta}=0 \tag{3.11}$$

taking advantage of (3.9), one recovers the dynamic equations in (3.8). It is not difficult to check that the relations (3.9) establish a one-to-one correspondence between the solutions of systems (3.1) and (3.10) (we ignore the pure gauge part since it has no influence for our discussion). Therefore, these two systems are equivalent and may be thought of as two different representations of the same theory which are usually called $(\xi\eta,\bar{\xi}\bar{\eta})$- and $\varphi\bar{\varphi}$-representations. It is worth mentioning here that, in this picture, the factorization (3.8) would provide an efficient and useful tool to illustrate the issues of the stability in the higher derivative dynamic systems as will be discussed below.

Now let us focus on the $(\xi\eta,\bar{\xi}\bar{\eta})$- system and it is well known from the Noether theorem that, if the action (3.10) is invariant with respect to the spacetime translations $x^\mu \to x^\mu-\varepsilon^\mu$, then the system will admit two independent conserved currents $J^\mu(\xi,\bar{\xi})$ and $J^\mu(\eta,\bar{\eta})$

$$\partial_\mu J^\mu(\xi,\bar{\xi})=-\varepsilon^\mu\partial_\mu\xi\frac{\delta S_1}{\delta\xi}-\varepsilon^\mu\partial_\mu\bar{\xi}\frac{\delta S_1}{\delta\bar{\xi}}, \quad \partial_\mu J^\mu(\eta,\bar{\eta})=-\varepsilon^\mu\partial_\mu\eta\frac{\delta S_1}{\delta\eta}-\varepsilon^\mu\partial_\mu\bar{\eta}\frac{\delta S_1}{\delta\bar{\eta}} \tag{3.12}$$

which can be expressible in the form of

$$\begin{aligned} J^\mu(\xi,\bar{\xi}) &=-\frac{1}{m_1^2-m_2^2}(\partial^\mu\bar{\xi}\partial_\nu\xi\varepsilon^\nu-\partial_\sigma\xi\partial^\sigma\bar{\xi}\varepsilon^\mu+\partial^\mu\xi\partial_\nu\bar{\xi}\varepsilon^\nu+m_1^2\xi\bar{\xi}\varepsilon^\mu), \\ J^\mu(\eta,\bar{\eta}) &=\frac{1}{m_1^2-m_2^2}(\partial^\mu\bar{\eta}\partial_\nu\eta\varepsilon^\nu-\partial_\sigma\eta\partial^\sigma\bar{\eta}\varepsilon^\mu+\partial^\mu\eta\partial_\nu\bar{\eta}\varepsilon^\nu+m_2^2\eta\bar{\eta}\varepsilon^\mu) \end{aligned} \tag{3.13}$$

in such a way, the canonical energy–momentum tensors are defined by the rule

$$J^\mu(\xi,\bar{\xi})=\Theta^\mu_\nu(\xi,\bar{\xi})\varepsilon^\nu, \qquad J^\mu(\eta,\bar{\eta})=\Theta^\mu_\nu(\eta,\bar{\eta})\varepsilon^\nu \tag{3.14}$$

more explicitly, we simply have

$$\begin{aligned} \Theta^\mu_\nu(\xi,\bar{\xi}) &=-\frac{1}{m_1^2-m_2^2}(\partial^\mu\bar{\xi}\partial_\nu\xi-\delta^\mu_\nu\partial_\sigma\xi\partial^\sigma\bar{\xi}+\partial^\mu\xi\partial_\nu\bar{\xi}+\delta^\mu_\nu m_1^2\xi\bar{\xi}), \\ \Theta^\mu_\nu(\eta,\bar{\eta}) &=\frac{1}{m_1^2-m_2^2}(\partial^\mu\bar{\eta}\partial_\nu\eta-\delta^\mu_\nu\partial_\sigma\eta\partial^\sigma\bar{\eta}+\partial^\mu\eta\partial_\nu\bar{\eta}+\delta^\mu_\nu m_2^2\eta\bar{\eta}) \end{aligned} \tag{3.15}$$

as explained in [14,18,19], every symmetry of primary theory will give rise to the n-parametric series of symmetries of the higher derivative theory together with a series of independent bounded-conserved quantities which is a linear combination of the above expressions

$$\Theta^{\mu}_{\nu} = \beta_1 \Theta^{\mu}_{\nu}(\xi, \bar{\xi}) + \beta_2 \Theta^{\mu}_{\nu}(\eta, \bar{\eta}) \tag{3.16}$$

here β_1, β_2 are some real constants. In this formalism, the component Θ^0_0 has the sense of the energy density of the theory and by this reason, the total energy of the system is provided by the integral

$$E = \int d^3x \Theta^0_0 \tag{3.17}$$

using the metric $g_{\mu\nu} = (1, -1, -1, -1)$ we have

$$\begin{aligned}\Theta^0_0 =& -\frac{\beta_1}{m_1^2 - m_2^2}\left(\partial_0\left(\frac{\Box\bar{\varphi} + m_2^2\bar{\varphi}}{m_2^2 - m_1^2}\right)\partial_0\left(\frac{\Box\varphi + m_2^2\varphi}{m_2^2 - m_1^2}\right)\right.\\ &\left.+\partial_i\left(\frac{\Box\varphi + m_2^2\varphi}{m_2^2 - m_1^2}\right)\partial_i\left(\frac{\Box\bar{\varphi} + m_2^2\bar{\varphi}}{m_2^2 - m_1^2}\right) + m_1^2\left(\frac{\Box\bar{\varphi} + m_2^2\bar{\varphi}}{m_2^2 - m_1^2}\right)\left(\frac{\Box\varphi + m_2^2\varphi}{m_2^2 - m_1^2}\right)\right)\\ &+\frac{\beta_2}{m_1^2 - m_2^2}\left(\partial_0\left(\frac{\Box\varphi + m_1^2\varphi}{m_1^2 - m_2^2}\right)\partial_0\left(\frac{\Box\bar{\varphi} + m_1^2\bar{\varphi}}{m_1^2 - m_2^2}\right)\right.\\ &\left.+\partial_i\left(\frac{\Box\varphi + m_1^2\varphi}{m_1^2 - m_2^2}\right)\partial_i\left(\frac{\Box\bar{\varphi} + m_1^2\bar{\varphi}}{m_1^2 - m_2^2}\right) + m_2^2\left(\frac{\Box\varphi + m_1^2\varphi}{m_1^2 - m_2^2}\right)\left(\frac{\Box\bar{\varphi} + m_1^2\bar{\varphi}}{m_1^2 - m_2^2}\right)\right)\end{aligned} \tag{3.18}$$

thus it is obvious to assert that its 00-component is bounded and positive if

$$-\frac{\beta_1}{m_1^2 - m_2^2} > 0, \qquad \frac{\beta_2}{m_1^2 - m_2^2} > 0 \tag{3.19}$$

with this choice in hand, we assert the previous higher derivative complex system is stable, though the canonical energy is unbounded from below.

