




Determining lump solutions for a combined soliton equation in (2+1)-dimensions

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Abstract

We consider a combined soliton equation involving three fourth-order nonlinear terms in (2+1)-dimensional dispersive waves and determine its lump solutions via symbolic computations. The combined equation is transformed into a Hirota bilinear equation under a logarithmic transformation and its lump solutions are computed explicitly in two cases of the coefficients in the model. Illustrative examples are presented, together with three-dimensional plots and contour plots of two specific lump solutions.

1 Introduction

Soliton theory provides effective methods to solve nonlinear partial differential equations (PDEs) [1, 2], including soliton equations generated from zero curvature equations. The Hirota bilinear method is particularly powerful in constructing soliton solutions [3, 4]. Those solutions are analytic and exponentially localized. Let a polynomial P determine a Hirota bilinear differential equation

$$P(D_x, D_y, D_t) f \cdot f = 0$$

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in (2+1)-dimensions, where D_x, D_y and D_t are Hirota’s bilinear derivatives [3]. The corresponding PDE with a dependent variable u is usually determined through one of the logarithmic transformations: $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$. Based on the Hirota bilinear form, the N -soliton solution is formulated as follows:

$$f = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \xi_i + \sum_{i < j} \mu_i \mu_j a_{ij} \right),$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for $\mu_1, \mu_2, \dots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are given by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \leq i \leq N,$$

and

$$e^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \leq i < j \leq N,$$

in which k_i, l_i and $\omega_i, 1 \leq i \leq N$, satisfy the corresponding dispersion relation and $\xi_{i,0}, 1 \leq i \leq N$, are arbitrary phase shifts.

Lump solutions are rational solutions, which are analytic and localized in all directions in space (see, e.g., [5–7]), and they can often be obtained from computing long wave limits of soliton equations (see, e.g., [8]). Various recent studies on (2+1)-dimensional soliton equations show the striking richness of lump solutions (see, e.g., [5,6]), which describe various dispersive wave phenomena. The KPI equation possesses diverse lump solutions (see, e.g., [9]), and special lump solutions are derived from soliton solutions [10]. Other soliton equations which possess lump solutions contain the three-dimensional three-wave resonant interaction [11], the BKP equation [12,13], the Davey-Stewartson II equation [8], the Ishimori-I equation [14], the KPI equation with a self-consistent source [15] and the mKPI equation [16]. A crucial step in constructing lump solutions is to determine positive quadratic function solutions to Hirota bilinear equations [5]. Then lump solutions to nonlinear PDEs are generated through the logarithmic transformations from positive quadratic function solutions.

In this paper, we would like to consider a combined fourth-order soliton equation in (2+1)-dimensional dispersive waves and determine its abundant lump solution structures. The Hirota bilinear form plays a key role in our discussion (see, e.g., [5,6,17,18]). We will formulate a combined fourth-order soliton equation including three fourth-order nonlinear terms and all second-order linear terms. To present lump solutions with Maple symbolic computations, we will discuss two cases of the combined model. Illustrative examples of the considered model will be presented and some three-dimensional plots and contour plots will be made for two particular lump solutions by using Maple. A conclusion and some remarks will be given at the end of the paper.

2 A combined soliton equation

We would like to consider a general combined fourth-order soliton equation as follows:

$$P(u) = \alpha(6u_x u_{xx} + u_{xxx}) + \beta[3(u_x u_t)_x + u_{xxx}] + \gamma[3(u_x u_y)_x + u_{xxx}] + \delta_1 u_{yt} + \delta_2 u_{xx} + \delta_3 u_{xt} + \delta_4 u_{xy} + \delta_5 u_{yy} + \delta_6 u_{tt} = 0, \tag{2.1}$$

where the constants α, β and γ are not all zero, but the constants $\delta_i, 1 \leq i \leq 6$, are all arbitrary. This equation contains three fourth-order nonlinear terms and all second-order linear terms, and it generalizes the standard KP equation.

