Eur. Phys. J. Plus (2020) 135:440
https://doi.org/10.1140/epjp/s13360-020-00439-z

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Thin-shell wormholes in $f(R)$-gravity coupled with nonlinear electrodynamics

S. Habib Mazharimousavi[a], **Mustafa Halilsoy**[b], **Khashayar Kianfar**[c]

Department of Physics, Faculty of Arts and Sciences, Eastern Mediterranean University, North Cyprus via Mersin 10, Famagusta, Turkey

Received: 8 October 2019 / Accepted: 12 May 2020 / Published online: 29 May 2020

Abstract We study the possibility of constructing thin-shell wormhole (TSW) in a particular $f(R)$-gravity model with nonconstant Ricci scalar and coupled minimally with nonlinear electromagnetic fields. In doing so, first we give a new static spherically symmetric solution of the theory. Then we apply the cut-and-paste method to construct the TSW. As $f'''(R) \neq 0$ we use the specific junction conditions to match the two spacetimes. We find the exact equilibrium radius of the shell from non-black hole solution and show that a linear perturbation leaves the TSW stable.

1 Introduction

In Einstein's general theory of relativity described by the Einstein–Hilbert (EH) action supplemented by an energy-momentum, which is in general exotic, construction of thin-shell wormholes (TSWs) has turned almost into a routine process [1–3]. The original idea by Visser [1–3] was to localize the nonphysical source on a thin-layer, leaving the rest of the bulk with a physical source. Similar constructions of TSWs in modified, highly nonlinear theories have also been attempted [4–7]. Among those modified theories $f(R)$ theory [8–10] has already been much popular during recent decades. In this approach, the action of EH is modified into an arbitrary function of the Ricci scalar R denoted by $f(R)$ -theory. In general, such a theory may attain the EH limit or not. For physical requirements, however, the $f(R)$ theory must reproduce all the experimental tests that Einstein's theory has successfully passed. Besides, the stability criterion, as well as the absence of ghosts conditions, must be satisfied before the $f(R)$-theory is considered feasible [11,12]. In this paper our aim is restricted by construction of TSWs in a particular $f(R)$-theory given by $f(R) = R + 2\alpha\sqrt{R + R_0} + R_1$ [13,14], in which α, R_0 and R_1 are dimensionful constant parameters of the theory. For $\alpha = 0$, the theory reduces to the EH form in which R_1 acts as a cosmological constant. Our choice of $f(R)$ relies on an exact solution in the presence of nonlinear electromagnetism. The extended source of our $f(R)$ is provided by a Lagrangian of nonlinear electrodynamics (NED) of the form $\mathcal{L} = -\frac{1}{4\pi}\left(\mathcal{F} + 2\beta\sqrt{-\mathcal{F}}\right)$, in which $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the electromagnetic invariant

[a] e-mail: habib.mazhari@emu.edu.tr (corresponding author)
[b] e-mail: mustafa.halilsoy@emu.edu.tr
[c] e-mail: khashayar.kianfar@emu.edu.tr

and β is a constant parameter. In this approach, our NED is powered by a pure electric field without a magnetic component so that from the outset, our problem is assumed static. Once obtain an exact solution of the model our next step is to search for the proper junction conditions required in the construction of TSWs. We reviewed that the junction conditions valid for general relativity proposed long ago by Israel [15–17], does not apply in the present problem without modifications. We searched for an extension of those conditions and arrived at the conditions [18], applicable whenever we have $f''' = \frac{d^3 f}{dR^3} \neq 0$. To construct TSWs we employ the exact solutions for black holes or nonblack holes. Our finding in the present problems is that although we obtain black hole solution it's event horizon lies outside the possible radius/ throat of the TSW. Since this is not admissible for such passage through the throat from one universe to the other we had to abandon the black hole solution and be satisfied only with the nonblack hole solution with a naked singularity. As a matter of fact this is the case that we encountered first: usually, in other models, it was possible to choose the radius of the shell arbitrary outside the event horizon of the available black hole. Now we face a situation that the thin-shell cannot be located arbitrarily. The possible location of the shell which is determined by the theory contains a naked singularity at the center instead of a black hole. Once we fix our thin-shell appropriately to serve as a throat our next task is to perturb the resulting TSW. We do and find out that for stability to be effective a non-barotropic equation of state must be imposed at the throat after the perturbation. This implies that the pressure p and the surface energy density σ on the shell are related by $p = \mathcal{P}(a, \sigma)$ where a stands for the time-dependent radius of the shell. Such a type of variable equation of state was proposed a before as a possibility [19,20] but here it arises in a natural way which can be considered an interesting result. Naturally, if our TSW was not stable it would collapse at the slightest perturbation to the central naked singularity. Fortunately, this does not happen for a tuned set of parameters we obtain a harmonic oscillatory motion about the equilibrium radius of our TSW.

Let us add that TSW in $f(R)$-gravity has already been considered in the literature. In [21], E. F. Eiroa and G. F. Aguirre have constructed TSWs in the spherically symmetric bulk solution in the quadratic $f(R)$-gravity coupled with linear Maxwell field and constant curvature, i.e., $R = R_0$. Furthermore, in [22] the authors considered charged thin-shell wormholes in black string solutions in $f(R)$-gravity. The current work is somehow in the same line as of the Ref. [21] with different $f(R)$-gravity whose Ricci scalar is not constant and with $f'''(R) \neq 0$.