3.2 Consistent interactions

In order to construct the consistent interactions in original free Lagrangian (3.1), firstly we notice that such system is invariant under the following symmetry transformations

$$\Delta A_\mu = \partial_\mu \lambda, \qquad \Delta\varphi = -i\varphi\xi, \qquad \Delta\bar{\varphi} = i\bar{\varphi}\xi \tag{3.20}$$

here ξ is a global constant. Similarly, by introducing a pair of complex auxiliary scalar fields $(Z, \bar{Z})$, the above Lagrangian can be recast as equivalently

$$\begin{aligned}\tilde{\mathcal{L}} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi\partial^\mu\bar{Z} + \partial^\mu\bar{\varphi}\partial_\mu Z) + \frac{1}{4}m^2\varphi\bar{\varphi} + \frac{1}{4}m^2 Z\bar{Z}\\ &-\frac{1}{4}m^2(\varphi\bar{Z} + \bar{\varphi}Z) - M^2\varphi\bar{\varphi}\end{aligned} \tag{3.21}$$

with the equations of motion

$$\begin{aligned}&\left(1 + 2\frac{\Box}{m^2}\right)\varphi = Z, \qquad \left(1 + 2\frac{\Box}{m^2}\right)\bar{\varphi} = \bar{Z},\\ &(2\Box + m^2)Z = (m^2 - 4M^2)\varphi, \qquad (2\Box + m^2)\bar{Z} = (m^2 - 4M^2)\bar{\varphi}\end{aligned} \tag{3.22}$$

It is immediately seen that this Lagrangian density also possesses two independent symmetries, the local gauge transformation and the rigid transformation

$$\Delta A_\mu = \partial_\mu \lambda, \quad \Delta\varphi = -i\varphi\xi, \quad \Delta\bar{\varphi} = i\bar{\varphi}\xi, \quad \Delta Z = -iZ\xi, \quad \Delta\bar{Z} = i\bar{Z}\xi \tag{3.23}$$

by analogy with the discussion of the real scalar field, it follows from (3.23) that the conservation law for complex scalar fields is

$$\partial_\mu J^\mu + \frac{\delta\tilde{\mathcal{L}}}{\delta\varphi}(-i\varphi) + \frac{\delta\tilde{\mathcal{L}}}{\delta\bar{\varphi}}(i\bar{\varphi}) + \frac{\delta\tilde{\mathcal{L}}}{\delta Z}(-iZ) + \frac{\delta\tilde{\mathcal{L}}}{\delta\bar{Z}}(i\bar{Z}) = 0 \tag{3.24}$$

in the present case, there is no difficulty in computing

$$\begin{aligned}
\frac{\delta\tilde{\mathcal{L}}}{\delta\varphi} &= -\frac{1}{2}\Box\bar{Z} + \frac{1}{4}m^2\bar{\varphi} - \frac{1}{4}m^2\bar{Z} - M^2\bar{\varphi}, \qquad \frac{\delta\tilde{\mathcal{L}}}{\delta Z} = -\frac{1}{2}\Box\bar{\varphi} + \frac{1}{4}m^2\bar{Z} - \frac{1}{4}m^2\bar{\varphi},\\
\frac{\delta\tilde{\mathcal{L}}}{\delta\bar{\varphi}} &= -\frac{1}{2}\Box Z + \frac{1}{4}m^2\varphi - \frac{1}{4}m^2 Z - M^2\varphi, \qquad \frac{\delta\tilde{\mathcal{L}}}{\delta\bar{Z}} = -\frac{1}{2}\Box\varphi + \frac{1}{4}m^2 Z - \frac{1}{4}m^2\varphi
\end{aligned} \tag{3.25}$$

together with the conserved current

$$J^\mu = \frac{1}{2}i(\partial^\mu Z\bar{\varphi} - \partial^\mu\bar{Z}\varphi + \partial^\mu\varphi\bar{Z} - \partial^\mu\bar{\varphi}Z) \tag{3.26}$$

as a result, from (2.28) and (2.30) we obtain

$$S_1 = \int d^4x i((\bar{\varphi}^*\bar{\varphi} - \varphi^*\varphi + \bar{Z}^*\bar{Z} - Z^*Z)\eta + \frac{1}{2}(\partial^\mu Z\bar{\varphi} - \partial^\mu\bar{Z}\varphi + \partial^\mu\varphi\bar{Z} - \partial^\mu\bar{\varphi}Z)A_\mu) \tag{3.27}$$

furthermore, in the consideration of the second-order deformation, a direct calculation shows

$$\begin{aligned}
(S_1, S_1) = &\int d^4x(Z\partial^\mu(\bar{\varphi}A_\mu) + \partial^\mu Z\bar{\varphi}A_\mu + \bar{Z}\partial^\mu(\varphi A_\mu) + \partial^\mu\bar{Z}\varphi A_\mu\\
&+ \varphi\partial^\mu(\bar{Z}A_\mu) + \partial^\mu\varphi\bar{Z}A_\mu + \bar{\varphi}\partial^\mu(ZA_\mu) + \partial^\mu\bar{\varphi}ZA_\mu)\eta\\
= &- s\left(\int d^4x(Z\bar{\varphi} + \bar{Z}\varphi)A_\mu A^\mu\right)
\end{aligned} \tag{3.28}$$