Upon taking $\alpha = 0, \beta = 1$ and $\gamma = 0$, and $\delta_1 = \delta_2 = 1$ and the other δ_i 's as zero, we obtain an integrable $(2 + 1)$ -dimensional extension of the Hirota-Satsuma equation [3], namely the Hirota-Satsuma-Ito (HSI) equation in $(2+1)$ -dimensions [19]:

$$3(u_x u_t)_x + u_{xxx} + u_{yt} + u_{xx} = 0, \tag{2.2}$$

which satisfies the Hirota three-soliton condition and possesses a Hirota bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$(D_x^3 D_t + D_y D_t + D_x^2) f \cdot f = 0. \tag{2.3}$$

This equation is called the bilinear HSI equation.

Upon taking $\alpha = 0, \beta = 0$ and $\gamma = 1$, and $\delta_3 = \delta_5 = 1$ and the other δ_i 's as zero, we obtain a generalized Calogero–Bogoyavlenskii–Schiff equation [20]:

$$3(u_x u_y)_x + u_{xxy} + u_{xt} + u_{yy} = 0, \tag{2.4}$$

which also possesses a Hirota bilinear form

$$(D_x^3 D_y + D_x D_t + D_y^2) f \cdot f = 0, \tag{2.5}$$

under $u = 2(\ln f)_x$, and whose lump solutions have been computed in [20].

Upon taking $\alpha = 1, \beta = 0$ and $\gamma = 1$, and $\delta_2 = \delta_3 = \delta_5 = 1$ and the other δ_i 's as zero, we obtain a generalized Bogoyavlensky–Konopelchenko equation [21]:

$$6u_x u_{xx} + u_{xxx} + 3(u_x u_y)_x + u_{xxy} + u_{xt} + u_{xx} + u_{yy} = 0, \tag{2.6}$$

whose Hirota bilinear form is given by

$$(D_x^4 + D_x^3 D_y + D_x D_t + D_x^2 + D_y^2) f \cdot f = 0, \tag{2.7}$$

under $u = 2(\ln f)_x$. It has lump solutions as well [21].

The general combined soliton equation (2.1) also presents a generalization of the nonlinear soliton equation in [22], and it has a Hirota bilinear form

$$B(f) = (\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + \delta_1 D_y D_t + \delta_2 D_x^2 + \delta_3 D_x D_t + \delta_4 D_x D_y + \delta_5 D_y^2 + \delta_6 D_t^2) f \cdot f = 0, \tag{2.8}$$

under the logarithmic transformation

$$u = 2(\ln f)_x = \frac{2f_x}{f}. \tag{2.9}$$

In fact, we have the relation between the nonlinear and bilinear equations: $P(u) = (\frac{B(f)}{f^2})_x$, when u and f satisfy the link (2.9).

3 Lump solutions

In this section, we would like to determine lump solutions to the combined fourth-order soliton equation (2.1) in $(2+1)$ -dimensions, through symbolic computations with Maple.

Let us begin with positive quadratic solutions to the combined Hirota bilinear equation (2.8):

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{3.1}$$

where the constant parameters $a_i, 1 \leq i \leq 9$, are to be determined, to present lump solutions to the combined fourth-order soliton equation (2.1). This is a general ansatz on lump solutions in (2+1)-dimensions, generated from quadratic functions [9].

3.1 The case of $\delta_6 = 0$

Let us first consider the case of $\delta_6 = 0$ for the combined soliton equation (2.1). A direct symbolic computation determines a set of solutions for the parameters, where

$$\begin{cases} a_3 = -\frac{b_1}{(a_2\delta_1 + a_1\delta_3)^2 + (a_6\delta_1 + a_5\delta_3)^2}, \\ a_7 = -\frac{b_2}{(a_2\delta_1 + a_1\delta_3)^2 + (a_6\delta_1 + a_5\delta_3)^2}, \\ a_9 = -\frac{3(a_1^2 + a_5^2)(\alpha b_3 - \beta b_4 + \gamma b_5)}{(a_1a_6 - a_2a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_3^2\delta_5)}, \end{cases} \tag{3.2}$$

and all other a_i 's are arbitrary. The involved five constants $b_i, 1 \leq i \leq 5$, are given by