The paper is organized as follows. In Sect. 2 we introduce our model of $f(R)$-gravity coupled with nonlinear electrodynamics and derive exact solutions. In Sect. 3 we construct TSW in the bulk solution and study its stability against linear radial perturbations. We summarize the paper in Sect. 4 with the conclusion.

2 TSW in a model of $f(R)$-gravity with nonlinear electromagnetism

The action of the $f(R)$ modified theory of gravity coupled with a NED Lagrangian, is given by

$$I = \int \sqrt{-g} d^4 x \left(\frac{f(R)}{2\kappa} + \mathcal{L}(\mathcal{F}) \right), \tag{1}$$

in which $\mathcal{L}(\mathcal{F})$ is NED Lagrangian given in (we choose $\kappa = 8\pi$ and $G = 1$)

$$\mathcal{L}(\mathcal{F}) = -\frac{1}{4\pi} \left(\mathcal{F} + 2\beta\sqrt{-\mathcal{F}} \right), \tag{2}$$

where $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant, β is a constant parameter and our $f(R)$ is

$$f(R) = R + 2\alpha\sqrt{R + R_0} + R_1. \tag{3}$$

Herein, α, R_0 and R_1 are dimensionful constants and $F_{\mu\nu}$ is the electromagnetic field tensor defined through

$$\mathbf{F} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{4}$$

Variation of the action (1) with respect to the metric tensor $g_{\mu\nu}$ yields the Einstein's field equations given by

$$f'(R) R_\mu^\nu + \left(\Box f'(R) - \frac{1}{2} f(R) \right) \delta_\mu^\nu - \nabla^\nu \nabla_\mu f'(R) = \kappa T_\mu^\nu, \tag{5}$$

in which $f'(R) = \frac{\mathrm{d}f}{\mathrm{d}R}$, and $\Box\psi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu) \psi$. Furthermore, the energy momentum tensor T_μ^ν is given by

$$T_\mu^\nu = \mathcal{L}(\mathcal{F}) \delta_\mu^\nu - F_{\mu\lambda} F^{\nu\lambda} \frac{\partial \mathcal{L}(\mathcal{F})}{\partial \mathcal{F}}. \tag{6}$$

In this study we choose the spacetime to be spherically symmetric and static whose line element is given by

$$\mathrm{d}s^2 = -\psi(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\psi(r)} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right). \tag{7}$$

The NED field equations are also found by the variation of the action with respect to A_μ, the vector potential, and is given by the exterior algebraic notation

$$\mathrm{d} \left(\tilde{\mathbf{F}} \frac{\partial \mathcal{L}(\mathcal{F})}{\partial \mathcal{F}} \right) = 0, \tag{8}$$

in which $\tilde{\mathbf{F}}$ is the dual of $\mathbf{F}$. The Maxwell field used in this study is a pure electric field given by the 2−form

$$\mathbf{F} = E(r) \mathrm{d}t \wedge \mathrm{d}r, \tag{9}$$

whose dual 2−form field is obtained to be

$$\tilde{\mathbf{F}} = E(r) r^2 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{10}$$

The NED's equation, then becomes

$$\mathrm{d} \left(E r^2 \frac{\partial \mathcal{L}}{\partial \mathcal{F}} \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \right) = 0 \tag{11}$$

which upon the choice $E = E(r)$ yields

$$E(r) \frac{\partial \mathcal{L}}{\partial \mathcal{F}} r^2 = C \tag{12}$$

where, C is an integration constant. On the other hand, we find

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} E^2 \tag{13}$$

and since

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = -\frac{1}{4\pi} \left(1 - \frac{\beta}{\sqrt{-\mathcal{F}}} \right), \tag{14}$$

upon substitution into the NED Eq. (11) yields

$$E = \sqrt{2}\beta + \frac{Q}{r^2} \tag{15}$$

where C is identified as $C = -Q$, the electric charge. Let's note that the NED's Lagrangian (2), at the limit $\beta \to 0$, reduces to the linear Maxwell Lagrangian, i.e.,

$$\lim_{\beta\to 0} \mathcal{L}(\mathcal{F}) = -\frac{1}{4\pi}\mathcal{F} \tag{16}$$

Hence, expectedly the electric field found in (15), reduces to the classical electric field $\frac{Q}{r^2}$ in the limit $\beta \to 0$. Having the closed form of the electric field and the Maxwell's invariant $\mathcal{F} = -\frac{1}{2}E^2$ one finds from (6)

$$T_r^r = T_t^t = \frac{1}{4\pi}\left(\mathcal{L} - 2\mathcal{F}\mathcal{L}_{\mathcal{F}}\right), \tag{17}$$

and

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{4\pi}\mathcal{L}. \tag{18}$$

Explicitly, one finds

$$T_t^t = T_r^r = -\frac{1}{8\pi}\left(\sqrt{2}\beta + \frac{Q}{r^2}\right)^2, \tag{19}$$

and

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{8\pi}\left(2\beta^2 - \frac{Q^2}{r^4}\right). \tag{20}$$

We note that, considering this energy momentum tensor as a perfect fluid, i.e., $T_\mu^\nu = \text{diag}\left[-\rho, P_r, P_\theta, P_\phi\right]$, one can see that in order to satisfy the weak energy conditions including $\rho \geq 0$ and $\rho + P_i \geq 0$ with $i = r, \theta, \phi$, one must impose $\beta \geq 0$.

