Eur. Phys. J. Plus (2020) 135:470
https://doi.org/10.1140/epjp/s13360-020-00447-z

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

# Hamilton–Jacobi analysis of the Freidel–Starodubtsev BF (A)dS gravity action

**G. B. de Gracia**[a], **B. M. Pimentel**[b]

Institute of Theoretical Physics, São Paulo State University (UNESP), P.O. Box 01140-070, São Paulo, SP, Brazil

Received: 2 November 2019 / Accepted: 12 May 2020 / Published online: 5 June 2020
© Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2020

**Abstract** In this work, we perform the Hamilton–Jacobi analysis of a modified gravity action, the so-called Freidel–Starodubtsev model. The complete set of involutive Hamiltonians that guarantee the system's integrability is obtained. The generalized Poisson brackets are calculated in the metric phase by means of a suitable constraint matrix inversion. We also present a discussion about the metric and non-metric degrees of freedom. From the fundamental differential we recover the equations of motion and explicitly obtain the generators of local Lorentz transformations and also diffeomorphisms for the tetrad and the spin connection fields.

## 1 Introduction

The $BF$ formulation has been employed to describe gravity from $D = 1 + 1$ to $D = 3 + 1$ dimensions [1–3]. The lower dimensional models [4] do not have any local degrees of freedom and constitutes a kind of laboratory to understand the four-dimensional gravity, for which BF theory has been used, for example, in the context of spin foam quantization [5,6]. The bidimensional case, a generalization of the Jackiw–Teitelboim action [7,8], is formulated by means of an auxiliary field and a curvature written in terms of a generalized connection that envolves a rotational and also a translational part. It can be visualized as connecting the different tangent spaces by rolling a ball through the manifold [9]. Those sectors are associated with the spin connection and the tetrad field, respectively. Unfortunately, canonical analysis reveals that they do not transform exactly as those geometric objects, because they are formulated as a generalized connection whose internal dimensionality is bigger than the space–time one but do not decompose exactly as local Poincaré internal group. If one employs the Inönu–Wigner contraction, the Killing metric vanishes and it is not possible to build the action [10]. The three-dimensional case has the $B$-field being interpreted as a tetrad and a gauge curvature associated with the spin connection. The problem now is the fact that there is freedom for additional shift symmetry [11]. Then, this extra symmetry avoids the complete identification of the aforementioned fields with the tangent space geometric variables. Moreover, this complicate symmetry structure turn, for example, its covariant quantization at least peculiar since it present couplings between different kinds of ghosts.

[a] e-mail: gb9950@gmail.com (corresponding author)
[b] e-mail: b.m.pimentel@gmail.com

Although it happens, it is possible to furnish an elegant one-loop exact quantization [12] and also a path integral one, that allows to discuss topological transition amplitudes [13].

Those BF models are highly symmetric such that their fields may be gauged away locally, see [14,15] for the four-dimensional case. So, a $BF$ description of gravity would demand a reduction in internal symmetry. We may cite the Plebanski formulation [16] which has the presence of an extra scalar field that imposes a constraint on the $B$-field allowing its description in terms of tetrads. Its quantization may reveal a possible fluctuation between a metrical and a topological phase due to radiative corrections to the mentioned constraint [17]. On the 70s, the Mac-Dowell action was proposed [18,19], what consisted in a four-dimensional $BF$ model with a five-dimensional $SO(1,4)$ internal structure with a term that breaks it down to $SO(1,3)$. It can be written in terms of geometrical fields and its equation of motion recovers the Einstein's ones without cosmological constant. The Freidel–Starodubtsev (FS) model [3,20], which is the modified gravity action we want to analyse here, is a generalization of this last one, now with the addition of a cosmological constant term.

This action has some very desirable features as the fact that it is formulated in terms of an extremely small dimensional parameter (which sounds well when one thinks about renormalizability), and also provide several topological invariant terms that although do not contribute to the equations of motion, guarantee the finiteness of Noether charges [21] without the need to add *ad-hoc* extra boundary terms such as the Gibbons–Hawking one. The BF structure also provides a Gaussian kinetic term which may be useful for a further quantization, see [22] for the BF formulation of QCD. Since it is written in terms of Cartan variables, it may admit the possibility of the unbroken phase, a situation contemplated in string theory that has vanishing tetrad average [23]. Now, after introducing some facts about the model to be studied, we focus on our specific paper analysis.

We want to give a precise canonical formulation to the Freidel–Starodubtsev model. Usually, this procedure is made with the Hamiltonian Dirac formalism [24–26]. The Hamiltonian analysis of Plebanski theory has been made in [27], the Freidel–Starodubtsev in [28], the two-dimensional polynomial $BF$ in [29] and the topological $BF$ with cosmological term in [30]. In this article, we want to employ the Hamilton–Jacobi analysis to investigate the Freidel–Starodubtsev model. Although the (FS) model were previously analysed in [28], here we intend to present it by another method. Also, we give a clear counting of its degrees of freedom in the metric and non-metric phases, and we do not skip to derive explicitly the reduced phase space generalized brackets. In [28], they applied the constraints in the strong form without constructing the Dirac brackets, which is not the most rigorous procedure. We must point out that the Hamilton–Jacobi method is mathematically consistent. It also do not need an *ad-hoc* procedure such as the Castelani one [31] to find local symmetries since (HJ) integrability requirement furnishes the generators of all canonical transformations of the system and, of course, its subset was represented by the Lagrangian symmetries. We also call attention to the fact that there are examples in which the Castellani method cannot be employed [32]. So, we are going to show, using (HJ) integrability, that the $(FS)$ model indeed presents the right symmetries that allow a correspondence between the theory's fields and the spin connection and tetrad variables.

The Hamilton–Jacobi formalism we present here comes from the approach of Güler [33], which is an extension of Caratheodory's equivalent Lagrangian method in the calculus of variations [34]. This formalism is characterized by a set of Hamilton–Jacobi differential equations called Hamiltonians. The dynamical evolution of the system is given in terms of a fundamental differential which depends on the time and other linear independent arbitrary parameters related to the involutive Hamiltonians [35,36], obtained from the Frobenius' integrability condition. The canonical transformations are obtained immediately from this fundamental differential when just the dynamics described by those arbitrary parameters are considered. Furthermore, the gauge transformations are the subgroup of those transformations that leave the Lagrangian invariant as mentioned in the previous paragraph. Therefore, we claim again that the Hamilton–Jacobi formalism is a way to illuminate the canonical origin of the gauge structure of the (FS) model. This approach was used to study several examples of gauge systems such as topologically massive theories [37], gravity models [38,39], and the two-dimensional, three-dimensional and four-dimensional $BF$ theories [40–42]. This formalism were also extended to higher-order Lagrangians and Berezinian systems [43–45].

The paper is organized as follows. In Sect. 2 we introduce the Hamilton–Jacobi formalism. Section 3 is devoted to present the gravitational interpretation of the Freidel–Starodubtsev model. Section 4 has the Hamilton–Jacobi analysis of the model's canonical structure, degree of freedom counting and the obtainment of the involutive and the non-involutive Hamiltonians. In Sect. 5, we derive explicitly the Poisson bracket in the reduced phase space restricted by the non-involutive Hamiltonians. In the next section, the generator of the internal Lorentz $SO(1,3)$ symmetry, resulting from the breaking of the pentadimensional symmetry is obtained and we show that it indeed generate the right transformations for the tetrad and the spin connection. Finally, in Sect. 7, we set down our conclusions.

## 2 The Hamilton–Jacobi formalism

In this section, we develop the Hamilton–Jacobi formalism for constrained systems, which are defined as the ones whose Lagrangian do not satisfy the Hessian condition.