$$\begin{cases} b_1 = [(a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2)\delta_2 + a_1(a_2^2 + a_6^2)\delta_4 + a_2(a_2^2 + a_6^2)\delta_5]\delta_1 \\ \quad + [a_1(a_1^2 + a_5^2)\delta_2 + a_2(a_1^2 + a_5^2)\delta_4 + (a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2)\delta_5]\delta_3, \\ b_2 = [(-a_1^2a_6 + 2a_1a_2a_5 + a_5^2a_6)\delta_2 + a_5(a_2^2 + a_6^2)\delta_4 + a_6(a_2^2 + a_6^2)\delta_5]\delta_1 \\ \quad + [a_5(a_1^2 + a_5^2)\delta_2 + a_6(a_1^2 + a_5^2)\delta_4 + (-a_2^2a_5 + 2a_1a_2a_6 + a_5a_6^2)\delta_5]\delta_3, \\ b_3 = (a_1^2 + a_5^2)[(a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2], \\ b_4 = (a_1^2 + a_5^2)(a_1a_2 + a_5a_6)(\delta_1\delta_2 + \delta_3\delta_4) + (a_1^2 + a_5^2)(a_2^2 + a_6^2)\delta_1\delta_4 \\ \quad + (a_1^2 + a_5^2)^2\delta_2\delta_3 + (a_2^2 + a_6^2)(a_1a_2 + a_5a_6)\delta_1\delta_5 \\ \quad + [(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2]\delta_3\delta_5, \\ b_5 = (a_1a_2 + a_5a_6)[(a_2\delta_1 + a_1\delta_3)^2 + (a_6\delta_1 + a_5\delta_3)^2]. \end{cases} \tag{3.3}$$

The expressions of a_3 and a_7 present diverse dispersion relations in (2+1)-dimensional dispersive waves.

3.2 The case of $\delta_5 = 0$

Let us second consider the case of $\delta_5 = 0$ for the combined nonlinear equation (2.1). A similar direct symbolic computation determines a set of solutions for the parameters, where

$$\begin{cases} a_2 = -\frac{c_1}{(a_3\delta_1 + a_1\delta_4)^2 + (a_7\delta_1 + a_5\delta_4)^2}, \\ a_6 = -\frac{c_2}{(a_3\delta_1 + a_1\delta_4)^2 + (a_7\delta_1 + a_5\delta_4)^2}, \\ a_9 = -\frac{3(a_1^2 + a_5^2)(\alpha c_3 - \gamma c_4 + \beta c_5)}{(a_1a_7 - a_3a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_4^2\delta_6)}, \end{cases} \tag{3.4}$$

and all other a_i 's are arbitrary. The involved five constants c_i , $1 \leq i \leq 5$, are given by

$$\left\{ \begin{aligned} c_1 &= [(a_1^2 a_3 + 2 a_1 a_5 a_7 - a_3 a_5^2) \delta_2 + a_1 (a_3^2 + a_7^2) \delta_3 + a_3 (a_3^2 + a_7^2) \delta_6] \delta_1 \\ &\quad + [a_1 (a_1^2 + a_5^2) \delta_2 + a_3 (a_1^2 + a_5^2) \delta_3 + (a_1 a_3^2 + 2 a_3 a_5 a_7 - a_1 a_7^2) \delta_6] \delta_4, \\ c_2 &= [(-a_1^2 a_7 + 2 a_1 a_3 a_5 + a_5^2 a_7) \delta_2 + a_5 (a_3^2 + a_7^2) \delta_3 + a_7 (a_3^2 + a_7^2) \delta_6] \delta_1 \\ &\quad + [a_5 (a_1^2 + a_5^2) \delta_2 + a_7 (a_1^2 + a_5^2) \delta_3 + (-a_3^2 a_5 + 2 a_1 a_3 a_7 + a_5 a_7^2) \delta_6] \delta_4, \\ c_3 &= (a_1^2 + a_5^2) [(a_1 \delta_4 + a_3 \delta_1)^2 + (a_5 \delta_4 + a_7 \delta_1)^2], \\ c_4 &= (a_1^2 + a_5^2) (a_1 a_3 + a_5 a_7) (\delta_1 \delta_2 + \delta_3 \delta_4) + (a_1^2 + a_5^2) (a_3^2 + a_7^2) \delta_1 \delta_3 \\ &\quad + (a_1^2 + a_5^2)^2 \delta_2 \delta_4 + (a_3^2 + a_7^2) (a_1 a_3 + a_5 a_7) \delta_1 \delta_6 \\ &\quad + [(a_1 a_3 + a_5 a_7)^2 - (a_1 a_7 - a_3 a_5)^2] \delta_4 \delta_6, \\ c_5 &= (a_1 a_3 + a_5 a_7) [(a_3 \delta_1 + a_1 \delta_4)^2 + (a_7 \delta_1 + a_5 \delta_4)^2]. \end{aligned} \right. \tag{3.5}$$