and we claim that

$$S_2 = \frac{1}{2}\int d^4x(Z\bar{\varphi} + \bar{Z}\varphi)A_\mu A^\mu \tag{3.29}$$

it is clear to see that $(S_1, S_2) = 0$ which then yields $S_i = 0$ for $i \geq 3$. Consequently, the solution of the deformation master equations, consistent to all orders of the deformation parameter g can be written as

$$\begin{aligned}
S = \int d^4x \Bigg(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi\partial^\mu\bar{Z} + \partial^\mu\bar{\varphi}\partial_\mu Z) + \frac{1}{4}m^2\varphi\bar{\varphi} + \frac{1}{4}m^2 Z\bar{Z}\\
&-\frac{1}{4}m^2(\varphi\bar{Z} + \bar{\varphi}Z) - M^2\varphi\bar{\varphi} + g(i(\bar{\varphi}^*\bar{\varphi} - \varphi^*\varphi + \bar{Z}^*\bar{Z} - Z^*Z)\eta\\
&+\frac{1}{2}i(\partial^\mu Z\bar{\varphi} - \partial^\mu\bar{Z}\varphi + \partial^\mu\varphi\bar{Z} - \partial^\mu\bar{\varphi}Z)A_\mu) + \frac{1}{2}g^2(Z\bar{\varphi} + \bar{Z}\varphi)A_\mu A^\mu\Bigg)
\end{aligned} \tag{3.30}$$

with the help of the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu, \qquad \bar{D}^\mu = \partial^\mu + igA^\mu \tag{3.31}$$

we are able to extract the antighost number zero part in the deformed master action in a precise way

$$\begin{aligned}\mathcal{L}' =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(D_\mu\varphi)(\bar{D}^\mu\bar{Z}) + (\bar{D}^\mu\bar{\varphi})(D_\mu Z)\right] + \frac{1}{4}m^2\varphi\bar{\varphi} + \frac{1}{4}m^2 Z\bar{Z}\\ &-\frac{1}{4}m^2(\varphi\bar{Z} + \bar{\varphi}Z) - M^2\varphi\bar{\varphi}\end{aligned} \tag{3.32}$$

taking advantage of the equations of motion of auxiliary fields $Z, \bar{Z}$

$$Z = \varphi + \frac{2}{m^2}D^\mu D_\mu\varphi, \qquad \bar{Z} = \bar{\varphi} + \frac{2}{m^2}\bar{D}^\mu\bar{D}_\mu\bar{\varphi} \tag{3.33}$$

along with the fact that for arbitrary functions f, g, the identities

$$\int d^4x f D_\mu g = -\int d^4x g \bar{D}_\mu f \tag{3.34}$$

hold, we show that the above system is equivalent to

$$\begin{aligned}\tilde{\mathcal{L}}' =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\varphi)\left(\bar{D}^\mu\bar{\varphi} + \frac{2}{m^2}\bar{D}^\mu\bar{D}_\nu\bar{D}^\nu\bar{\varphi}\right) + \frac{1}{2}(\bar{D}_\mu\bar{\varphi})(D^\mu\varphi\\ &+\frac{2}{m^2}D^\mu D_\nu D^\nu\varphi) + \frac{1}{m^2}(\bar{D}_\nu\bar{D}^\nu\bar{\varphi})(D_\mu D^\mu\varphi) - M^2\varphi\bar{\varphi}\\ =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)(\bar{D}^\mu\bar{\varphi}) - \frac{1}{m^2}(\bar{D}_\nu\bar{D}^\nu\bar{\varphi})(D_\mu D^\mu\varphi) - M^2\varphi\bar{\varphi}\end{aligned} \tag{3.35}$$

clearly, this Lagrangian is invariant under the following local gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda, \quad \varphi \to \varphi' = e^{-ig\lambda}\varphi, \quad \bar{\varphi} \to \bar{\varphi}' = e^{ig\lambda}\bar{\varphi} \tag{3.36}$$

a simple observation then tells us that the consistent interactions added in the resulting Lagrangian can be received from the original theory just through replacing the ordinary partial derivative ∂_μ by the covariant derivatives D_μ, $\bar{D}_\mu$ both in first and higher derivative terms. Certainly, this interacting theory describes the couplings between the Abelian gauge field and complex scalar field and the gauge symmetry (3.36) originates from the gaugings of the rigid invariance of the matter fields.

In what follows, we ignore the pure gauge part in Lagrangian (3.35) and mainly focus the attentions on the remaining coupling part. In order to analyze the stability of the coupling system, firstly, we derive the Euler–Lagrange equations of motions from (3.35)

$$\begin{aligned}\frac{\delta S}{\delta\varphi} =& -\bar{D}_\mu\bar{D}^\mu\bar{\varphi} - \frac{1}{m^2}\bar{D}_\mu\bar{D}^\mu\bar{D}_\nu\bar{D}^\nu\bar{\varphi} - M^2\bar{\varphi} = 0,\\ \frac{\delta S}{\delta\bar{\varphi}} =& -D_\mu D^\mu\varphi - \frac{1}{m^2}D_\mu D^\mu D_\nu D^\nu\varphi - M^2\varphi = 0,\\ \frac{\delta S}{\delta A_\mu} =& ig(\bar{\varphi}D^\mu\varphi - \varphi\bar{D}^\mu\bar{\varphi})\\ &+\frac{1}{m^2}ig(\bar{D}_\nu\bar{D}^\nu\bar{\varphi}D^\mu\varphi - \varphi\bar{D}^\mu\bar{D}_\nu\bar{D}^\nu\bar{\varphi} - \bar{D}^\mu\bar{\varphi}D_\nu D^\nu\varphi + \bar{\varphi}D^\mu D_\nu D^\nu\varphi) = 0\end{aligned} \tag{3.37}$$

in a similar way, we introduce the differential operators $\mathcal{P}$, $\mathcal{Q}$, $\bar{\mathcal{P}}$, $\bar{\mathcal{Q}}$ in the form of