Let us consider a physical system whose Lagrangian has the form $L = L(x^i,\ \dot{x}^i,\ t)$ where the Latin indices run from 1 to $n$, which is the dimension of the configuration space. The system is called constrained or singular if it does not satisfy the Hessian condition $\det W_{ij} \neq 0$ with the matrix $W_{ij}$ given by $W_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$. If the Hessian condition is satisfied, the transformation that leads the configuration space to the phase space is invertible. If it is not, some of the conjugated momenta $p_i = \frac{\partial L}{\partial \dot{x}^i}$ are not invertible on velocities and we are led to equations of the form $\Phi(q, p) = 0$ which constrains the phase space. Now, if we consider $k$ non-invertible momenta and $m = n - k$ invertible momenta, we have

$$p_z - \frac{\partial L}{\partial \dot{x}^z} = 0, \tag{1}$$

where $z = 1, \ldots, k$. Then, the above equation defines the primary constraints of the theory. By using the definition $H_z \equiv -\frac{\partial L}{\partial \dot{x}^z}$ we can rewrite the above equation as

$$H'_z \equiv p_z + H_z = 0. \tag{2}$$

These constraints are called Hamiltonians. If we define $p_0 \equiv \frac{\partial S}{\partial t}$, the Hamilton–Jacobi equation can be written as

$$H'_0 \equiv p_0 + H_0 = 0. \tag{3}$$

The canonical Hamiltonian function $H_0 = p_a \dot{x}^a + p_z \dot{x}^z - L$ with $a = 1, \ldots, m$, is independent of the non-invertible velocities $\dot{x}^z$ if the constraints are implemented. The unified notation is given by

$$H'_\alpha \equiv p_\alpha + H_\alpha, \tag{4}$$

where $\alpha = 0, \ldots, k$. The Cauchy's method [34] is employed to find the characteristic equations related to the above first-order equations

$$\mathrm{d}x^a = \frac{\partial H'_\alpha}{\partial p_a}\mathrm{d}t^\alpha, \qquad \mathrm{d}p^a = -\frac{\partial H'_\alpha}{\partial x_a}\mathrm{d}t^\alpha, \qquad \mathrm{d}S = (p_a \mathrm{d}x^a - H_\alpha \mathrm{d}t^\alpha). \tag{5}$$

The differentials written above depend on $t^\alpha = (t^0, t^z \equiv x^z)$ independent variables or parameters. The name Hamiltonians used for the constraints is now justified, once that $H_z$ generates flows parameterized by $t^z$ in analogy with the temporal evolution generated by $H_0$. From the characteristic equations one can use the Poisson brackets defined on the extended phase space $(x^a, t^\alpha, p_a, p_\alpha)$ to express in a concise form the evolution of any function $f = f(x^a, t^\alpha, p_a, p_\alpha)$:

$$\mathrm{d}f = \left\{f, H'_\alpha\right\}\mathrm{d}t^\alpha. \tag{6}$$

This is the fundamental differential, from where we identify the Hamiltonians as the generators of the dynamical evolution of the phase space functions.

Let us define the operator

$$X_\alpha[f] = \sum_I \left\{\gamma^I, H'_\alpha\right\}\frac{\delta f}{\delta \gamma^I} \quad ; \quad \gamma^I = (x^a, t^\alpha, p_a, p_\alpha), \tag{7}$$

where $X_\alpha[*]$ can be interpreted as $k$ vectors whose $2(n+1)$ components are $\{\gamma^I, H'_\alpha\}$. The fundamental differential can be expressed in terms of this operator as

$$\mathrm{d}f = X_\alpha[f]\mathrm{d}t^\alpha. \tag{8}$$

The Frobenius' integrability condition (IC) ensures the system of equations (8) is integrable. The IC can be expressed as

$$\left[X_\alpha, X_\beta\right](x^a, p^a) \equiv X_\alpha[x^a]X_\beta[p^a] - X_\beta[x^a]X_\alpha[p^a] = -\left\{H'_\alpha, H'_\beta\right\} = 0. \tag{9}$$

The above condition can be generalized to

$$\left\{H'_\alpha, H'_\beta\right\} = C^\gamma_{\alpha\beta} H'_\gamma, \tag{10}$$

where $C^\gamma_{\alpha\beta}$ are structure coefficients. Therefore, the IC ensures that the Hamiltonians close an involutive algebra. In terms of the fundamental differential (6), the IC (10) can be written as

$$\mathrm{d}H'_\alpha = \left\{H'_\alpha, H'_\beta\right\}\mathrm{d}t^\beta = C^\gamma_{\alpha\beta} H'_\gamma \mathrm{d}t^\beta = 0. \tag{11}$$

The Hamiltonians that satisfy the IC are called involutives. However, not all Hamiltonians from a physical system satisfy this condition identically. Therefore, we must define new Hamiltonians.

Let us suppose we have a set of non-involutive Hamiltonians $H'_{\bar{a}}$. Then

$$\mathrm{d}H'_{\bar{a}} = \left\{H'_{\bar{a}}, H'_0\right\}\mathrm{d}t + \left\{H'_{\bar{a}}, H'_{\bar{b}}\right\}\mathrm{d}x^{\bar{b}}. \tag{12}$$

Once that we impose $\mathrm{d}H'_{\bar{a}} = 0$, we can define a matrix with components $M_{\bar{a}\bar{b}} \equiv \left\{H'_{\bar{a}}, H'_{\bar{b}}\right\}$. If this matrix is invertible, we can write $\mathrm{d}x^{\bar{b}} = -M^{-1}_{\bar{a}\bar{b}}\left\{H'_{\bar{a}}, H'_0\right\}\mathrm{d}t$, i.e. there is a dependence between the parameters related to the non-involutive Hamiltonians. Replacing in the fundamental differential, we have

$$\mathrm{d}F = \left[\left\{F, H_0'\right\} - \left\{F, H_{\bar{a}}'\right\}M^{-1}_{\bar{a}\bar{b}}\left\{H_{\bar{b}}', H_0'\right\}\right]\mathrm{d}t. \tag{13}$$

Therefore, we can define generalized brackets (GB) as:

$$\left\{A, B\right\}^* \equiv \left\{A, B\right\} - \left\{A, H_{\bar{a}}'\right\}\left(M^{-1}\right)_{\bar{a}\bar{b}}\left\{H_{\bar{a}}', B\right\}, \tag{14}$$

which redefine the dynamic of the constrained system reducing its phase space, once that $\mathrm{d}F = \left\{F, H_0'\right\}^* \mathrm{d}t$. This procedure is the result of the integrability condition and, as shown in [35], it allows the possibility that the matrix $M_{ab}$ is non-invertible, or that the system has involutive and non-involutive Hamiltonians as well.

The dynamical evolution described by the resulting arbitrary parameters can be understood as canonical transformations, with the involutive Hamiltonians as generators. To understand this, we need to check that the variation $\delta\gamma^I = \delta t^\alpha X_\alpha[\gamma^I]$ is generated by $g = 1 + \delta t^\alpha X_\alpha$, also preserves the symplectic structure $\mathrm{d}x^a \wedge \mathrm{d}p_a + \mathrm{d}t^\alpha \wedge \mathrm{d}p_\alpha + \mathrm{d}H_\alpha \wedge \mathrm{d}t^\alpha$. with fixed $\mathrm{d}t^0$. In order to relate canonical transformations with the gauge ones, we need to restrict the study to fixed times $\mathrm{d}t^0 = 0$. Then, the transformation on any variable $\gamma^I$ is

$$\delta\gamma^I = \left\{\gamma^I, H_z'\right\}^* \delta t^z. \tag{15}$$

The Hamiltonians must be involutives, then $\left\{H_x', H_y'\right\}^* = C^z_{xy} H_z'$. However, the IC ensures that $\left\{H_x', H_y'\right\}^* = C^0_{xy} H_0' + C^z_{xy} H_z'$. To conciliate these equations, we must consider whether $C^0_{xy} = 0$ or $H_0' = 0$. The condition $C^0_{xy} = 0$ is almost never satisfied. On the other hand, the condition $H_0' = 0$ constrains the phase space. Under this assumption, we define the generator of gauge transformations as

$$G^{\mathrm{can}} \equiv H_z' \delta t^z, \tag{16}$$

since $\delta\gamma^I = \left\{\gamma^I, G^{\mathrm{can}}\right\}^*$. More details on the role of involutive Hamiltonians in the HJ formalism can be found in [36].