All the above expressions for wave frequencies and wave numbers in (3.2), (3.3), (3.4) and (3.5) have been obtained with some direct simplifications with Maple. Based on those solution expressions, we need two basic conditions:

$$\delta_1^2 + \delta_3^2 \neq 0, \tag{3.6}$$

in the case of $\delta_6 = 0$, and

$$\delta_1^2 + \delta_4^2 \neq 0, \tag{3.7}$$

in the case of $\delta_5 = 0$, to present lump solutions.

In the case of $\delta_5 = 0$, to check when the set of the resulting parameters presents lumps, we work out

$$\begin{aligned} &a_1 a_6 - a_2 a_5 \\ &= \frac{(a_1 a_7 - a_3 a_5) [(a_1^2 + a_5^2) (\delta_1 \delta_2 - \delta_3 \delta_4) - (a_3^2 + a_7^2) \delta_1 \delta_6 - 2(a_1 a_3 + a_5 a_7) \delta_4 \delta_6]}{(a_3 \delta_1 + a_1 \delta_4)^2 + (a_7 \delta_1 + a_5 \delta_4)^2}. \end{aligned}$$

Therefore, we can see that the condition $a_1 a_6 - a_2 a_5 \neq 0$, which guarantees the existence of lumps, holds if and only if besides (3.7), the following two additional conditions are satisfied:

$$\left\{ \begin{aligned} &a_1 a_7 - a_3 a_5 \neq 0, \\ &(a_1^2 + a_5^2) (\delta_1 \delta_2 - \delta_3 \delta_4) - (a_3^2 + a_7^2) \delta_1 \delta_6 - 2(a_1 a_3 + a_5 a_7) \delta_4 \delta_6 \neq 0. \end{aligned} \right. \tag{3.8}$$

Together with $a_9 > 0$, these three conditions ensure that the corresponding set of the parameters will lead to lump solutions.

4 Illustrative examples of the combined model

We present diverse examples of the considered combined soliton equation, on the base of the presented solution expressions above in the two cases of solutions.

4.1 The case of $\delta_1 = \delta_2 = 1$

Let $\delta_1 = \delta_2 = 1$ and the other δ_i 's be zero. Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_y D_t + D_x^2) f \cdot f = 0. \tag{4.1}$$

The subcase of $\alpha = \beta = 0$ and $\gamma = 1$ gives the dimensionally reduced Jimbo–Miwa equation with $z = x$ [23].

The subcase of $\beta = 1$ and $\alpha = \gamma = 0$ gives us the original HSI equation in (2+1)-dimensions (2.2). The two solution classes in this case are equivalent to each other. The function f by (3.1) with (3.2) and (3.3) presents a class of lump solutions to the HSI equation (2.2), where

$$\begin{cases} a_3 = -\frac{a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2}{a_2^2 + a_6^2}, \\ a_7 = \frac{a_1^2 a_6 - 2a_1 a_2 a_5 - a_5^2 a_6}{a_2^2 + a_6^2}, \\ a_9 = \frac{3(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \end{cases} \tag{4.2}$$

and all other a_i 's are arbitrary; and the function f by (3.1) with (3.4) and (3.5) presents another class of lump solutions to the HSI equation (2.2), where [24]

$$\begin{cases} a_2 = -\frac{a_1^2 a_3 + 2a_1 a_5 a_7 - a_3 a_5^2}{a_3^2 + a_7^2}, \\ a_6 = \frac{a_1^2 a_7 - 2a_1 a_3 a_5 - a_5^2 a_7}{a_3^2 + a_7^2}, \\ a_9 = -\frac{3(a_1^2 + a_5^2)(a_3^2 + a_7^2)(a_1 a_3 + a_5 a_7)}{(a_1 a_7 - a_3 a_5)^2}. \end{cases} \tag{4.3}$$

Obviously, we see that

$$a_1 a_3 + a_5 a_7 = -\frac{(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{a_2^2 + a_6^2}, \tag{4.4}$$

and

$$a_1 a_6 - a_2 a_5 = \frac{(a_1^2 + a_5^2)(a_1 a_7 - a_3 a_5)}{a_3^2 + a_7^2}. \tag{4.5}$$