2.1 Black hole solution

In this section we use the energy momentum tensor's components found above to solve the Einstein's field Eq. (5). We start with the tt, and rr components of the Einstein's field equations, which read

$$f'(R)R_t^t + \left(\Box f'(R) - \frac{1}{2}f\right) - \nabla^t\nabla_t f'(R) = \kappa T_t^t \tag{21}$$

and

$$f'(R)R_r^r + \left(\Box f'(R) - \frac{1}{2}f\right) - \nabla^r\nabla_r f'(R) = \kappa T_r^r. \tag{22}$$

Herein,

$$R_t^t = R_r^r = -\frac{r\psi'' + 2\psi'}{2r} \tag{23}$$

in which a prime denotes derivative with respect to r. Subtracting (21) from (22) one gets

$$\nabla^t\nabla_t f'(R) = \nabla^r\nabla_r f'(R) \tag{24}$$

and since we have, $R_t^t = R_r^r$ and $T_t^t = T_r^r$ this implies

$$\psi f'_{,r,r} = 0 \tag{25}$$

or equivalently $f'_{,r,r} = 0$. This suggests that the second derivative of $f'(R)$ with respect to r must be zero. Hence, a general solution for $f'(R)$ is found to be

$$f'(R) = c_0 + c_1 r, \tag{26}$$

in which c_0 and c_1 are two integration constants. On the other hand , $f(R)$ is given in (3), therefore

$$\frac{\mathrm{d}}{\mathrm{d}R}(R + 2\alpha\sqrt{R + R_0} + R_1) = c_0 + c_1 r. \tag{27}$$

Hence, we can solve the latter equation to find R, which amounts to

$$R = -R_0 + \frac{\alpha^2}{(c_1 r + c_0 - 1)^2}. \tag{28}$$

For the sake of simplicity we set $c_0 = 1$, such that the Ricci scalar becomes

$$R = -R_0 + \frac{\alpha^2}{c_1^2 r^2}. \tag{29}$$

Moreover, the explicit form of the Ricci scalar in terms of the metric function ψ is given by

$$R = -\frac{r^2\psi'' + 5r\psi' + 2(\psi - 1)}{r^2}. \tag{30}$$

Equating (29) and (30) gives a second-order DE which can be solved for ψ. The solution is given by

$$\psi = 1 - \frac{6\alpha^2}{c_1^2} + \frac{R_0}{12}r^2 + \frac{c_2}{r} + \frac{c_3}{r^2}, \tag{31}$$

in which c_2 and c_3 are new integration constants. Let's add that the metric function found in (31) has to satisfy all the gravitational field equations and through those we find the nature of the parameters and integration constants. Once more we look at the tt and rr components of the Einstein's field equations. Knowing $f'(R) = 1 + c_1 r$ one finds

$$(1 + c_1 r)R_t^t + \left(\frac{c_1}{r}(2\psi + r\psi') - \frac{1}{2}f\right) - \frac{c_1}{2}\psi' = \kappa\left(-\frac{1}{8\pi}\right)\left(\sqrt{2}\beta + \frac{Q}{r^2}\right)^2. \tag{32}$$

To find the closed form of f in terms of r we apply the chain rule, i.e.,

$$\frac{\mathrm{d}f}{\mathrm{d}R} = \frac{\frac{\mathrm{d}f}{\mathrm{d}r}}{\frac{\mathrm{d}R}{\mathrm{d}r}} = 1 + c_1 r. \tag{33}$$

Having R given in (29) one finds

$$\frac{\mathrm{d}f}{\mathrm{d}r} = -(1 + c_1 r)\frac{2\alpha^2}{c_1^2 r^3} \tag{34}$$

which admits a solution for $f(r)$ given by

$$f = \frac{\alpha^2}{c_1^2 r^2} + \frac{2\alpha^2}{c_1 r} + c_4 \tag{35}$$

where c_4 is an integration constant. Comparing (35) with (3) reveals that $c_4 = R_1 - R_0$. Finally, satisfying Eq. (32) reveals the following relations between the constant parameters,

$$2\beta^2 - \frac{R_0}{4} = \frac{c_4}{2}, \tag{36}$$

$$c_1^2 - \alpha^2 = 0, \tag{37}$$

$$4\sqrt{2}Q\beta c_1^2 + 3c_2c_1^3 - \alpha^2 = 0, \tag{38}$$

and

$$Q^2 - c_3 = 0. \tag{39}$$

Imposing, $c_1 = \alpha$, $c_2 = \frac{1-4\sqrt{2}\beta Q}{3\alpha}$, $c_3 = Q$ and $c_4 = 4\beta^2 - \frac{R_0}{2}$ yields

$$\psi(r) = \frac{1}{2} + \frac{R_0}{12}r^2 - \frac{4\sqrt{2}\beta Q - 1}{3\alpha r} + \frac{Q^2}{r^2}, \tag{40}$$

and

$$f(R) = \frac{1}{r^2} + \frac{2\alpha}{r} + 4\beta^2 - \frac{R_0}{2}. \tag{41}$$