## 3 The gravitational interpretation of the model

As previously mentioned, the Freidel–Starodubtsev model can be formulated as a BF theory with explicit internal symmetry breaking. The internal symmetry is broken from $SO(1, 4)$ down to the $SO(1, 3)$ group , while the action is given in a four-dimensional space–time manifold. The theory is formulated by means of 2 form field $B = \frac{1}{2} B^{IJ}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu M_{IJ}$, where $M_{IJ}$ are the generators of the $SO(1, 4)$ group and a curvature $F = \frac{1}{2} F^{IJ}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu M_{IJ}$ written in terms of a 1-form field $A = A^{IJ}_\mu \mathrm{d}x^\mu M_{IJ}$ as $F \equiv \mathrm{d}A + A \wedge A$. The action is given below [3,20]

$$S_{FS} = \frac{1}{16\pi} \int \left(F^{IJ} \wedge B_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} \epsilon^{abcd4} B_{ab} \wedge B_{cd}\right) ;$$
$$I = 0, \ldots, 4 \;\; ; \;\; a = 0, \ldots, 3 \tag{17}$$

where the internal indices run from $I = 0, \ldots, 4$ and $a = 0, \ldots, 3$. The symmetry breaking term is proportional to $\alpha$ since it has a five-dimensional Levi-Civitta symbol with a fixed 4 coordinate. With regard to the $\alpha$ and $\beta$ constants, they are defined as $\frac{G\Lambda}{3(1+\gamma^2)}$, $\frac{\gamma G\Lambda}{3(1+\gamma^2)}$, respectively, and $\gamma$ denotes the Imirizi parameter that appears in the definition of Ashtekar variables [46]. Its value is estimated by loop quantum gravity numerical simulations as $0.2 < \gamma < 0.3$, [47].

Since the 2 form field $B^{IJ}$ provides a polynomial constraint to the curvature $F^{IJ}$, the action may be rewritten just in terms of the 1-form $A^{IJ} = A^{IJ}_\mu \mathrm{d}x^\mu$ generalized connection. According to the internal action symmetry breaking, we decompose this field in a rotational sector, related to the four-dimensional residual Lorentz symmetry, and a translational one due to the remaining broken generators. So, in analogy to the BF representation of the Jackiw–Teitelboim model [7], we have the 1 form decomposition in terms of the tetrad and the spin connection

$$A^{a4}_\mu \equiv \frac{1}{l} e^a_\mu \tag{18}$$

$$A^{ab}_\mu \equiv \omega^{ab}_\mu \tag{19}$$

Accordingly, the curvature $F = \mathrm{d}A + A \wedge A$ is divided as

$$\begin{aligned} F^{ab} &= \mathrm{d}\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b + \frac{e^a \wedge e^b}{l^2} = R^{ab} + \frac{e^a \wedge e^b}{l^2} \\ F^{a4} &= \frac{1}{l}\mathrm{d}e^a + \omega^{ac} \wedge e_c = \frac{T^a}{l} \end{aligned} \tag{20}$$

where the first equation refers to a Riemman curvature associated with a cosmological constant term and the second one to the torsion. The $l$ parameter has length dimension and is related to the cosmological constant as $\frac{\Lambda}{3} = -\frac{1}{l^2}$.

The $B$ field furnishes the following curvature polynomial constraints

$$F^{a4} = \beta B^{a4} \tag{21}$$

$$F^{ab} = \left(\beta \Delta^{ab}{}_{cd} + \frac{\alpha}{2}\epsilon^{ab}{}_{cd}\right) B^{cd} \tag{22}$$

This constraint may be solved by the inversion of the matrix[1] $\left(\beta \Delta^{ab}{}_{cd} + \frac{\alpha}{2}\epsilon^{ab}{}_{cd}\right)$:

$$B^{ab} = \frac{1}{\left(\alpha^2 + \beta^2\right)} \left(\beta F^{ab} - \frac{\alpha}{2}\epsilon^{ab}{}_{cd} F^{cd}\right) \tag{23}$$

By plugging this result to the action, we can represent it by means of the geometrical fields:

$$\begin{aligned} 32\pi G S_{(F-S)} = &\int R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} + \frac{1}{2l^2}\int e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd} \\ &+ \frac{2}{\gamma}\int R^{ab} \wedge e_a \wedge e_b \\ &+ \frac{l^2}{2}\int R^{ab} \wedge R^{cd}\epsilon_{abcd} - l^2\gamma \int R^{ab} \wedge R_{ab} \\ &+ 2\frac{\left(\gamma^2+1\right)}{\gamma}\int \left(T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b\right) \end{aligned}$$

The generalization due to this model is related to the addition of topological boundary terms that do not affect the equations of motion but improve the action renormalizing properties. Namely, its Noether charges [21] are totally finite, differently from the conventional

[1] Where $\Delta^{ab}{}_{cd} \equiv \frac{(\delta^a_c \delta^b_d - \delta^b_c \delta^a_d)}{2}$.

approach. The topological terms are the Holst, the Euler, the Pontryagin and the Nieh-Yan topological invariants

$$H = R^{ab} \wedge e_b \wedge e_a, \quad E_4 = R^{ab} \wedge R^{cd} \epsilon_{abcd}$$
$$P_4 = R^{ab} \wedge R_{ab}, \quad NY_4 = T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b \tag{24}$$

Since the topological invariant terms do not change the equations of motion, the Einstein's equations with vanishing torsion are derived from it

$$0 = \left( R^{ab} \wedge e^c + \frac{1}{l^2} e^a \wedge e^b \wedge e^c \right)$$
$$0 = \mathrm{d}e^a + \omega^{ac} \wedge e_c \tag{25}$$

## 4 The Hamilton–Jacobi analysis

In order to apply the Hamilton–Jacobi formalism to the Freidel–Starodubtsev (FS) model we foliate the space–time as $\mathcal{M} = \mathcal{R} \times \mathcal{M}_3$ with $\mathcal{M}_3$ being a constant time hipersurface. The Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \left( B_{IJ\mu\nu} F^{IJ}_{\alpha\beta} - \frac{\beta}{2} B^{IJ}_{\mu\nu} B_{\alpha\beta IJ} - \frac{\alpha}{4} \epsilon_{4IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\alpha\beta} \right) \tag{26}$$

where $F^{IJ}_{\mu\nu} = \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + f^{IJ}{}_{KLMN} A^{KL}_\mu A^{MN}_\nu$ and the $SO(1, 4)$ structure constants are defined by means of its generators $M_I{}^J$ commutator

$$[M_{IJ}, M_{KL}] = \eta_{IL} M_{JK} - \eta_{IK} M_{JL} + \eta_{JK} M_{IL} - \eta_{JL} M_{IK} \equiv 2 f_{IJKL}{}^{MN} M_{MN} \tag{27}$$

With $\eta_{IK}$ being an internal five-dimensional Minkowski metric.
The Euler–Lagrange equations are given below

$$0 = \epsilon^{\mu\nu\alpha\beta} \left( F^{IJ}_{\alpha\beta} - \beta B^{IJ}_{\alpha\beta} - \frac{\alpha}{2} \epsilon^{4IJ}{}_{KL} B^{KL}_{\alpha\beta} \right) \tag{28}$$
$$0 = \epsilon^{\mu\nu\alpha\beta} D_\nu B^{IJ}_{\alpha\beta} \tag{29}$$

The covariant derivative is defined as $D_\nu B^{IJ}_{\alpha\beta} = \partial_\nu B^{IJ}_{\alpha\beta} + f^{IJMNKL} A^{MN}_\nu B^{KL}_{\alpha\beta}$.