Therefore, the conditions of

$$a_1 a_2 + a_5 a_6 > 0, \quad a_1 a_6 - a_2 a_5 \neq 0, \tag{4.6}$$

under which f by (3.1) with (4.2) will present lump solutions to (2.2), are equivalent to the conditions of

$$a_1 a_3 + a_5 a_7 < 0, \quad a_1 a_7 - a_3 a_5 \neq 0, \tag{4.7}$$

under which f by (3.1) with (4.3) will present lump solutions to (2.2). To conclude, the two classes of lump solutions presented in the two solution cases are the same. They can be derived from each other.

4.2 The case of $\delta_3 = \delta_5 = 1$

Let $\delta_3 = \delta_5 = 1$ and the other δ_i 's be zero. Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_x D_t + D_y^2) f \cdot f = 0. \tag{4.8}$$

The subcase of $\alpha = \beta = 1$ and $\gamma = 0$ provides a new model, for which a specific lump will be presented in the next section. The subcase of $\alpha = \beta = 0$ and $\gamma = 1$ is the generalized Calogero–Bogoyavlenskii–Schiff equation discussed previously in [20].

4.3 The case of $\delta_4 = \delta_6 = 1$

Let $\delta_4 = \delta_6 = 1$ and the other δ_i 's be zero. Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_x D_y + D_t^2) f \cdot f = 0. \tag{4.9}$$

The condition $a_1 a_6 - a_2 a_5 \neq 0$ for the existence of lumps requires

$$a_1 a_7 - a_3 a_5 \neq 0, \quad a_1 a_3 + a_5 a_7 \neq 0; \tag{4.10}$$

and thus, $a_9 > 0$ requires

$$\alpha(a_1^2 + a_5^2)^2 + \beta(a_1 a_3 + a_5 a_7)(a_1^2 + a_5^2) + \gamma[(a_1 a_7 - a_3 a_5)^2 - (a_1 a_3 + a_5 a_7)^2] < 0. \tag{4.11}$$

Therefore, we see that if $\beta = \gamma = 0$, we need to require $\alpha < 0$ to present lumps. A specific lump in the subcase of $\alpha = \beta = \gamma = 1$ will be computed and plotted in the next section.

4.4 The case of $\delta_1 = \delta_3 = \delta_4 = 1$

Let $\delta_1 = \delta_3 = \delta_4 = 1$ and the other δ_i 's be zero. Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_y D_t + D_x D_t + D_x D_y) f \cdot f = 0, \tag{4.12}$$

which generate two new combined soliton equations possessing lump solutions, when $\alpha = \beta = 1$ and $\gamma = 0$ or when $\beta = 0$ and $\alpha = \gamma = 1$.

Two classes of lump solutions determined by (3.2) with (3.3) and (3.4) with (3.5) are essentially equivalent to each other. In other words, one can be generated from the other as in the first case. The first class of parameters by (3.2) with (3.3) reads

$$\begin{cases} a_3 = -\frac{a_1(a_2^2 + a_6^2) + a_2(a_1^2 + a_5^2)}{(a_1 + a_2)^2 + (a_5 + a_6)^2}, \\ a_7 = -\frac{a_5(a_2^2 + a_6^2) + a_6(a_1^2 + a_5^2)}{(a_1 + a_2)^2 + (a_5 + a_6)^2}, \\ a_9 = \frac{3(a_1^2 + a_5^2)(\alpha b_3 - \beta b_4 + \gamma b_5)}{(a_1 a_6 - a_2 a_5)^2}, \end{cases} \tag{4.13}$$

where

$$\begin{cases} b_3 = (a_1^2 + a_5^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2], \\ b_4 = (a_1^2 + a_5^2)[(a_1 a_2 + a_5 a_6) + (a_2^2 + a_6^2)], \\ b_5 = (a_1 a_2 + a_5 a_6)[(a_1 + a_2)^2 + (a_5 + a_6)^2]. \end{cases} \tag{4.14}$$