Rewriting $f(R)$ in terms of R, one finds

$$f(R) = R + 2\alpha\sqrt{R + R_0} + 4\beta^2 + \frac{R_0}{2}, \tag{42}$$

which upon comparison with the original form given in (3) yields

$$R_1 = 4\beta^2 + \frac{R_0}{2}. \tag{43}$$

The last two equations to be checked are the $\theta\theta$ and $\phi\phi$ components of the Einstein's field equations. Due to the symmetry $\theta\theta$ and $\phi\phi$ components of the Einstein's equation are identical. Let's concentrate on $\theta\theta$ component. From (5) we find

$$f' R_\theta^\theta + \left(\Box f' - \frac{1}{2}f\right) - \nabla^\theta\nabla_\theta f' = \kappa T_\theta^\theta, \tag{44}$$

in which we have

$$\nabla^\theta\nabla_\theta f' = \frac{\psi}{r}f'_{,r}. \tag{45}$$

Explicitly, we obtain

$$(1 + \alpha r)R_\theta^\theta + \left(\frac{\alpha}{r}(2\psi + r\psi') - \frac{1}{2}f\right) - \frac{\alpha\psi}{r} = -\frac{\kappa}{8\pi}\left(2\beta^2 - \frac{Q^2}{r^4}\right). \tag{46}$$

Putting ψ and f from (40) and (41), respectively, and considering

$$R_\theta^\theta = -\frac{r\psi' - 1 + \psi}{r^2}. \tag{47}$$

one finds that (44) is satisfied. Let's add that the solution for $\psi(r)$ given in (40) whose Ricci scalar becomes

$$R = \frac{1}{r^2} - R_0 \tag{48}$$

is singular at $r = 0$. Moreover, $\frac{R_0}{12}$ in $\psi(r)$ plays the role of an effective cosmological constant. In the sequel we set $R_0 = 0$ such that the solution becomes rather simpler

$$\psi = \frac{1}{2} - \frac{\mu}{3\alpha r} + \frac{Q^2}{r^2}, \tag{49}$$

in which $\mu = 4Q\sqrt{2}\beta - 1$ and

$$f(R) = R + 2\alpha\sqrt{R} + 4\beta^2, \tag{50}$$

with the NED Lagrangian given in (2). The solution for $\psi(r)$ given in (49) admits, black hole with two horizons or a single double horizon, and naked singularity, depending on whether $\left(\frac{18\alpha^2 Q^2}{\mu^2}\right)$ is less than, equal to or greater than one, respectively. The two horizons are given by

$$r_{\pm} = \frac{\mu}{3\alpha}\left(1 \pm \sqrt{1 - \frac{18\alpha^2 Q^2}{\mu^2}}\right). \tag{51}$$

while the double horizon is found to be at

$$r_D = \frac{\mu}{3\alpha}. \tag{52}$$

Note also that if $\mu^2 < 18\alpha^2 Q^2$ there is no horizon at all, and the solution becomes naked singular.

3 TSW in $f(R) = R + 2\alpha\sqrt{R} + 4\beta^2$ gravity

Let's start with the line element (7) with $\psi(r)$ given in (49) where the solution of the $f(R)$-gravity (50) is coupled with the nonlinear electrodynamics (2). Using the standard method of cut-and-paste introduced by Matt Visser in [1–3] and applying the generalized Israel junction conditions, in this chapter we construct a TSW. First, we cut-out the region $r < a(\tau)$ from the bulk spacetime (7) and make two identical copies of the rest of the manifold and call them $\mathcal{M}^+$ and $\mathcal{M}^-$. $\mathcal{M}^{\pm}$ are individually incomplete but after we paste them at their identical boundary $r = a(\tau)$, the resultant Manifold $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ is a complete manifold.
The two submanifolds $\mathcal{M}^+$ and $\mathcal{M}^-$ are connected with a thin-shell (timelike) defined by $r = a(\tau)$. This spherical timelike thin-shell is called the throat between the two submanifolds. In other words, assume a traveler is going toward the center of the spacetime $\mathcal{M}^+$. When she reaches at $r = a$, without noticing, enters the second spacetime $\mathcal{M}^-$. Hence $r = a(\tau)$ which is a timelike hypersurface plays the role of a gate (or throat). In principle we chose $a(\tau) > r_h$ in which r_h is the event horizon of the bulk spacetime. Therefore, the traveler never encounters a horizon in her journey from $\mathcal{M}^+$ toward $\mathcal{M}^-$ or in opposite. The hypersurface $\Sigma^{\pm} := r^{\pm} - a(\tau) = 0$, is one of the boundary of each submanifold and we glue them at $\Sigma = r - a(\tau) = 0$. In each submanifold one writes

$$\mathrm{d}s_{\pm}^2 = -\psi(r_{\pm})\mathrm{d}t^2 + \frac{\mathrm{d}r_{\pm}^2}{\psi(r_{\pm})} + r_{\pm}^2\left(\mathrm{d}\theta_{\pm}^2 + \sin^2\theta_{\pm}\mathrm{d}\phi_{\pm}^2\right). \tag{53}$$