The canonical momenta $\pi^{IJ}_\mu$, $\Pi^{IJ}_{\mu\nu}$ conjugated to the fields $A^{IJ}_\mu$ e $B^{IJ}_{\mu\nu}$ are given by

$$\pi^\mu_{IJ} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A^{IJ}_\mu)} = \epsilon^{0\mu}{}_{\alpha\beta} B^{\alpha\beta}_{IJ} \tag{30}$$
$$\Pi^{\mu\nu}_{IJ} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 B^{IJ}_{\mu\nu})} = 0 \tag{31}$$

Those canonical momenta do not depend on the velocities $\partial_0 A^{IJ}_\mu$ e $\partial_0 B^{IJ}_{\mu\nu}$, so they generate constraints in the phase space. The canonical Hamiltonian is given by a Legendre transform

$$\mathcal{H}_0 = -\varepsilon^{0\mu\nu\beta} \left( D_\mu B^{IJ}_{\nu\beta} A_{0IJ} + B^{IJ}_{0\mu} F_{\nu\beta IJ} - \beta B^{IJ}_{0\mu} B_{\nu\beta IJ} - \frac{\alpha}{2} \epsilon_{4IJKL} B^{IJ}_{0\mu} B^{KL}_{\nu\beta} \right) \tag{32}$$

According to the Hamilton–Jacobi (HJ) formalism, we may define $\pi \equiv \partial_0 S$ where $S$ is the action of the system. This definition allows us to write all the (HJ) partial differential equation in an unified form

$$\mathcal{H}' \equiv \pi + \mathcal{H} = 0 \tag{33}$$

$$\mathcal{A}^{0'}_{IJ} \equiv \pi^{0}_{IJ} = 0 \tag{34}$$

$$\mathcal{A}^{k'}_{IJ} \equiv \pi^{k}_{IJ} - \epsilon^{0k}{}_{\alpha\beta} B^{\alpha\beta}_{IJ} = 0 \tag{35}$$

$$\mathcal{B}^{\mu\nu'}_{IJ} \equiv \Pi^{\mu\nu}_{IJ} = 0 \tag{36}$$

The first Hamiltonian is related to the dynamical evolution parameterized by the time $t \equiv x_0$, while the remaining ones will be related to generators of canonical transformations or to the definition of a generalized Poisson brackets, depending on their nature.

So, in order to characterize them as involutive or non-involutive we must first define the fundamental Poisson brackets of the theory

$$\left\{A^{IJ}_{\mu}(x), \pi^{\nu}_{KL}(y)\right\} = \delta^{\nu}_{\mu} \Delta^{IJ}{}_{KL} \delta^{3}(x-y) \tag{37}$$

$$\left\{B^{IJ}_{\mu\nu}(x), \Pi^{\alpha\beta}_{KL}(y)\right\} = \Delta^{IJ}{}_{KL} \Delta^{\mu\nu}{}_{\alpha\beta} \delta^{3}(x-y) \tag{38}$$

where $\Delta^{IJ}{}_{KL} \equiv \frac{1}{2}\left(\delta^{I}_{K}\delta^{J}_{L} - \delta^{I}_{L}\delta^{J}_{K}\right)$ and $\Delta^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right)$.

The fundamental differential furnishes the evolution of all phase space variable $f(x)$ parameterized by the time and the canonical transformation local parameters

$$\mathrm{d}f(x) = \int \left(\left\{f(x), \mathcal{H}'(y)\right\}\mathrm{d}t + \left\{f(x), \mathcal{A}^{\mu'}_{IJ}(y)\right\} d\lambda^{IJ}_{\mu}(y) + \left\{f(x), \mathcal{B}^{\mu\nu'}_{IJ}(y)\right\} d\beta^{IJ}_{\mu\nu}(y)\right) d^{3}y \tag{39}$$

The Hamiltonians whose Poisson brackets with itself and with all the remaining ones are vanishing or a linear combination of them are called involutive. It means that it is not necessary to impose constraints between the evolution parameters of the theory to ensure its integrability conditions. In the present case, the involutive ones are $\mathcal{A}^{0'}_{IJ}$ and $\mathcal{B}^{0k}_{IJ}$. To guarantee the integrability of the remaining set of Hamiltonians, relations between the parameters must be imposed which means that the phase space is governed by a generalized Poisson brackets constructed by means of the matrix $M^{AB}_{IJKL}(x,y) = \left\{h^{A}_{IJ}(x), h^{B}_{KL}(y)\right\}$, where, in this case, the $A$ and $B$ indices run from 1 to 6 since they count the number of non-involutive Hamiltonians. The $h^{x}_{IJ}$ are defined as

$$\begin{gathered} \Pi^{12}_{IJ} \equiv h^{1}_{IJ}\ ;\ \Pi^{13}_{IJ} \equiv h^{2}_{IJ}\ ;\ \Pi^{23}_{IJ} \equiv h^{3}_{IJ}\ ; \\ \pi^{1}_{IJ} - 2B^{23}_{IJ} \equiv h^{4}_{IJ}\ ;\ \pi^{2}_{IJ} + 2B^{13}_{IJ} \equiv h^{5}_{IJ}\ ;\ \pi^{3}_{IJ} - 2B^{12}_{IJ} \equiv h^{6}_{IJ} \end{gathered} \tag{40}$$

The matrix built by the Poisson brackets of the Hamiltonians is given below

$$M^{AB}_{IJKL}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^{3}(x-y)\Delta^{IJ}{}_{KL} \tag{41}$$

The inverse matrix is

$$[M^{-1}(x,y)]^{AB}_{IJKL} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{\delta^3(x-y)}{2} \Delta^{IJ}{}_{KL} \tag{42}$$

The generalized Poisson brackets (GPB) between two-phase space functions are defined as

$$\Big\{f(x), g(y)\Big\}^* = \Big\{f(x), g(y)\Big\} - \int \Big\{f(x), h_a^{IJ}(z)\Big\}[M^{-1}(z,w)]^{ab}_{IJKL} \Big\{h_b^{KL}(w), g(y)\Big\} d^3z\, d^3w \tag{43}$$

The non-vanishing fundamental (GPB) are

$$\Big\{A^i{}_{IJ}(x), B^{KL}_{lm}(y)\Big\}^* = \Delta^{IJ}{}_{KL}\frac{\delta^3(x-y)}{2}\epsilon^i{}_{lm} \tag{44}$$

$$\Big\{A^\mu_{IJ}(x), \pi_\nu^{KL}(y)\Big\}^* = \Big\{A^\mu_{IJ}(x), \pi_\nu^{KL}(y)\Big\} \tag{45}$$

$$\Big\{B^{0k}_{IJ}(x), \Pi^{KL}_{0i}(y)\Big\}^* = \Big\{B^{0k}_{IJ}(x), \Pi^{KL}_{0i}(y)\Big\} \tag{46}$$