The second class of parameters by (3.4) with (3.5) reads

$$\begin{cases} a_2 = -\frac{a_1(a_3^2 + a_7^2) + a_3(a_1^2 + a_5^2)}{(a_1 + a_3)^2 + (a_5 + a_7)^2}, \\ a_6 = -\frac{a_5(a_3^2 + a_7^2) + a_7(a_1^2 + a_5^2)}{(a_1 + a_3)^2 + (a_5 + a_7)^2}, \\ a_9 = \frac{3(a_1^2 + a_5^2)(\alpha c_3 - \gamma c_4 + \beta c_5)}{(a_1 a_7 - a_3 a_5)^2}, \end{cases} \tag{4.15}$$

where

$$\begin{cases} c_3 = (a_1^2 + a_5^2)[(a_1 + a_3)^2 + (a_5 + a_7)^2], \\ c_4 = (a_1^2 + a_5^2)[(a_1 a_3 + a_5 a_7) + (a_3^2 + a_7^2)], \\ c_5 = (a_1 a_3 + a_5 a_7)[(a_1 + a_3)^2 + (a_5 + a_7)^2]. \end{cases} \tag{4.16}$$

These two classes of parameters are the same, since they can be solved from each other. Moreover, one has

$$a_1 a_7 - a_3 a_5 = -\frac{(a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5)}{(a_1 + a_2)^2 + (a_5 + a_6)^2},$$

and therefore, two classes present the exactly same set of values for the parameters.

4.5 The case of $\delta_1 = \delta_3 = \delta_5 = 1$

Let $\delta_1 = \delta_3 = \delta_5 = 1$ and the other δ_i 's be zero, Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_y D_t + D_x D_t + D_y^2) f \cdot f = 0. \tag{4.17}$$

Observe that $a_1 a_6 - a_2 a_5 \neq 0$ leads to $(a_1 + a_2)^2 + (a_5 + a_6)^2 \neq 0$, which ensures that a_3 and a_7 in (3.2) are well defined. Therefore, besides $a_1 a_6 - a_2 a_5 \neq 0$, the condition for guaranteeing lumps is

$$\begin{aligned} & \alpha(a_1^2 + a_5^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2] \\ & - \beta[(a_2^2 + a_6^2)(a_1 a_2 + a_5 a_6) + (a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2] \\ & + \gamma(a_1 a_2 + a_5 a_6)[(a_1 + a_2)^2 + (a_5 + a_6)^2] < 0, \end{aligned} \tag{4.18}$$

which guarantees that a_9 defined in (3.2) is positive.

4.6 The case of $\delta_1 = \delta_4 = \delta_6 = 1$

Let $\delta_1 = \delta_4 = \delta_6 = 1$ and the other δ_i 's be zero. Then the combined Hirota bilinear equation (2.8) becomes

$$(\alpha D_x^4 + \beta D_x^3 D_t + \gamma D_x^3 D_y + D_y D_t + D_x D_y + D_t^2) f \cdot f = 0. \tag{4.19}$$

Observe that $a_1 a_7 - a_3 a_5 \neq 0$ leads to $(a_1 + a_3)^2 + (a_5 + a_7)^2 \neq 0$, which ensures that a_2 and a_6 in (3.4) are well defined. Also, it is easy to see that in this case, we have

$$a_1 a_6 - a_2 a_5 = \frac{(a_1 a_7 - a_3 a_5)[(a_3^2 + a_7^2) + 2(a_1 a_3 + a_5 a_7)]}{(a_1 + a_3)^2 + (a_5 + a_7)^2}.$$

Therefore, the conditions for guaranteeing lumps are

$$a_1 a_7 - a_3 a_5 \neq 0, (a_3^2 + a_7^2) + 2(a_1 a_3 + a_5 a_7) \neq 0, \tag{4.20}$$

and

$$[\alpha(a_1^2 + a_5^2) + \beta(a_1 a_3 + a_5 a_7)][(a_1 + a_3)^2 + (a_5 + a_7)^2] - \gamma[(a_3^2 + a_7^2)(a_1 a_3 + a_5 a_7) + (a_1 a_3 + a_5 a_7)^2 - (a_1 a_7 - a_3 a_5)^2] < 0. \tag{4.21}$$

This guarantees, together with the first condition in (4.20), that a_9 defined in (3.4) is positive.