$$\begin{aligned} \mathcal{P} &= \frac{D_\mu D^\mu + m_1^2}{m_1^2 - m_2^2}, & \mathcal{Q} &= \frac{D_\mu D^\mu + m_2^2}{m_2^2 - m_1^2}, \\ \bar{\mathcal{P}} &= \frac{\bar{D}_\mu \bar{D}^\mu + m_1^2}{m_1^2 - m_2^2}, & \bar{\mathcal{Q}} &= \frac{\bar{D}_\mu \bar{D}^\mu + m_2^2}{m_2^2 - m_1^2} \end{aligned} \tag{3.38}$$

as well as the fields

$$\xi = \mathcal{Q}\varphi, \qquad \eta = \mathcal{P}\varphi, \qquad \bar{\xi} = \bar{\mathcal{Q}}\bar{\varphi}, \qquad \bar{\eta} = \bar{\mathcal{P}}\bar{\varphi} \tag{3.39}$$

as is seen from above, the derivatives of the original fields φ, $\bar{\varphi}$ are absorbed into these new dynamical variables. Analogously, the equations of motion can be derived from the least action principle for the action functional

$$S_1[\xi(x), \eta(x); \bar{\xi}(x), \bar{\eta}(x)] = \int (\bar{\xi}\mathcal{P}\xi + \bar{\eta}\mathcal{Q}\eta) d^4x, \tag{3.40}$$

and the explicit expressions are given by

$$\begin{aligned} &\frac{\delta S_1}{\delta \xi} = \bar{\mathcal{P}}\bar{\xi} = 0, \quad \frac{\delta S_1}{\delta \bar{\xi}} = \mathcal{P}\xi = 0, \quad \frac{\delta S_1}{\delta \eta} = \bar{\mathcal{Q}}\bar{\eta} = 0, \quad \frac{\delta S_1}{\delta \bar{\eta}} = \mathcal{Q}\eta = 0, \\ &\frac{\delta S_1}{\delta A_\mu} = \frac{ig}{m_1^2 - m_2^2}(\xi \bar{D}^\mu \bar{\xi} - \bar{\xi} D^\mu \xi - \eta \bar{D}^\mu \bar{\eta} + \bar{\eta} D^\mu \eta) = 0 \end{aligned} \tag{3.41}$$

it is simple to check that the relations (3.39) establish a one-to-one correspondence between solutions of both the systems by substituting (3.39) into (3.41) and thus the above two systems are equivalent. In this new set, as demonstrated previously, the conserved energy–momentum tensors turn out to be

$$\begin{aligned} \Theta^\mu_\nu(\xi, \bar{\xi}) &= -\frac{1}{m_1^2 - m_2^2}(\bar{D}^\mu \bar{\xi} D_\nu \xi - \delta^\mu_\nu D_\sigma \xi \bar{D}^\sigma \bar{\xi} + D^\mu \xi \bar{D}_\nu \bar{\xi} + \delta^\mu_\nu m_1^2 \xi \bar{\xi}), \\ \Theta^\mu_\nu(\eta, \bar{\eta}) &= \frac{1}{m_1^2 - m_2^2}(\bar{D}^\mu \bar{\eta} D_\nu \eta - \delta^\mu_\nu D_\sigma \eta \bar{D}^\sigma \bar{\eta} + D^\mu \eta \bar{D}_\nu \bar{\eta} + \delta^\mu_\nu m_2^2 \eta \bar{\eta}) \end{aligned} \tag{3.42}$$

if the gauge field A_μ and the scalar fields η, ξ meet the dynamic equations. In this way, the series of the energy–momentum tensors for the non-linear system can be expressed as

$$\Theta^\mu_\nu = \beta_1 \Theta^\mu_\nu(\xi, \bar{\xi}) + \beta_2 \Theta^\mu_\nu(\eta, \bar{\eta}) \tag{3.43}$$

also the 00-component has the structure of

$$\begin{aligned} \Theta^0_0 = &-\frac{\beta_1}{m_1^2 - m_2^2}\left(D_0\left(\frac{\bar{D}_\tau \bar{D}^\tau \bar{\varphi} + m_2^2 \bar{\varphi}}{m_2^2 - m_1^2}\right) D_0\left(\frac{D_\tau D^\tau \varphi + m_2^2 \varphi}{m_2^2 - m_1^2}\right)\right. \\ &+ D_i\left(\frac{D_\tau D^\tau \varphi + m_2^2 \varphi}{m_2^2 - m_1^2}\right) D_i\left(\frac{\bar{D}_\tau \bar{D}^\tau \bar{\varphi} + m_2^2 \bar{\varphi}}{m_2^2 - m_1^2}\right) \\ &\left.+ m_1^2\left(\frac{\bar{D}_\tau \bar{D}^\tau \bar{\varphi} + m_2^2 \bar{\varphi}}{m_2^2 - m_1^2}\right)\left(\frac{D_\tau D^\tau \varphi + m_2^2 \varphi}{m_2^2 - m_1^2}\right)\right) \\ &+ \frac{\beta_2}{m_1^2 - m_2^2}\left(D_0\left(\frac{\bar{D}_\tau \bar{D}^\tau \bar{\varphi} + m_1^2 \bar{\varphi}}{m_1^2 - m_2^2}\right) D_0\left(\frac{D_\tau D^\tau \varphi + m_1^2 \varphi}{m_1^2 - m_2^2}\right)\right. \end{aligned}$$

$$+D_i\left(\frac{D_\tau D^\tau\varphi+m_1^2\varphi}{m_1^2-m_2^2}\right)D_i\left(\frac{\bar{D}_\tau\bar{D}^\tau\bar{\varphi}+m_1^2\bar{\varphi}}{m_1^2-m_2^2}\right)$$
$$+m_2^2\left(\frac{\bar{D}_\tau\bar{D}^\tau\bar{\varphi}+m_1^2\bar{\varphi}}{m_1^2-m_2^2}\right)\left(\frac{D_\tau D^\tau\varphi+m_1^2\varphi}{m_1^2-m_2^2}\right)\Bigg)\Bigg) \tag{3.44}$$

evidently, it is bounded and positive if

$$-\frac{\beta_1}{m_1^2-m_2^2}>0,\qquad \frac{\beta_2}{m_1^2-m_2^2}>0 \tag{3.45}$$

this brings us the conclusion that the coupling system (3.35) including the consistent interactions should be stable.