The fundamental differential can be written in terms of the generalized brackets as:

$$\mathrm{d}f(x) = \int \bigg(\Big\{f(x), \mathcal{H}'(y)\Big\}^* \mathrm{d}t + \Big\{f(x), \mathcal{A}^{0'}_{IJ}(y)\Big\}^* d\lambda_0^{IJ}(y) + \Big\{f(x), \mathcal{B}^{0k'}_{IJ}(y)\Big\}^* d\beta_{0k}^{IJ}(y)\bigg) d^3y \tag{47}$$

Now, we find the integrability conditions for the involutive Hamiltonians $\mathcal{A}^{0'}_{IJ}(y)$ and $\mathcal{B}^{0k}_{IJ}(y)$. We see that in order to impose $\mathrm{d}\mathcal{A}^{0'}_{IJ}(y) = 0$ and $\mathrm{d}\mathcal{B}^{0k}_{IJ}(y) = 0$ is necessary to introduce the following new Hamiltonians

$$C^{IJ} \equiv \epsilon^{0kij} D_k B^{IJ}_{ij} \tag{48}$$

$$D^{kIJ} \equiv \epsilon^{0kij}\bigg(F^{IJ}_{ij} - \beta B^{IJ}_{ij} - \frac{\alpha}{2}\epsilon^{4IJ}{}_{KL} B^{KL}_{ij}\bigg) \tag{49}$$

It is possible to write the canonical Hamiltonian in terms of them

$$\mathcal{H}_0 = -A_0^{IJ} C_{IJ} - B^{IJ}_{0k} D^k_{IJ} \tag{50}$$

To classify the Hamiltonians $C_{IJ}$ and $D^k_{IJ}$ as involutive or non-involutive we should calculate their (GPB)

$$\Big\{D^k_{IJ}(x), D^m_{MN}(y)\Big\}^* = \alpha\epsilon^{kmi}\Big(f_{IJRSLK} A_i^{RS} \epsilon_{MN}{}^{LK} + f_{MNBAOP} A_i^{BA} \epsilon_{IJ}{}^{OP}\Big)\delta^3(x-y) \tag{51}$$

where the definition $\epsilon^{0kln} \equiv \epsilon^{kln}$ is used throughout.

The (GPB) between $C_{IJ}(x)$ and $D^k_{MN}(y)$ is given below

$$\left\{C_{IJ}(x), D^{k}_{MN}(y)\right\}^{*} = f_{IJMNOP}D^{OPk} - \frac{\alpha}{2}\epsilon^{kmn} \left(f_{IJOPLK}\epsilon_{MN}^{\ \ OP}B^{LK}_{mn} - f_{IJMNOP}\epsilon^{\ \ OP}_{RS}B^{RS}_{mn}\right) \tag{52}$$

To finish, we calculate $\left\{C^{IJ}(x), C^{KL}(y)\right\}^{*}$

$$\left\{C_{IJ}(x), C_{MN}(y)\right\}^{*} = f_{IJMN}^{\ \ \ \ \ OP}C_{OP}\,\delta^{3}(x-y) \tag{53}$$

We see that different from the topological four-dimensional BF theory, those Hamiltonians are not all involutive and the difference from its results, besides the internal symmetry dimensionality, is due to the parameter $\alpha$ which is extremely small but guarantee the (FS) model gravitational interpretation.

In order to identify which are the involutive and non-involutive Hamiltonians, we focus just on the contributions proportional to $\alpha$ and use the explicit form of the structure constant

$$\begin{aligned}
&\left\{C_{IJ}(x), C_{MN}(y)\right\}^{*} = f_{IJMN}^{\ \ \ \ \ OP}C_{OP}\,\delta^{3}(x-y)\\
&\quad\left\{D^{k}_{IJ}(x), D^{m}_{MN}(y)\right\}^{*}\\
&= -\alpha\epsilon^{kmi}\left(\epsilon_{MNJK4}A^{K}_{i\ I} - \epsilon_{MNIK4}A^{K}_{i\ J} + \epsilon_{IJNP4}A^{P}_{i\ M} - \epsilon_{IJMP4}A^{P}_{i\ N}\right)\delta^{3}(x-y)\\
&\quad\left\{C_{IJ}(x), D^{k}_{MN}(y)\right\}^{*}\\
&= f_{IJMN}^{\ \ \ \ \ OP}D^{k}_{OP}\,\delta^{3}(x-y) - \frac{\alpha}{2}\epsilon^{kmn}\left(\epsilon_{MNLI4}B_{Jmn}^{\ \ \ \ L} - \epsilon_{MNLJ4}B_{Imn}^{\ \ \ \ L}\right)\delta^{3}(x-y)\\
&\quad+\frac{\alpha}{4}\left(\eta_{IN}\epsilon_{JMRS4} - \eta_{IM}\epsilon_{JNRS4} + \eta_{JM}\epsilon_{INRS4} - \eta_{JN}\epsilon_{IMRS4}\right)B^{RS}_{mn}\,\epsilon^{kmn}\,\delta^{3}(x-y)
\end{aligned} \tag{54}$$

It can be decomposed into the involutive sector

$$\left\{C_{ab}(x), C_{cd}(y)\right\}^{*} = f_{abcd}^{\ \ \ \ \ ef}C_{ef}\delta^{3}(x-y) \tag{55}$$

$$\left\{C_{ab}(x), D^{k}_{cd}(y)\right\}^{*} = f_{abcd}^{\ \ \ \ \ OP}D^{k}_{OP}\delta^{3}(x-y) \tag{56}$$

where the internal indexes $a$, $b$ run from 0 to 3.

And the non-involutive sector

$$\left\{C_{a4}(x), D^{k}_{cd}(y)\right\}^{*} = f_{a4cd}^{\ \ \ \ \ ef}D^{k}_{ef}\delta^{3}(x-y) - \frac{\alpha}{4}\epsilon^{kmn}\left(\epsilon_{cdau4}B_{mn}^{\ \ \ u4}\right)\delta^{3}(x-y) \tag{57}$$

$$\left\{C_{a4}(x), D^{k}_{b4}(y)\right\}^{*} = -\frac{\alpha}{8}\epsilon^{kmn}\epsilon_{abcd4}B^{cd}_{mn}\delta^{3}(x-y) \tag{58}$$

$$\left\{D^{i}_{a4}(x), D^{j}_{bc}(y)\right\}^{*} = \alpha\,\epsilon_{abcu4}A^{u}_{m\ 4}\epsilon^{ijm}\delta^{3}(x-y) \tag{59}$$

As we can see, the $C_{ab}$ and $C_{a4}$ are the involutive Hamiltonians. The involutive character of $C_{a4}$ is due to the fact that we are considering the physical configuration of the system, so that the 2-form $B$ field is algebraically related to the torsion and the generalized curvature and both of them vanishes.

Therefore, these Hamiltonians generate the model's local canonical transformations. So, from a given subset of the complete set of generators, we intend to find the correct local lorentz transformations for the tetrad $e^a_\mu = l A^{a4}_\mu$ and the spin connection $A^{ab}_\mu \equiv \omega^{ab}_\mu$ and also their diffeomorphisms. Regarding the others Hamiltonians, since they are non-involutive, their role is to enter into the definition of a double generalized Poisson bracket (DGPB). Due to the algebra of $C_{ab}$ and $C_{a4}$ with all the other Hamiltonians, we see that the differentials $\mathrm{d}C_{ab}$ and $\mathrm{d}C_{a4}$, considering the physical configuration, are proportional to the others Hamiltonians and it is sufficient to state that they are integrable. So, all the involutive Hamiltonians, the generators of local transformations, are found.

The system's degrees of freedom are counted by imposing that the local freedom to perform canonical transformations must be eliminated in order to have a well-defined time evolution. In order to break this local freedom it is necessary to impose one extra constraint for each of the canonical transformation generators in order to turn them into non-involutive ones defining a new reduced unique phase space.