5 Two particular lumps

Let us first take

$$\alpha = 1, \beta = 1, \gamma = 0, \delta_3 = \delta_5 = 1, \delta_1 = \delta_2 = \delta_4 = \delta_6 = 0, \tag{5.1}$$

which leads to a special combined soliton equation

$$u_{xxxx} + 6u_x u_{xx} + u_{xxxt} + 3(u_x u_t)_x + u_{xt} + u_{yy} = 0. \tag{5.2}$$

This has a Hirota bilinear form

$$(D_x^4 + D_x^3 D_t + D_x D_t + D_y^2) f \cdot f = 0,$$

under the logarithmic transformation (2.9). Upon further taking

$$a_1 = 2, a_2 = -6, a_4 = 1, a_5 = 2, a_6 = -5, a_8 = 2, \tag{5.3}$$

the transformation (2.9) with (3.1) presents a lump solution to the special combined soliton equation (5.2):

$$u_1 = \frac{2(-120t + 16x - 44y + 12)}{(-\frac{71}{4}t + 2x - 6y + 1)^2 + (-\frac{49}{4}t + 2x - 5y + 2)^2 + 2496}. \tag{5.4}$$

Three-dimensional plots and contour plots of this lump solution at three different times are made by using Maple in Fig. 1.

Let us second take

$$\alpha = 1, \beta = 1, \gamma = 1, \delta_4 = \delta_6 = 1, \delta_1 = \delta_2 = \delta_3 = \delta_5 = 0, \tag{5.5}$$

which leads to another special combined soliton equation

$$u_{xxxx} + 6u_x u_{xx} + u_{xxxt} + 3(u_x u_t)_x + u_{xxxy} + 3(u_x u_y)_x + u_{xy} + u_{tt} = 0. \tag{5.6}$$

It has a Hirota bilinear form

$$(D_x^4 + D_x^3 D_t + D_x^3 D_y + D_x D_y + D_t^2) f \cdot f = 0,$$

under the logarithmic transformation (2.9). Upon further taking

$$a_1 = 2, a_3 = -6, a_4 = 5, a_5 = 2, a_7 = -8, a_8 = 3, \tag{5.7}$$

the transformation (2.9) with (3.1) gives a lump solution to the special combined soliton equation (5.6):

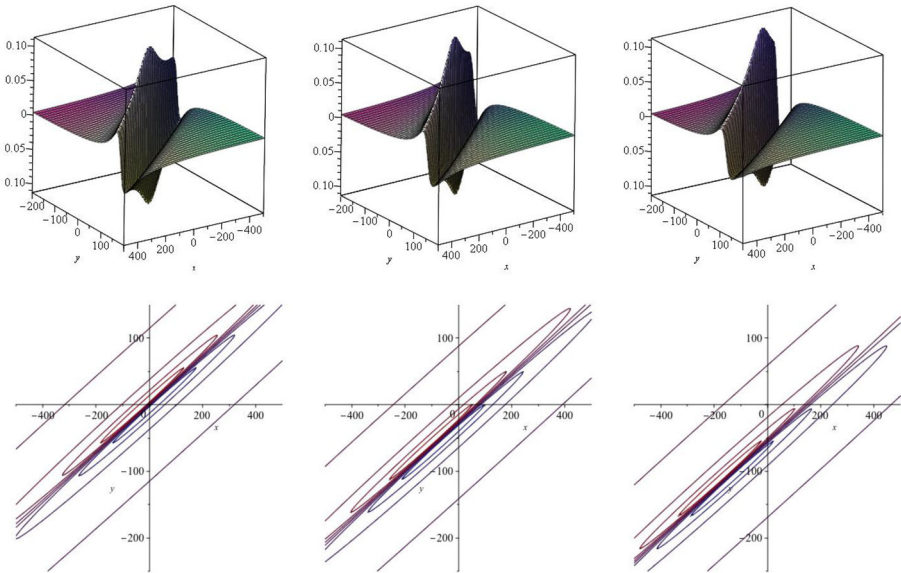


Fig. 1 Profiles of u_1 when $t = 0, 10, 20$: 3d plots (top) and contour plots (bottom)