4 Massive Dirac spinor field

4.1 Stability

We consider the following free Lagrangian density between the Abelian gauge field A_μ and the Dirac spinor fields $(\psi^\alpha, \bar{\psi}_\alpha)$ with higher derivative term

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}_\alpha((\gamma^\mu\partial_\mu)^\alpha_\beta-M\delta^\alpha_\beta)\psi^\beta+(\partial_\mu\bar{\psi}_\sigma\gamma^{\mu\sigma}_\alpha)((\gamma^\nu\partial_\nu)^\alpha_\tau(\gamma^\mu\partial_\mu)^\tau_\beta)\psi^\beta \tag{4.1}$$

here γ^μ is the standard Dirac's gamma matrices fulfilling $\{\gamma^\mu,\gamma^\nu\}=2\delta^{\mu\nu}$. The equations of motion are given by

$$(\gamma^\mu\partial_\mu)^3\psi-\gamma^\mu\partial_\mu\psi+M\psi=0,\qquad \bar{\psi}((\overleftarrow{\partial}_\mu\gamma^\mu)^3-\overleftarrow{\partial}_\mu\gamma^\mu-M)=0 \tag{4.2}$$

which can be decomposed into

$$\begin{aligned}(\gamma^\omega\partial_\omega+m_1)(\gamma^\mu\partial_\mu+m_2)(\gamma^\nu\partial_\nu+m_3)\psi=0,\\ \bar{\psi}(\overleftarrow{\partial}_\omega\gamma^\omega-m_1)(\overleftarrow{\partial}_\mu\gamma^\mu-m_2)(\overleftarrow{\partial}_\nu\gamma^\nu-m_3)=0\end{aligned} \tag{4.3}$$

here for the sake of simplicity, we assume that the m_1, m_2, m_3 are three real different roots of the characteristic polynomial

$$z^3-z-M=0 \tag{4.4}$$

then we define the new dynamic fields

$$\begin{aligned}\xi_1&=(\gamma^\mu\partial_\mu+m_2)(\gamma^\nu\partial_\nu+m_3)\psi,\quad \xi_2=(\gamma^\mu\partial_\mu+m_1)(\gamma^\nu\partial_\nu+m_3)\psi,\\ \xi_3&=(\gamma^\mu\partial_\mu+m_1)(\gamma^\nu\partial_\nu+m_2)\psi,\quad \bar{\xi}_1=\bar{\psi}(\overleftarrow{\partial}_\mu\gamma^\mu-m_2)(\overleftarrow{\partial}_\nu\gamma^\nu-m_3),\\ \bar{\xi}_2&=\bar{\psi}(\overleftarrow{\partial}_\mu\gamma^\mu-m_1)(\overleftarrow{\partial}_\nu\gamma^\nu-m_3),\quad \bar{\xi}_3=\bar{\psi}(\overleftarrow{\partial}_\mu\gamma^\mu-m_1)(\overleftarrow{\partial}_\nu\gamma^\nu-m_2)\end{aligned} \tag{4.5}$$

and the corresponding action functional is given by

$$S=\int d^4x\left[\bar{\xi}_1(\gamma^\mu\partial_\mu+m_1)\xi_1+\bar{\xi}_2(\gamma^\mu\partial_\mu+m_2)\xi_2+\bar{\xi}_3(\gamma^\mu\partial_\mu+m_3)\xi_3\right] \tag{4.6}$$

which determines the equations of motion of the following form

$$(\gamma^\mu\partial_\mu+m_i)\xi_i=0,\qquad \bar{\xi}_i(\overleftarrow{\partial}_\mu\gamma^\mu-m_i)=0,\qquad i=1,2,3 \tag{4.7}$$

as pointed out in the previous section, the relations (4.5) establish a one-to-one correspondence between solutions of the systems (4.1) and (4.6). Taking into account of the Noether theorem, the conserved currents $J^\mu(\xi_i, \bar{\xi}_i)$ can be derived from

$$\partial_\mu J^\mu(\xi_i, \bar{\xi}_i) = -\varepsilon^\mu \frac{\delta S_1}{\delta \xi_i}\partial_\mu \xi_i - \varepsilon^\mu \partial_\mu \bar{\xi}_i \frac{\delta S_1}{\delta \bar{\xi}_i} \tag{4.8}$$

which leads to

$$J^\mu(\xi_i, \bar{\xi}_i) = \frac{1}{2}(\bar{\xi}_i \gamma^\mu \partial_\nu \xi_i \varepsilon^\nu - \partial_\nu \bar{\xi}_i \gamma^\mu \xi_i \varepsilon^\nu - \bar{\xi}_i \gamma^\sigma \partial_\sigma \xi_i \varepsilon^\mu + \partial_\sigma \bar{\xi}_i \gamma^\sigma \xi_i \varepsilon^\mu - 2m_i \bar{\xi}_i \xi_i \varepsilon^\mu) \tag{4.9}$$

together with the energy–momentum tensors

$$\Theta^\mu_\nu(\xi_i, \bar{\xi}_i) = \frac{1}{2}(\bar{\xi}_i \gamma^\mu \partial_\nu \xi_i - \partial_\nu \bar{\xi}_i \gamma^\mu \xi_i - \delta^\mu_\nu \bar{\xi}_i \gamma^\sigma \partial_\sigma \xi_i + \delta^\mu_\nu \partial_\sigma \bar{\xi}_i \gamma^\sigma \xi_i - 2\delta^\mu_\nu m_i \bar{\xi}_i \xi_i) \tag{4.10}$$

now the bounded 3-parameter conserved quantity reads

$$\Theta^\mu_\nu = \beta_1 \Theta^\mu_\nu(\xi_1, \bar{\xi}_1) + \beta_2 \Theta^\mu_\nu(\xi_2, \bar{\xi}_2) + \beta_3 \Theta^\mu_\nu(\xi_3, \bar{\xi}_3) \tag{4.11}$$

furthermore, modulo a total derivative term which has no influence for the integral, the 00-component is given by

$$\Theta^0_0 = -\sum_{i=}^{3}\sum_{j=1}^{3} \beta_i \bar{\xi}_i (\gamma^j \partial_j + m_i)\xi_i \tag{4.12}$$

this is a standard energy form of the usual low derivative Dirac theory and making choice of $\beta_i < 0, m_i > 0$, we give the illustration of the stability of the higher derivative system of spinor fields (4.1).