So, in order to count degrees of freedom, we note that the phase space generated by (GPB) have 140 degrees of freedom $\{A^{IJ}_\mu, \pi^{IJ}_0, B^{IJ}_{\mu\nu}, \Pi^{IJ}_{0i}\}$. Regarding the Hamiltonians, we have 30 non-involutives $D^{ab}_i$, $D^a_i$ and 50 involutive ones $\{\Pi^{IJ}_{0i}, C^{ab}, C^{a4}, \pi^{IJ}_0\}$. It must be emphasized that in the present case with $\alpha \neq 0$ there is no reducibility relations [48] between the constraints as happened to the topological four-dimensional BF case [42]. So, the total degrees of freedom are $140 - 2(50) - 30 = 10$, a different value that is expected in the conventional metric gravity. The point is that when the tetrad is not inversible, a situation that can be naturally contemplated by the (FS) model, the spin connection arises as an independent variable. For now on, we focus on a given class of solutions characterized by metric tetrads, the inversible ones. They must obey the following additional 6 relations $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$ where $g_{\mu\nu}$ is the space–time metric. It gives us 4 phase space degrees of freedom, the right ones for general relativity.

In order to facilitate the task of obtaining the reduced phase space we, instead of fixing the Schwinger [49] gauge directly at the Lagrangian level as done in [38], find it more consistent to consider a class of solutions with $e^0_i = 0$ (exploiting the internal rotational invariance of the system) but allowing the time tetrad component to have non-vanishing gradients in the phase space.

## 5 The obtainment of the reduced phase space

In order to obtain the reduced phase space of the theory, it is necessary to build the non-involutive Hamiltonian matrix and then invert it. It is convenient to divide them between their spatial and temporal internal indices.

To build the Hamiltonian matrix we first organize them in a generalized vector $\Phi_A(x) = \left(D^k_{\bar{a}\bar{b}}(x), D^k_{\bar{a}0}(x), D^k_{\bar{a}4}(x), D^k_{04}(x),\right)$. The spatial sector of the internal indexes are being denoted as $\bar{a} = 1, 2, 3$.

The non-involutive Hamiltonian matrix reads

$$
\begin{aligned}
&\left\{\Phi^A(x), \Phi^B(y)\right\} \\
&\quad = \begin{pmatrix}
0 & 0 & -\epsilon^{kln}\epsilon_{\bar{a}\bar{b}\bar{d}} A^{04}_n \alpha & \alpha\epsilon^{kln}\epsilon_{\bar{a}\bar{b}\bar{d}} A^{\bar{d}4}_n \\
0 & 0 & -\alpha\epsilon_{\bar{a}\bar{d}\bar{e}}\epsilon^{kln} A^{\bar{e}4}_n & 0 \\
-\alpha\epsilon_{\bar{d}\bar{c}\bar{a}}\epsilon^{lkn} A^{04}_n & -\alpha\epsilon_{\bar{d}\bar{a}\bar{e}}\epsilon^{kln} A^{\bar{e}4}_n & 0 & 0 \\
-\alpha\epsilon^{lkn}\epsilon_{\bar{d}\bar{c}\bar{e}} A^{\bar{e}4}_n & 0 & 0 & 0
\end{pmatrix} \delta^3(x-y)
\end{aligned}
\tag{60}
$$

Since we are considering a class of solutions with $e_i^0 = 0$, the remaining matrix elements are just the anti diagonal ones. Then, in order to invert this matrix, it is just required to know the inverse of each of its elements. Denoting the inverse as $G^{\bar{b}\bar{c}}_{lm}$, we get

$$\alpha\epsilon^{kln}\epsilon_{\bar{a}\bar{b}\bar{e}}A^{\bar{e}}_n G^{\bar{b}\bar{c}}_{lm} = \delta^k_m \delta^{\bar{c}}_{\bar{a}} \tag{61}$$

We must use the useful identity [50]

$$\epsilon^{0kln}\epsilon_{0\bar{a}\bar{b}\bar{e}}e^{\bar{e}}_n = -e\Big(e^0_0(e^k_{\bar{a}}e^l_{\bar{b}} - e^l_{\bar{a}}e^k_{\bar{b}}) - e^k_0(e^0_{\bar{a}}e^l_{\bar{b}} - e^l_{\bar{a}}e^0_{\bar{b}}) + e^l_0(e^0_{\bar{a}}e^k_{\bar{b}} - e^k_{\bar{a}}e^0_{\bar{b}})\Big) \tag{62}$$

where $e$ is equal to the tetrad determinant.

Then, considering the aforementioned class of solutions and also the projecting tetrad properties, we get

$$-\alpha e\, e^0_0(e^k_{\bar{z}}G^{l\bar{c}}_{lm} - G^{k\bar{c}}_{\bar{z}m}) = l\delta^{\bar{c}}_{\bar{z}}\delta^k_m \tag{63}$$

where $l$ is the length parameter related to the cosmological constant which enters in the relation between the 1-form gauge field and the tetrad.

Contracting the tetrad $e^{\bar{z}}_k$ with the above expression gives

$$G^{l\bar{c}}_{lm} = -l\frac{e^{\bar{c}}_m}{2\alpha e e^0_0} \tag{64}$$

It allows us to find

$$G^{\bar{b}\bar{c}}_{lm} = -\frac{l}{2\alpha e^0_0 e}\Big(-2e^{\bar{c}}_l e^{\bar{b}}_m + e^{\bar{b}}_l e^{\bar{c}}_m\Big) \tag{65}$$

The matrix $G^{\bar{b}\bar{c}}_{lm}$ is the inverse of a quantity that envolves different kind of indexes. We can show that it is also a left inverse

$$G^{\bar{b}\bar{c}}_{lm}\epsilon^{mng}\epsilon_{\bar{c}\bar{a}\bar{e}}\alpha A^e_k = \delta^{\bar{b}}_{\bar{a}}\delta^n_l \tag{66}$$

Since we know the inverse of each of the matrix elements, we can invert it

$$M^{AB}_{k\,l}(x, y) \equiv \begin{pmatrix} 0 & 0 & 0 & G^{kl}_{\bar{a}\bar{b}} \\ 0 & 0 & -G^{kl}_{\bar{a}\bar{d}} & 0 \\ 0 & G^{kl}_{\bar{d}\bar{c}} & 0 & 0 \\ -G^{kl}_{\bar{d}\bar{c}} & 0 & 0 & 0 \end{pmatrix}\delta^3(x-y) \tag{67}$$

where the $A$, $B$ indexes run from 1 to 4 and counts the number of Hamiltonians involved in its computation.

Then, the double generalized Poisson brackets (DGPB) related to the spin connection and the tetrad reads

$$\begin{aligned}\Big\{A^{IJ}_i(x), A^{MN}_j(y)\Big\}^{**} &= -\int d^3w d^3z \Big\{A^{IJ}_i(x), \Phi_A(w)\Big\}^* M^{AB}_{k\,l}(w,z)\Big\{\Phi_B(z), A^{MN}_j(y)\Big\}^* \\ &= \Big[\Big(\hat{O}^{IJ}_{\bar{a}\bar{b}}G^{\bar{a}\bar{b}}_{ij}\hat{O}^{MN}_{04} - \hat{O}^{IJ}_{04}G^{\bar{a}\bar{b}}_{ij}\hat{O}^{MN}_{\bar{a}\bar{b}}\Big) \\ &\quad + \Big(\hat{O}^{IJ}_{\bar{a}4}G^{\bar{a}\bar{b}}_{ij}\hat{O}^{MN}_{\bar{b}0} - \hat{O}^{MN}_{\bar{a}0}G^{\bar{a}\bar{b}}_{ij}\hat{O}^{IJ}_{\bar{b}4}\Big)\Big]\end{aligned} \tag{68}$$

where $\hat{O}^{MN}_{JK}$ denotes the internal symmetry breaking matrix

$$\hat{O}^{MN}_{JK} = \Big(\beta\Delta^{MN}_{JK} + \frac{\alpha}{2}\epsilon^{4MN}_{JK}\Big) \tag{69}$$

## 6 Characteristic equations and canonical transformations

In this section, we use the previously derived brackets to determine the characteristic equations and also the canonical transformations by means of the fundamental differential.