Following, the Israel junction conditions in $f(R)$ -gravity [11, 14], the first boundary condition is to have the induced metric continuous across the throat. Using the definition of the induced metric for $\mathcal{M}^+$ and $\mathcal{M}^-$ one finds;

$$h_{\mathrm{ij}}^{\pm} = g_{\alpha\beta}^{\pm}\frac{\partial x_{\pm}^{\alpha}}{\partial\xi_{\pm}^{i}}\frac{\partial x_{\pm}^{\beta}}{\partial\xi_{\pm}^{j}}, \tag{54}$$

in which $\alpha, \beta = \{t, r, \theta, \phi\}$ while $i, j = \{t, \theta, \phi\}$. Explicitly, this means that, $r_{\pm} = a(\tau)$, $\theta_{\pm} = \theta$, $\phi_{\pm} = \phi$ and

$$\dot{t}_{\pm}^2 = \dot{t}^2 = \frac{1}{\psi(a)}\left(1 + \frac{\dot{a}^2}{\psi(a)}\right). \tag{55}$$

which yields

$$ds_{\pm}^2 = ds_{\Sigma}^2 = -d\tau^2 + a^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{56}$$

where a dot implies derivative with respect to the proper time. From now on we refer to (56), as the induced metric of the throat in which τ stands for the proper time on the throat. To apply the other boundary conditions we must introduce the normal four vectors $n_{\gamma}^{\pm}$ on the throat from both manifolds' perspective. The standard definition of $n_{\gamma}^{\pm}$ are given by

$$n_{\gamma}^{\pm} = \frac{\pm 1}{\sqrt{\Delta^{\pm}}} \frac{\partial \Sigma^{\pm}}{\partial x^{\gamma}}, \tag{57}$$

in which $\Sigma^{\pm} := r^{\pm} - a(\tau) = 0$ and $\Delta^{\pm}$is the coefficient that makes $n_{\gamma}^{\pm} n^{\pm\gamma} = 1$. The positive direction is chosen to be from the throat toward $\mathcal{M}^{+}$ which makes the negative direction from the throat toward $\mathcal{M}^{-}$. Therefore, one finds

$$n_{\gamma}^{\pm} = \pm \frac{\sqrt{\psi + \dot{a}^2}}{\psi} \left(-\frac{\dot{a}}{\dot{t}}, 1, 0, 0 \right), \tag{58}$$

or using $\dot{t} = \frac{\sqrt{\psi + \dot{a}^2}}{\psi}$ one may write

$$n_{\gamma}^{\pm} = \pm \left(-\dot{a}, \dot{t}, 0, 0 \right). \tag{59}$$

The next quantity to be calculated is the second fundamental form $K_{\rm ij}^{\pm}$. According to the definition

$$K_{\rm ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^i} \right) \Bigg|_{\pm}, \tag{60}$$

which explicitly yields

$$K_i^{j\,\pm} = \mathrm{diag} \left(\frac{\psi' + 2\ddot{a}}{2\sqrt{\psi + \dot{a}^2}}, \frac{\sqrt{\psi + \dot{a}^2}}{a}, \frac{\sqrt{\psi + \dot{a}^2}}{a} \right). \tag{61}$$

In $f(R)$-gravity with $f'''(R) \neq 0$, the following two conditions should be satisfied

$$\left[K_i^i \right] = K_i^{i+} - K_i^{i-} = 0, \tag{62}$$

and

$$[R] = R^{+} - R^{-} = 0. \tag{63}$$

We note that all quantities are evaluated at the throat. The second one is trivially satisfied because the two submanifolds $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ are identical and therefore $R^{+} = R^{-}$. The first condition, however, implies

$$\frac{\psi' + 2\ddot{a}}{2\sqrt{\psi + \dot{a}^2}} + \frac{2}{a}\sqrt{\psi + \dot{a}^2} = 0. \tag{64}$$

This condition effectively gives a dynamic equation for the throat's radius which we shall consider in the stability analysis. Let's assume an equilibrium radius for the throat such that $\dot{a} = \ddot{a} = 0$ and $a = a_0$ where the latter equation becomes

$$\frac{\psi'(a_0)}{2\sqrt{\psi(a_0)}} + \frac{2}{a_0}\sqrt{\psi(a_0)} = 0, \tag{65}$$

or

$$a_0 \psi'(a_0) + 4\psi(a_0) = 0. \tag{66}$$

This is a restrictive constraint on the equilibrium radius which gives the location of the throat uniquely. Finally we apply the last junction condition given by

$$\kappa S_i^j = -f'(R)[K_i^j] + f''(R)\left[n^\gamma \nabla_\gamma R\right]\delta_i^j, \tag{67}$$

in which $S_i^j = (-\sigma, p, p)$ is the matter energy-momentum tensor on the throat. Explicitly, we find $[n^\gamma \nabla_\gamma R] = 2\sqrt{\psi + \dot{a}^2}R'(a)$ and upon considering (67), we obtain ($\kappa = 8\pi G$, $G = 1$)

$$\sigma = \frac{1}{8\pi}\left(f'\frac{\psi' + 2\ddot{a}}{2\sqrt{\psi + \dot{a}^2}} - 2f''\sqrt{\psi + \dot{a}^2}R'\right) \tag{68}$$

and

$$p = \frac{1}{8\pi}\left(-2f'\frac{\sqrt{\psi + \dot{a}^2}}{a} + 2f''\sqrt{\psi + \dot{a}^2}R'\right). \tag{69}$$