4.2 Consistent interactions

Now for the purpose of gauging the Dirac spinor fields, it is helpful to introduce the auxiliary fields $Z = (Z^\alpha)$, $\bar{Z} = (\bar{Z}_\alpha)^T$ to convert the free Lagrangian (4.1) into the following form

$$\tilde{\mathcal{L}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu \bar{\psi}\gamma^\mu)(\gamma^\nu \partial_\nu)Z + \bar{Z}(\gamma^\mu \partial_\mu)\psi + \bar{\psi}(\gamma^\mu \partial_\mu)\psi + \bar{Z}Z - M\bar{\psi}\psi \tag{4.13}$$

varying the action gives us the equations of motion for $(\psi, \bar{\psi})$ and the auxiliary fields $(Z, \bar{Z})$

$$\begin{aligned} &(\gamma^\mu \partial_\mu)\psi + Z = 0, \qquad (\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu)Z + (\gamma^\mu \partial_\mu)\psi - M\psi = 0, \\ &(\partial_\mu \partial_\nu \bar{\psi})\gamma^\mu \gamma^\nu + \bar{Z} = 0, \qquad \partial_\mu \bar{Z}\gamma^\mu + \partial_\mu \bar{\psi}\gamma^\mu + M\bar{\psi} = 0 \end{aligned} \tag{4.14}$$

plugging these equations into (4.13), it is easy to return to the original Lagrangian density and in this form, the gauge symmetry transformation and rigid one-parameter symmetry are given by

$$\Delta A_\mu = \partial_\mu \lambda, \quad \Delta\psi^\alpha = -i\psi^\alpha \xi, \quad \Delta\bar{\psi}_\alpha = i\bar{\psi}_\alpha \xi, \quad \Delta Z^\alpha = -iZ^\alpha \xi, \quad \Delta\bar{Z}_\alpha = i\bar{Z}_\alpha \xi, \tag{4.15}$$

respectively, and taking into account of the conservation law

$$\partial_\mu J^\mu + \frac{\delta\tilde{\mathcal{L}}}{\delta\psi}(-i\psi) + (i\bar{\psi})\frac{\delta\tilde{\mathcal{L}}}{\delta\bar{\psi}} + \frac{\delta\tilde{\mathcal{L}}}{\delta Z}(-iZ) + (i\bar{Z})\frac{\delta\tilde{\mathcal{L}}}{\delta\bar{Z}} = 0 \tag{4.16}$$

here

$$\begin{aligned} &\frac{\delta\tilde{\mathcal{L}}}{\delta\psi} = -\partial_\mu\bar{Z}\gamma^\mu - \partial_\mu\bar{\psi}\gamma^\mu - M\bar{\psi}, \qquad \frac{\delta\tilde{\mathcal{L}}}{\delta\bar{\psi}} = (\gamma^\mu\partial_\mu)(\gamma^\nu\partial_\nu)Z + (\gamma^\mu\partial_\mu)\psi - M\psi, \\ &\frac{\delta\tilde{\mathcal{L}}}{\delta Z} = (\partial_\mu\partial_\nu\bar{\psi})\gamma^\mu\gamma^\nu + \bar{Z}, \qquad \frac{\delta\tilde{\mathcal{L}}}{\delta\bar{Z}} = (\gamma^\mu\partial_\mu)\psi + Z \end{aligned} \tag{4.17}$$

we gain the conserved current

$$J^\mu = i\left[(\partial_\nu\bar{\psi}\gamma^\nu)\gamma^\mu Z - (\bar{\psi}\gamma^\mu)(\gamma^\nu\partial_\nu Z) - \bar{Z}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\psi\right] \tag{4.18}$$

as a consequence, the first-order deformation takes the form of

$$\begin{aligned} S_1 = \int d^4x (&i(\bar{\psi}\bar{\psi}^* - \psi^*\psi + \bar{Z}\bar{Z}^* - Z^*Z)\eta + i((\partial_\nu\bar{\psi}\gamma^\nu)\gamma^\mu Z \\ &- (\bar{\psi}\gamma^\mu)(\gamma^\nu\partial_\nu Z) - \bar{Z}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\psi)A_\mu) \end{aligned} \tag{4.19}$$

from this formula it is readily seen that

$$\begin{aligned} (S_1, S_1) =& -2\int d^4x((\partial_\nu\bar{\psi}\gamma^\nu)\gamma^\mu Z A_\mu + \bar{\psi}\gamma^\nu\gamma^\mu\partial_\nu(ZA_\mu) + (\bar{\psi}\gamma^\mu)(\gamma^\nu\partial_\nu Z)A_\mu \\ &+ \partial_\nu(\bar{\psi}A_\mu)\gamma^\mu\gamma^\nu Z)\eta \\ =& -2s\left(\int d^4x\bar{\psi}\gamma^\nu\gamma^\mu Z A_\mu A_\nu\right) \end{aligned} \tag{4.20}$$

hence we assert

$$S_2 = \int d^4x(\bar{\psi}\gamma^\nu\gamma^\mu Z A_\mu A_\nu) \tag{4.21}$$

besides, one can examine $(S_1, S_2) = 0$ which leads to $S_i = 0$ for $i \geq 3$. Assembling all of these together, we get the total consistent expression of the deformed master action

$$\begin{aligned} S = \int d^4x \Big(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\bar{\psi}\gamma^\mu)(\gamma^\nu\partial_\nu)Z + \bar{Z}(\gamma^\mu\partial_\mu)\psi + \bar{\psi}(\gamma^\mu\partial_\mu)\psi + \bar{Z}Z \\ &-M\bar{\psi}\psi + g(i(\bar{\psi}\bar{\psi}^* - \psi^*\psi + \bar{Z}\bar{Z}^* - Z^*Z)\eta + i((\partial_\nu\bar{\psi}\gamma^\nu)\gamma^\mu Z \\ &-(\bar{\psi}\gamma^\mu)(\gamma^\nu\partial_\nu Z) - \bar{Z}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\psi)A_\mu) + g^2(\bar{\psi}\gamma^\nu\gamma^\mu Z A_\mu A_\nu)\Big) \end{aligned} \tag{4.22}$$

employing the covariant derivatives, the antighost number zero part can be organized as