### 6.1 Equations of motion

For obtaining the system's equations of motion we fix the local parameters and consider just the time variations. The fact that the Hamiltonian $D^{k}_{IJ}$ vanishes in the reduced phase space is also considered and it is equivalent to solve the polynomial constraint, expressing the $B$-field in terms of the gauge curvature, as done in Sect. 3. The fundamental differential, in this case, reads

$$\mathrm{d}F = \int d^3y \Big\{ F, \mathcal{H}_c \Big\}^{**} \mathrm{d}t = -\int \Big\{ F, A_0^{IJ} C_{IJ} \Big\}^{**} d^3y \mathrm{d}t \tag{70}$$

The structure above can be used to find

$$\mathrm{d}A_{\mu}^{IJ} = \delta_i^{\mu} \Big[ D_{\mu} A_0^{IJ} \Big] \mathrm{d}t \tag{71}$$

where we have used the fact that since $C^{IJ}$ is involutive, we can use the $\Big\{ , \Big\}^{*}$ brackets to compute the equations of motion.

Then, the differential equations for the physical fields (tetrad and spin connection) are

$$\partial_0 A_{\mu}^{IJ} = \delta_i^{\mu} \Big[ D_{\mu} A_0^{IJ} \Big] \tag{72}$$

We conclude that $A_0^{IJ}$ do not have dynamics, as it can be inferred by the fact that it appear as Lagrange multiplier in the canonical Hamiltonian. The equation above makes sense since it states that the torsion and the d$S$ (de-Sitter.) curvature must vanish, a result in accordance with the gravitational interpretation of the model given in Sect. 3 in which the polynomial constraint was solved.

### 6.2 Canonical transformations

The canonical transformations are generated by the fundamental differential with a fixed time coordinate and considering just variations with relation to the local arbitrary parameters. By the proper definition of the involutive Hamiltonians, using the $\Big\{ , \Big\}^{*}$ bracket is enough if we consider the physical configurations and the fact that they are involutive with all the other Hamitonians, but we must pay attention to the fact that the Hamiltonian $D^{k}_{IJ}(x)$ should be considered to strongly vanish. So, the generator of canonical transformations is given below

$$\begin{aligned} G = \int d^3x \Big[ \Big\{ A_l^{ab}, C^{cd} \Big\}^{*} \delta\varepsilon_{cd} + \Big\{ A_l^{ab}, C^{c4} \Big\}^{*} \delta\gamma_c \\ + \Big\{ A_l^{ab}, \pi^0_{IJ} \Big\}^{*} \delta w^{IJ} + \Big\{ A_l^{ab}, \Pi^{0i}_{IJ} \Big\}^{*} \delta\Lambda_i^{IJ} \Big] \end{aligned} \tag{73}$$

With regard to the expression above, we have changed the local parameter notation from $d\lambda_0^{IJ}(y)$ and $\mathcal{B}^{0k'}_{IJ}(y)$ to $\delta\Lambda_i^{IJ}$ and $\delta w^{IJ}$. Moreover, we associate the parameter $\delta\varepsilon_{cd}$ to the involutive Hamiltonian $C^{ab}(x)$ and $\delta\gamma_a$ to $C^{a4}(x)$.

From a subset of these canonical transformations with $\delta\gamma_a = 0$, we may find the local Lorentz symmetry.

The canonical transformation for the spatial part of the spin connection is given by

$$\delta\omega_l^{ab} = -D_l\delta\varepsilon^{ab} \tag{74}$$

where $D_l\delta\varepsilon^{ab} \equiv \partial_l\delta\varepsilon^{ab} + \omega_l^{ac}\delta\varepsilon_c{}^b - \omega_l^{bc}\delta\varepsilon_c{}^a$ denotes the covariant derivative.

Analogously, for the tetrad field, we obtain

$$\delta e_i^a = \int d^3x\{e_i^a, C^{cd}\}^*\delta\varepsilon_{cd} = \delta\varepsilon^a{}_b e_i^b \tag{75}$$

We have $a \neq 0$ due to the Schwinger gauge.

For the time component for the tetrads and the spin connection, we have

$$\delta A_0^{IJ} = \delta w^{IJ} \tag{76}$$

Having found the canonical transformations, the next step is to obtain its subset that represent a Lagrangian symmetry. Then, we must impose relations between the arbitrary local parameters. The form of the $\delta w^{IJ}$ parameters is fixed by the requirement that the fields must transform as tensors with relation to its internal and space–time symmetries. It gives $w_{ab} = D_0\delta\varepsilon_{ab}$, $\delta w_{a4} = \delta\varepsilon_{ab}A_0^{b4}$. We show that using this requirement is enough to find the Lagrangian internal symmetry. They are the local Lorentz transformations

$$\delta e_\mu^a = \delta\varepsilon^{ab}e_\mu^b \tag{77}$$

$$\delta\omega_\mu^{ab} = \partial_\mu\delta\varepsilon^{ab} + \omega_\mu^{ac}\delta\varepsilon_c{}^b - \omega_\mu^{bc}\delta\varepsilon_c{}^a \tag{78}$$

The local Lorentz symmetry generator then reads

$$G = \int d^3x\left[C^{cd}\delta\varepsilon_{cd} + \pi_{ab}^0 D_0\delta\varepsilon^{ab} + \pi_{a4}^0\delta\varepsilon^a{}_b A_0^{b4}\right] \tag{79}$$

In order to find the diffeomorphism generator, we must consider a subset of the canonical transformations obtained by fixing the parameters as [51] $\varepsilon_{ab} \equiv \beta^\mu A_\mu^{ab}$ and $\gamma_a \equiv \beta^\mu A_\mu^{a4}$, while the others vanish. $\beta^\mu$ is a general vector field. It then gives

$$\begin{aligned}\delta A_i^{MN} &= \int d^3x\left[\left\{A_i^{MN}, C_{ab}\right\}^*\varepsilon_{ab} + \left\{A_i^{MN}, C_{a4}\right\}^*\gamma_a\right] = \tilde{D}_i\left(\beta^\mu A_\mu^{MN}\right) \equiv L_\beta A_i^{MN}\\ &= \beta^\nu\partial_\nu A_i^{MN} + \partial^\nu\beta_i A_\nu^{MN}\end{aligned} \tag{80}$$

where use has been made of the vanishing of the curvature.

The $L_\beta$ denotes a Lie derivative with respect to the $\beta_\mu$ vector [2] and the equation above is being considered in the physical configuration. The symbol $\left[\tilde{D}_\mu\right]^{IJ}_{MN} \equiv \partial_\mu\Delta^{IJ}_{MN} + f^{IJ}_{KLMN}A_\mu^{KL}$ denote a covariant derivative with relation to the rotational and also the translational sectors of the gauge connection.

For the time variable, we have

$$\delta A_0^{MN} = \int d^3x\left\{A_0^{MN}, \pi_0^{IJ}\right\}^*\tau^{IJ} = L_\beta A_0^{MN} \tag{81}$$

with $\tau^{IJ} = L_\beta A_0^{MN}$.