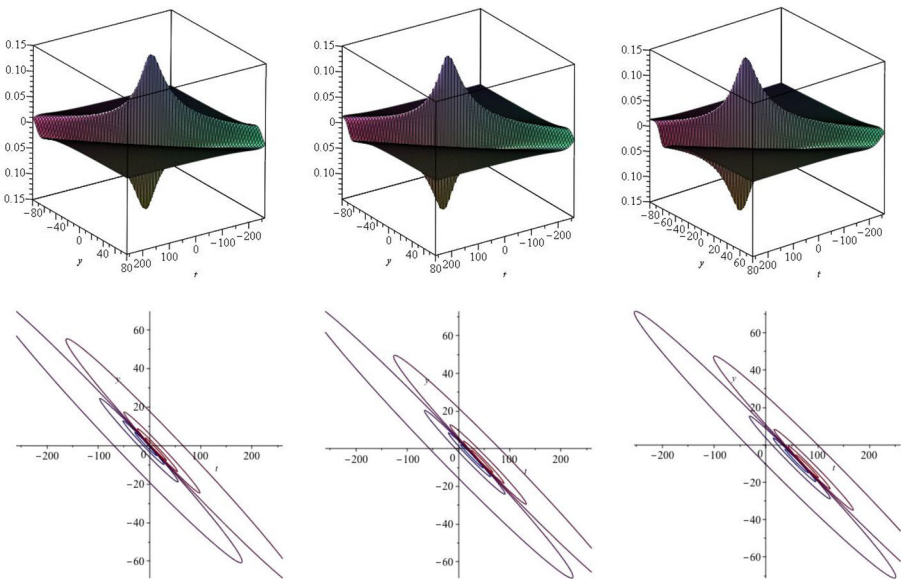


Fig. 2 Profiles of u_2 when $x = 0, 60, 120$: 3d plots (top) and contour plots (bottom)

$$u_2 = \frac{2(-56t + 16x - 192y + 32)}{(-6t + 2x - 17y + 5)^2 + (-8t + 2x - 31y + 3)^2 + 1392}. \tag{5.8}$$

Three three-dimensional plots and contour plots of this lump solution at three different values of x are made through Maple in Fig. 2.

6 Conclusion and remarks

With Maple symbolic computations, we have formulated a combined fourth-order soliton equation in $(2+1)$ -dimensions and determined its lump solutions in terms of the coefficients in the combined model. The presented results supplement the existing lumps and solitons in dispersive waves. Some three-dimensional plots and contour plots of two particular lump solutions were made by using Maple.

We remark that all exactly presented solutions add valuable insights into related studies on soliton solutions and dromion-type solutions in the continuous and discrete cases, achieved through effective techniques such as the generalized bilinear method (see, e.g., [25]), the Wronskian technique (see, e.g., [26,27]), Darboux transformations (see, e.g., [28,29]), the Riemann-Hilbert approach (see, e.g., [30]), the multiple-wave expansion method (see, e.g., [31,32]), symmetry reductions (see, e.g., [33,34]), and symmetry constraints (see, e.g., [35, 36] for the continuous case and [37,38] for the discrete case). The combined equation has theoretical interest itself, and is an essential step to obtain a significant model in multiple-scale and singular perturbation theory close to integrable equations [39].

We also remark that many nonlinear equations possess lump solutions, which include generalized KP, BKP, KP-Boussinesq, Sawada-Kotera, Calogero-Bogoyavlenskii-Schiff and Bogoyavlenskii-Konopelchenko equations in $(2+1)$ -dimensions [20,21,40–43]. Moreover, recent studies exhibit the remarkable richness of lump solutions to linear PDEs [32,44,45] and nonlinear PDEs in $(2+1)$ -dimensions (see, e.g., [46–50]) and $(3+1)$ -dimensions (see, e.g., [23,51–56]). A new kind of lumps with higher-order rational dispersion relations has been discovered as well [57]. Abundant lump solutions amend the existing theories of solutions through different kinds of combinations (see, e.g., [58–61]), and can generate interesting Lie-Bäcklund symmetries by taking derivatives with respect to the involved parameters. These symmetries can also be used to formulate conservation laws by pairs of symmetries and adjoint symmetries [62–64], and classify lump solutions based on an idea of determining optimal systems of solutions [65,66]. Moreover, diverse interaction solutions [42,67,68] have been constructed for different soliton equations in $(2+1)$ -dimensions, including homoclinic interaction solutions (see, e.g., [69–71]) and heteroclinic interaction solutions (see, e.g., [72–75]). Based on the Hirota bilinear form and the generalized bilinear forms, some general formulations have been presented for lump solutions [5,6,76].

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Compliance with ethical standards

Conflicts of interest The authors declare that there is no conflict of interest.

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