For the specific $f(R)$-gravity under consideration one finds

$$f' = 1 + \frac{\alpha}{\sqrt{R}} \quad \text{and} \quad f'' = \frac{-\alpha}{2(R)^{\frac{3}{2}}}. \tag{70}$$

with $R = \frac{1}{a^2}$ it yields

$$R' = \frac{-2}{a^3}, \quad f' = 1 + \alpha a, \quad f'' = -\frac{\alpha}{2}a^3, \tag{71}$$

and consequently

$$\sigma = \frac{1}{8\pi}\frac{(1 + \alpha a)(\psi' + 2\ddot{a})}{\sqrt{\psi + \dot{a}^2}} - \frac{\alpha}{4\pi}\sqrt{\psi + \dot{a}^2} \tag{72}$$

with

$$p = -\frac{1}{a}\frac{\sqrt{\psi + \dot{a}^2}}{4\pi}. \tag{73}$$

Furthermore, imposing (64) to (72) implies

$$\sigma = -\left(\frac{2}{a} + 3\alpha\right)\frac{\sqrt{\psi + \dot{a}^2}}{4\pi}. \tag{74}$$

Therefore, from (73) and (74), the equation of state (EoS) of the matter on the shell is found to be

$$p = \omega\sigma \tag{75}$$

in which

$$\omega = \frac{1}{2 + 3a\alpha}. \tag{76}$$

It is remarkable to observe that, naturally, the equation of state turns out to be of the non-barotropic type, i.e., $p = p(a, \sigma)$ [19,20]. Furthermore, at the equilibrium state where $a = a_0$ and $\dot{a} = \ddot{a} = 0$ one finds

$$\sigma_0 = -\frac{1}{4\pi}\left(\frac{2}{a_0} + 3\alpha\right)\sqrt{\psi_0} \tag{77}$$

and

$$p_0 = -\frac{1}{4\pi}\frac{1}{a_0}\sqrt{\psi_0}. \tag{78}$$

From Eqs. (77) and (78), we observe that for $\alpha > 0$, $\sigma_0 < 0$ and $p_0 < 0$. However, for $\alpha < 0$ such that $|\alpha| > \frac{3}{a_0}$, not only σ_0 but also $\sigma_0 + p_0$ become positive. In other words, the induced energy momentum tensor on the throat satisfies the weak energy conditions. Moreover, for satisfying the strong energy conditions by the energy momentum tensor one has to consider $|\alpha| > \frac{4}{a_0}$. Looking carefully at the model of $f(R)$-gravity considered in this study, i.e., Eq. (50), we see that α is originated from the modified theory of gravity such that $\alpha = 0$ takes the theory to the Einstein's R-gravity. Hence, having the energy conditions satisfied, is a feature of constructing TSW in $f(R)$-gravity. Nevertheless we should add that, having $\alpha < 0$ causes that, $f'(R)$ become negative for $r > -\frac{1}{\alpha}$ which is an indication of having ghost in the theory.

3.1 The radius of the throat

Our metric function is given in (49) from which one finds the equilibrium equation explicitly as

$$2\alpha {a_0}^2 - \mu a_0 + 2Q^2\alpha = 0. \tag{79}$$

There are two roots for this equation given by

$$a_0^{\pm} = \frac{\tilde{\mu}}{4}\left(1 \pm \sqrt{1 - \left(\frac{4Q}{\tilde{\mu}}\right)^2}\right) \tag{80}$$

in which $\tilde{\mu} = \frac{\mu}{\alpha}$ has to be positive. On the other hand, the horizon of the solution (49) is given by

$$\psi(r_h) = 0 \tag{81}$$

in which

$$r_h^{\pm} = \frac{\tilde{\mu}}{3}\left(1 \pm \sqrt{1 - \left(\frac{\sqrt{18}Q}{\tilde{\mu}}\right)^2}\right). \tag{82}$$

It is revealed from (80) and (82) that $a_0^{\pm}$ are smaller than the event horizon r_h^{+}. Under this condition in order to construct TSW the only alternative left for this solution is the non-black hole case, i.e., $\psi(r) \neq 0$. This condition, combined with the existence of the shell at equilibrium (80) amounts to

$$16\, Q^2 < \tilde{\mu}^2 < 18 Q^2. \tag{83}$$

Finally the unique pair of the equilibrium radii of the TSW are found to be those given in (80). In other words, for fixed Q, and $\tilde{\mu}$ there are two possible radii for the TSW denoted as $a_0^{\pm}$.

3.2 Linear stability analysis

In this section we study the dynamical stability of the TSW solution, constructed in the previous section. In doing this, we apply a linear-radial perturbation to the TSW and upon that $\dot{a}$ and $\ddot{a}$ are not zero. The radius of the throat $a(\tau)$ after the perturbation should satisfy (64). One may rewrite (64) as

$$\psi' + 2\ddot{a} = \frac{-4}{a}(\psi + \dot{a}^2), \tag{84}$$

which after applying the chain-rule, i.e., $\ddot{a} = \frac{\mathrm{d}\dot{a}}{\mathrm{d}a}\dot{a}$ one simply finds

$$a^4\psi' + 4a^3\psi = a^4\frac{\mathrm{d}}{\mathrm{d}a}(\dot{a}^2) + 4a^3(\dot{a}^2). \tag{85}$$