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\bar{D}_\mu\bar{\psi}\gamma^\mu)(\gamma^\nu D_\nu)Z + \bar{Z}(\gamma^\mu D_\mu)\psi + \bar{\psi}(\gamma^\mu D_\mu)\psi + \bar{Z}Z - M\bar{\psi}\psi \tag{4.23}$$

by variation with respect to Z and $\bar{Z}$, we simply deduce the algebraic equations of motion of auxiliary fields

$$(\gamma^\mu D_\mu)\psi + Z = 0, \qquad \bar{D}_\nu(\bar{D}_\mu\bar{\psi}\gamma^\mu)\gamma^\nu + \bar{Z} = 0 \tag{4.24}$$

and inserting these into (4.23), we immediately obtain the following equivalent Lagrangian density

$$\tilde{\mathcal{L}}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\gamma^{\mu}D_{\mu})\psi + (\bar{D}_{\mu}\bar{\psi}\gamma^{\mu})(\gamma^{\nu}D_{\nu})(\gamma^{\mu}D_{\mu})\psi - M\bar{\psi}\psi \tag{4.25}$$

with the local gauge transformations

$$A_{\mu} \to A'_{\mu} = A_{\mu} + \partial_{\mu}\lambda, \quad \psi \to \psi' = e^{-ig\lambda}\psi, \quad \bar{\psi} =\to \bar{\psi}' = e^{ig\lambda}\bar{\psi} \tag{4.26}$$

in the same spirit, the above gauge symmetry in the resulting Lagrangian comes from the rigid invariance of the spinor fields by gaugings and through replacing the ordinary partial derivative ∂_{μ} by the covariant derivatives D_{μ}, $\bar{D}_{\mu}$ both in first and higher second derivative terms of the matter field, we are able to acquire the consistent interactions that can be put into the original theory and are compatible with the BRST deformations procedure.

Finally, we address the problem of the stability of this higher derivative coupling system (4.25) along the lines of the previous section. At beginning, after some algebraic manipulations (ignore the pure gauge part), we arrive at the equations of motion

$$\begin{aligned}
\frac{\delta S}{\delta \psi} &= -\bar{\psi}(\overleftarrow{\bar{D}}_{\mu}\gamma^{\mu}) + \bar{\psi}(\overleftarrow{\bar{D}}_{\mu}\gamma^{\mu})^3 - M\bar{\psi} \\
&= \bar{\psi}(\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_1)(\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_2)(\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_3) = 0, \\
\frac{\delta S}{\delta \bar{\psi}} &= \gamma^{\mu}D_{\mu}\psi - (\gamma^{\mu}D_{\mu})^3\psi - M\psi \\
&= -(\gamma^{\mu}D_{\mu} + m_1)(\gamma^{\mu}D_{\mu} + m_2)(\gamma^{\mu}D_{\mu} + m_3)\psi = 0, \\
\frac{\delta S}{\delta A_{\mu}} &= ig\Big[-\bar{\psi}\gamma^{\mu}\psi + \bar{\psi}(\gamma^{\nu}\overleftarrow{\bar{D}}_{\nu})^2\gamma^{\mu}\psi - (\bar{D}_{\omega}\bar{\psi})\gamma^{\omega}\gamma^{\mu}\gamma^{\nu}D_{\nu}\psi \\
&\quad + \bar{\psi}\gamma^{\mu}(\gamma^{\nu}D_{\nu})^2\psi\Big] = 0
\end{aligned} \tag{4.27}$$

similarly, the new dynamic fields can be chosen in the form of

$$\begin{aligned}
&\xi_1 = (\gamma^{\mu}D_{\mu} + m_2)(\gamma^{\nu}D_{\nu} + m_3)\psi, \qquad \xi_2 = (\gamma^{\mu}D_{\mu} + m_1)(\gamma^{\nu}D_{\nu} + m_3)\psi, \\
&\xi_3 = (\gamma^{\mu}D_{\mu} + m_1)(\gamma^{\nu}D_{\nu} + m_2)\psi, \qquad \bar{\xi}_1 = \bar{\psi}(\overleftarrow{\bar{D}}_{\mu}\gamma^{\mu} - m_2)(\overleftarrow{\bar{D}}_{\nu}\gamma^{\nu} - m_3), \\
&\bar{\xi}_2 = \bar{\psi}(\overleftarrow{\bar{D}}_{\mu}\gamma^{\mu} - m_1)(\overleftarrow{\bar{D}}_{\nu}\gamma^{\nu} - m_3), \qquad \bar{\xi}_3 = \bar{\psi}(\overleftarrow{\bar{D}}_{\mu}\gamma^{\mu} - m_1)(\overleftarrow{\bar{D}}_{\nu}\gamma^{\nu} - m_2)
\end{aligned} \tag{4.28}$$

then we introduce the following collection of differential operators

$$\begin{aligned}
&\mathcal{P}_1 = \frac{\gamma^{\mu}D_{\mu} + m_1}{(m_1 - m_2)(m_1 - m_3)}, \quad \mathcal{P}_2 = \frac{\gamma^{\mu}D_{\mu} + m_2}{(m_2 - m_1)(m_2 - m_3)}, \\
&\mathcal{P}_3 = \frac{\gamma^{\mu}D_{\mu} + m_3}{(m_3 - m_1)(m_3 - m_2)}, \\
&\mathcal{Q}_1 = \frac{\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_1}{(m_1 - m_2)(m_1 - m_3)}, \quad \mathcal{Q}_2 = \frac{\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_2}{(m_2 - m_1)(m_2 - m_3)}, \\
&\mathcal{Q}_3 = \frac{\gamma^{\mu}\overleftarrow{\bar{D}}_{\mu} - m_3}{(m_3 - m_1)(m_3 - m_2)}
\end{aligned} \tag{4.29}$$