[2] It acts in a $p$-form as $L_\beta\Omega = i_\beta \mathrm{d}\omega + \mathrm{d}i_\beta\Omega$ where $i_\beta$ is the contraction operator with relation to the vector $\beta_\mu$ and $d$ denotes the exterior derivative.

The diffeomorphism generator is given below

$$G = \int d^3x \Big[ C_{ab}\beta^{\mu} A_{\mu}^{ab} + C_{a4}\beta^{\mu} A_{\mu}^{a4} + \pi^{0}_{IJ} L_{\beta} A_0^{IJ} \Big] \tag{82}$$

## 7 Conclusions and perspectives

In this article we have furnished a Hamilton–Jacobi analysis of the Freidel–Starodubtsev model. Section 2 was devoted to introduce the method which was applied in Sect. 4.

We have analysed the physical content of the Freidel–Starodubtsev model in Sect. 3. We solved the polynomial constraint and then wrote the action just in terms of the physical fields. Although the model has several renormalizing topological terms, they do not change the equations of motion and then, the torsionless Einstein's equations with cosmological constant were recovered.

We did not skip to find the reduced phase space for the system in Sect. 5. In general, one fixes a gauge to help in this procedure. The point is that the Schwinger gauge would turn the involutive Hamiltonians into non-involutive ones and then we could not investigate the system's symmetry transformations. We have opted to consider a class of solutions with $e_i^0 = 0$ but allowing this field to have non-vanishing gradients in the phase space. So, that is different than fixing a gauge. It then helped us to invert the matrix of non-involutive Hamiltonians and then obtain a simple reduced phase space structure.

In Sect. 6, we found the equations of motion by means of the fundamental differential with fixed local parameters written in the reduced phase space which was equivalent to solving the polynomial constraint. The solution was a torsionless system with vanishing d$S$ curvature which is in accordance with the solution found in Sect. 3. In order to obtain the canonical transformations generators, the fundamental differential with fixed time variable was used, of course considering $D^k_{IJ} = 0$. Considering a given specific relation between the local parameters we were able to find the local Lorentz symmetry and also the diffeomorphism generators.

Regarding the future perspectives, we intend to investigate the present model in the presence of a boundary, namely an asymptotic one, where the Wald mass charge is calculated. We intend to calculate fluctuations between the mass eigenvalue and the asymptotic configuration of the fields.

**Acknowledgements** B.M. Pimentel thanks CNPq for partial support and G.B. de Gracia thanks CAPES for the financial support.

**Data Availability Statement** This manuscript has associated data in a data repository. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

## References

1. A.H. Chamseddine, D. Wyler, Phys. Lett. B **228**, 1 (1989)
2. N. Ikeda, JHEP **11**, 9 (2000)
3. L. Freidel, S. Speziale, SIGMA **8**, 032 (2012)
4. J.D. Brown, *Lower Dimensional Gravity* (World Scientific, Singapore, 1988)
5. J.C. Baez, Lect. Notes Phys. **543**, 25 (2000)
6. A. Perez, Living Rev. Rel. **16**, 3 (2013)
7. C. Teiltelboim, Phys. Lett. B **126**, 41 (1983)
8. R. Jackiw, Nucl. Phys. B **252**, 343 (1985)

9. D.K. Wise, Class. Quant. Grav. **27**, 155010 (2010)
10. C.P. Constantinids, J.A. Lourenço, I. Morales, O. Piguet, A. Rios, Class. Quant. Grav. **25**, 12 (2008)
11. A. Escalante, P. Cavildo-Sánchez, Adv. Math. Phys. **2018**, 3474760 (2018)
12. I. Oda, S. Yahikozawa, Class. Quant. Grav. **11**, 2653–2666 (1994)
13. E. Witten, Nucl. Phys. B **323**, 113 (1989)
14. J.C. Baez, Lett. Math. Phys. **38**, 129 (1996)
15. M. Mondragon, M. Montesinos, J. Math. Phys. **47**, 022301 (2006)
16. J.F. Plebanski, J. Math. Phys. **18**, 2511 (1977)
17. K. Krasnov, (2006). arXiv:hep-th/0611182
18. S.W. MacDowell, F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977)
19. S.W. MacDowell, F. Mansouri, Phys. Rev. Lett. **38**, 739, (1977)., Erratum-ibid.38:1376 (1977)
20. L. Freidel, A. Starodubtsev, (2005). arXiv:hep-th/0501191
21. R. Durka, J. Kowalski-Glikman, Phys. Rev. D **83**, 124011 (2011)
22. M. Martelini, M. Zeni, Phys. Lett. B **401**, 62 (1997)
23. E. Witten, Phil. Trans. R. Soc. Lond. A **329**, 349–357 (1989)
24. P.A.M. Dirac, Can. J. Math. **2**, 129 (1950)
25. P.A.M. Dirac, Can. J. Math. **3**, 1 (1951)
26. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964)
27. E. Buffenoir, M. Henneaux, K. Noui, Ph Roche, Class. Quant. Grav. **21**, 5203 (2004)
28. R. Durka, J. Kowalski-Glikman, Class. Quant. Grav. **27**, 185008 (2010)
29. C.E. Valcárcel, Gen. Rel. Grav. **49**, 11 (2017)
30. A. Escalante, I. Rubalcava-García, Int. J. Geom. Methods Mod. Phys. **9**, 1250053 (2012)
31. L. Castellani, Ann. Phys. **143**, 357 (1982)
32. D.M. Gitman, I.V. Tyutin, Int. J. Mod. Phys. A **21**, 327 (2006)
33. Y. Güler, Il Nuovo cimento B **107**, 1398 (1992)
34. C. Caratheódory, *Calculus of Variations and Partial Diferential Equations of the First Order*, 3rd edn. (American Mathematical Society, Providence, 1999)
35. M.C. Bertin, B.M. Pimentel, C.E. Valcárcel, Ann. Phys. **323**, 3137 (2008)
36. M.C. Bertin, B.M. Pimentel, C.E. Valcárcel, J. Math. Phys. **55**, 112901 (2014)
37. M.C. Bertin, B.M. Pimentel, C.E. Valcárcel, G.R. Zambrano, J. Math. Phys. **55**, 042902 (2014)
38. B.M. Pimentel, P.J. Pompèia, J.F. da Rocha-Neto, Il Nuovo cimento B **120**, 981 (2005)
39. M.C. Bertin, B.M. Pimentel, P.J. Pompeia, Ann. Phys. **325**, 2499 (2010)
40. M.C. Bertin, B.M. Pimentel, C.E. Valcárcel, J. Math. Phys. **53**, 102901 (2012)
41. N.T. Maia, B.M. Pimentel, C.E. Valcárcel, Class. Quant. Grav. **32**, 185013 (2015)
42. G.B. de Gracia, C.E. Valcárcel, B.M. Pimentel, Eur. Phys. J. Plus **132**, 438 (2017)
43. B.M. Pimentel, R.G. Teixeira, J.L. Tomazelli, Ann. Phys. **267**, 75 (1998)
44. M.C. Bertin, B.M. Pimentel, P.J. Pompeia, Mod. Phys. Lett. A **20**, 2873 (2005)
45. M.C. Bertin, B.M. Pimentel, P.J. Pompeia, Ann. Phys. **323**, 527 (2008)
46. A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986)
47. G. Immirzi, Class. Quant. Grav. **14**, L177 (1997)
48. L.A. Batalin, E.S. Fradkin, Phys. Lett. B **122**, 2 (1983)
49. J. Schwinger, Phys. Rev. D **130**, 1253 (1963)
50. V. de Sabbata, M. Gasperini, *Introduction to Gravitation* (World Scientific Publishing Co, Singapore, 1985)
51. C.P. Constantinids, F. Gieres, O. Piguet, M.S. Sarandy, JHEP **201**, 17 (2002)