Both sides are total derivatives, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}a}(a^4\psi) = \frac{\mathrm{d}}{\mathrm{d}a}(a^4(\dot{a}^2)), \tag{86}$$

which after an integration yields

$$a^4\psi = a^4(\dot{a}^2) + c, \tag{87}$$

in which c in the integration constant. To find c, we recall that at $a = a_0$, $\dot{a} = 0$. Hence, we obtain

$$c = a_0^4\psi(a_0) = a_0^4\psi_0. \tag{88}$$

Taking back c into (87) we get finally

$$\dot{a}^2 + \frac{a_0^4}{a^4}\psi_0 - \psi = 0. \tag{89}$$

This is the equation of motion of the throat after the perturbation. Rewriting this equation in the standard form

$$\dot{a}^2 + V(a) = 0, \tag{90}$$

one finds the effective potential

$$V(a) = \frac{a_0^4}{a^4}\psi_0 - \psi. \tag{91}$$

As we have already assumed, let's keep the equilibrium point to be at $a = a_0$ where $\dot{a}_0 = \ddot{a}_0 = 0$. Then, a Taylor expansion of the potential $V(a)$ about $a = a_0$ implies

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + \mathcal{O}((a - a_0)^3), \tag{92}$$

in which $V(a_0) = V'(a_0) = 0$ identically, while

$$V''(a_0) = \frac{20}{a_0^2}\psi_0 - \psi_0''. \tag{93}$$

On the other hand, the linear equation of motion of the TSW after the perturbation becomes

$$\dot{a}^2 + \frac{1}{2}V''(a_0)(a - a_0)^2 \simeq 0. \tag{94}$$

Therefore, the nature of the solution of (94) depends on the sign of $V''(a_0)$. If $V''(a_0) > 0$ then the solution after the perturbation is oscillatory which is an indication for stability. Otherwise if $V''(a_0) < 0$ the motion becomes of exponential type which implies that the TSW is unstable. Therefore, we shall look for the possible values for the parameters such that the expansion for $V''(a_0)$, becomes positive. Considering ψ given in (49) with $R_0 = 0$ and $\tilde{\mu} = \frac{4\sqrt{2}\beta Q - 1}{\alpha} > 0$ one finds

$$V''(a_0) = \frac{6\tilde{\mu}a_0 - 14Q^2 - 10a_0^2}{a_0^4}. \tag{95}$$

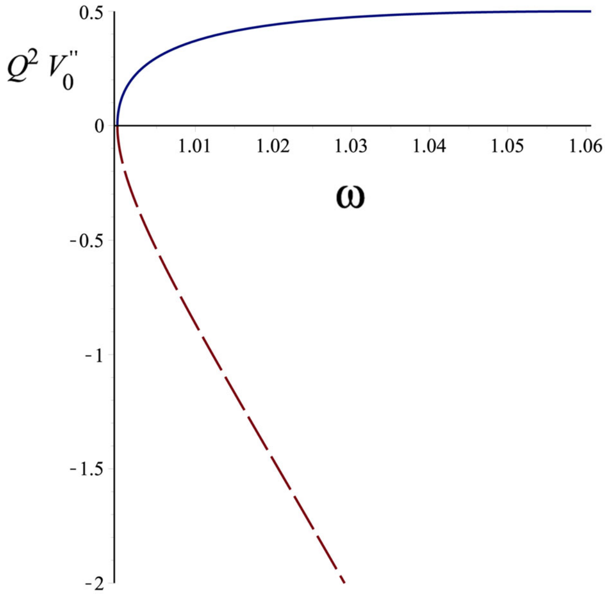

Fig. 1 $Q^2 V_0''(\omega)$ in terms of ω for both radii of equilibrium $a_0^{\pm}$. The solid-blue curve and the brown-dashed curve correspond to a_0^+ and a_0^-, respectively

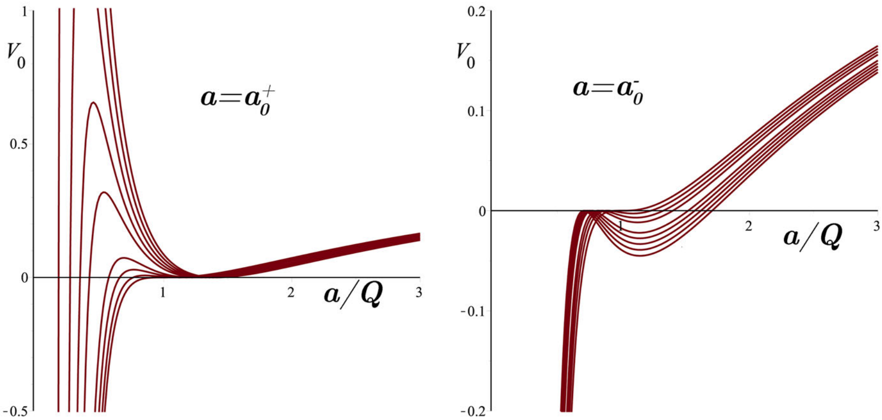

Fig. 2 V_0 in terms of a/Q for $\omega = 1..\frac{3\sqrt{2}}{4}$. The left panel and the right panel are for $a_0 = a_0^+$ an $a_0 = a_0^-$, respectively. The value of ω varies with equal intervals from below / above in left / right panel. The minimum and maximum of the potential are seen clearly for $a_0 = a_0^+$ an $a_0 = a_0^-$, respectively