and the action functional acquires the standard formulation

$$S_1[\xi_i(x), \bar{\xi}_i(x), A_{\nu}] = \int (\bar{\xi}_1\mathcal{P}_1\xi_1 + \bar{\xi}_2\mathcal{P}_2\xi_2 + \bar{\xi}_3\mathcal{P}_3\xi_3)d^4x \tag{4.30}$$

a direct calculation gives rise to the equations of motion

$$\frac{\delta S_1}{\delta\bar{\xi}_i} = \mathcal{P}_i\xi_i = 0, \quad \frac{\delta S_1}{\delta\xi_i} = \bar{\xi}_i\mathcal{Q}_i = 0, \qquad i = 1, 2, 3$$
$$\frac{\delta S_1}{\delta A_\mu} = ig\left(\frac{\bar{\xi}_1\gamma^\mu\xi_1}{(m_1-m_2)(m_1-m_3)} + \frac{\bar{\xi}_2\gamma^\mu\xi_2}{(m_2-m_3)(m_2-m_1)}\right.$$
$$\left. + \frac{\bar{\xi}_3\gamma^\mu\xi_3}{(m_3-m_1)(m_3-m_2)}\right) = 0 \tag{4.31}$$

at this stage, taking advantage of the relations

$$m_1 + m_2 + m_3 = 0, \quad m_1m_2 + m_2m_3 + m_3m_1 = -1, \quad m_1m_2m_3 = M \tag{4.32}$$

one can guarantee that the above two systems are equivalent and the solutions of the dynamic equations are related by the formulae (4.28). In the same manner, the conserved currents $J^\mu(\xi_i, \bar{\xi}_i)$ in this case can be found by the following receipt

$$J^\mu(\xi_i, \bar{\xi}_i) = \frac{1}{2}\prod_{j\neq i, j=1}^{3}\frac{1}{m_i - m_j}(\bar{\xi}_i\gamma^\mu D_\nu\xi_i\varepsilon^\nu - \bar{D}_\nu\bar{\xi}_i\gamma^\mu\xi_i\varepsilon^\nu - \bar{\xi}_i\gamma^\sigma D_\sigma\xi_i\varepsilon^\mu$$
$$+ \bar{D}_\sigma\bar{\xi}_i\gamma^\sigma\xi_i\varepsilon^\mu - 2m_i\bar{\xi}_i\xi_i\varepsilon^\mu) \tag{4.33}$$

with the aid of these expressions, we are able to formulate the energy–momentum tensors as

$$\Theta^\mu_\nu(\xi_i, \bar{\xi}_i) = \frac{1}{2}\prod_{j\neq i, j=1}^{3}\frac{1}{m_i - m_j}(\bar{\xi}_i\gamma^\mu D_\nu\xi_i - \bar{D}_\nu\bar{\xi}_i\gamma^\mu\xi_i - \delta^\mu_\nu\bar{\xi}_i\gamma^\sigma D_\sigma\xi_i$$
$$+ \delta^\mu_\nu\bar{D}_\sigma\bar{\xi}_i\gamma^\sigma\xi_i - 2\delta^\mu_\nu m_i\bar{\xi}_i\xi_i) \tag{4.34}$$

in particular, modulo a total derivative term and by introducing a series of parameters β_i, the 00-component of the conserved quantity can be written in the form of

$$\Theta^0_0 = -\sum_{i=1}^{3}\prod_{j\neq i, j=1}^{3}\frac{\beta_i}{m_i - m_j}\sum_{k=1}^{3}\bar{\xi}_i(\gamma^k D_k + m_i)\xi_i \tag{4.35}$$

now under the assumption

$$-\prod_{j\neq i, j=1}^{3}\frac{\beta_i}{m_i - m_j} > 0, \qquad m_i > 0, \qquad i = 1, 2, 3 \tag{4.36}$$

we confirm the stability of the coupling system between the gauge field and Dirac spinor fields with higher derivative term.

5 Conclusion and discussion

In this paper, we investigate the stability of the class of higher derivative matter field theories from the viewpoint of the n-parameter series of conserved quantities, which can be connected with the spacetime translations by appropriate Lagrange anchor. With the aid of factorization, it is possible to build up such conserved quantities that might be positive and bounded while the canonical energy usually is not positive definite for the real, complex and Dirac free systems. Then we mainly learn the consistent couplings between Abelian gauge field and the higher derivative matter fields. In our construction, the key idea is that by means of the auxiliary fields, we are capable of reducing the higher-order term to the low one which is more convenient to handle. Following the standard procedure in the BRST deformations, we acquire the correction terms at different orders by solving the deformation master equations. Next, extracting the antighost number zero part in the deformed master action and making use of the equations of motion of the auxiliary fields, we can convert this action into an equivalent one with higher derivative terms. By a comparison with the original free Lagrangian, it is easy to obtain the consistent interactions order by order. After that, we apply the factorization method again to study the classical stability in the higher derivative coupling systems with consistent interactions and in fact, such stability is still true at the quantum level. The next generalization of this work is of course the analysis of the stability and the construction of BRST deformations in the non-Abelian case. More specifically, we introduce a set of U(1) gauge fields together with a collection of matter fields and now that the Lagrangian may include the Podolsky's generalized electrodynamics which is a natural extension of the ordinary Maxwell's dynamics. Following the same lines in our discussion and through the analysis of the local BRST cohomology, we are able to derive the consistent interactions between gauge and matter fields expressed in terms of the non-Abelian curvatures and covariant derivatives in the free systems and the new feature will arise from the effect of the self-interactions among the gauge fields. Furthermore, the stabilities in these free and interacting systems can be demonstrated following the discussions of this paper in a similar way. The main difference comes from an additional nonlinear term included into the factorization equation which represents the self-interactions of gauge fields. Now as long as both the free factors in the 00-component of energy–momentum tensors are stable, even if this self-interaction term is not positive definite, the energy can still have a local minimum in a neighborhood of zero solution and such theories are also considered as physically acceptable models. By this reason, the factorization method can still be efficient for keeping track of stability in the non-Abelian coupling higher derivative dynamics. All of these would be interesting to exploit in future.

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