Furthermore, a_0 is given in Eq. (80) which upon a substitution in (95) we find

$$V_0''(a_0^{\pm}) = \frac{32\left(7\tilde{\mu}^2 \pm 2\tilde{\mu}\sqrt{\xi} - 112Q^2 - 5\xi\right)}{\left(\tilde{\mu} \pm \sqrt{\xi}\right)^4}, \tag{96}$$

in which $\xi = \left(\tilde{\mu}^2 - 16Q^2\right) > 0$. We recall that $\tilde{\mu}$ has been bounded due to other condition given in (83). Let's introduce new variable such that $\tilde{\mu} = 4Q\omega$ upon which V_0'' becomes

$$V_0''(\omega) = \frac{4\left(\omega^2 \pm \omega\sqrt{\omega^2-1} - 1\right)}{Q^2(\omega \pm \sqrt{\omega^2-1})^4} \tag{97}$$

and

$$a_0(\omega) = Q\left(\omega \pm \sqrt{\omega^2-1}\right). \tag{98}$$

In Fig. 1 we plot $V_0''(\omega)$ in terms of $\omega = 1..\frac{3\sqrt{2}}{4}$ for both $a_0^{\pm}$. Figure 1 shows clearly that in the domain of ω (and consequently μ), $V_0'' > 0$ for $a_0 = a_0^+$ and $V_0'' < 0$ for $a_0 = a_0^-$. It means that the constructed TSW is stable for $a_0 = a_0^+$ and is unstable if $a_0 = a_0^-$. To have a better picture of the meaning of stability, we plot the potential V_0 itself in terms of a/Q for various values of ω and $a_0 = a_0^{\pm}$ in different frames in Fig. 2. At $a = a_0^+$ the potential (left panel) admits a local minimum which states that the corresponding TSW is stable. In contrast, the potential for $a = a_0^-$ (right panel) possesses a local maximum at this point which indicates that the corresponding TSW is unstable.

4 Conclusion

Due to the stringent junction conditions construction of TSWs in $f(R)$ -gravity in contrast to Einstein's general relativity, is a rather difficult operation. This originates from the tough conditions imposed on the first and second fundamental forms. We overcome the difficulty by considering a class of $f(R)$ model coupled with NED whose third derivative, i.e., $f'''(R) \neq 0$ so that it satisfies the generalized junction conditions. An exact solution is obtained which is supported by an external static field within the context of NED. It admits electric black holes which, however, does not serve our purpose of TSWs. The reason is simple: the existence of the event horizon is not compatible with the radius of the thin-shell. We follow therefore a different route. We choose the non-black hole branch of the solution which allows us to locate the shell. The shell's radius becomes bounded from above which is stable against linear radial perturbations. The fluid energy-momentum emerging at the throat upon perturbation satisfies naturally a non-barotropic equation of state. If the shell was not stable then it would collapse at the slightest perturbation to the naked singularity at the center.

To complete our conclusion we would like to compare our work with the earlier works such as [21–24]. In [21] the explicit form of $f(R)$ is given by

$$f(R) = R + \alpha R^2 \tag{99}$$

which is clearly in a different class in comparison with our model given in (50) in the sense that in (99) in third derivative $f'''(R) = 0$ while in (50) it is non-zero. This in turn implies different junction conditions. On the other hand in [21], the Ricci scalar was considered to be constant, i.e., $R = R_0$ while in this current work R is not a constant as seen in Eq. (48). Similarly, in Ref. [22], and [23] the authors considered different models of $f(R)$-gravity in constant curvature background. Finally, we would like to mention a rather different study of Eiroa and Aguirre [24], where they considered the background curvature in each side of the wormhole to be constant but different from each other. In other words, at each side of the throat, the geometry has a different constant curvature. In short, the main difference between the current study and the other papers on TSW in $f(R)$ gravity, is the structure of the background curvature. While in the other studies the background curvature considered to be constant, we have considered a non-constant background curvature. Let's note that, having non-constant background curvature implies non-trivial consequences due to the junction conditions such as occurrence of the non-barotropic EoS, i.e., $p = p(a, \sigma)$ given in Eq. (75).

References

1. M. Visser, Phys. Rev. D **39**, 3182 (1989)
2. M. Visser, Nucl. Phys. B **328**, 203 (1989)
3. M. Visser, *Lorentzian Wormholes* (AIP Press, New York, 1996)
4. S.H. Mazharimousavi, M. Halilsoy, Z. Amirabi, Class. Quant. Grav. **28**, 025004 (2011)
5. Z. Amirabi, M. Halilsoy, S.H. Mazharimousavi, Phys. Rev. D **88**, 124023 (2013)
6. M.H. Dehghani, M.R. Mehdizadeh, Phys. Rev. D **85**, 024024 (2012)
7. M.R. Mehdizadeh, M.K. Zangeneh, F.S.N. Lobo, Phys. Rev. D **92**, 044022 (2015)
8. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**, 59 (2011)
9. A. De Felice, S. Tsujikawa, Living Rev. Relat. **13**, 3 (2010)
10. T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010)
11. A. de la Cruz-Dombriz, A. Dobado, A.L. Maroto, Phys. Rev. D **80**, 124011 (2009)
12. A. de la Cruz-Dombriz, A. Dobado, A.L. Maroto, Phys. Rev. D **83**, 029903(E) (2011)
13. S.H. Mazharimousavi, M. Halilsoy, T. Tahamtan, Eur. Phys. J. C **72**, 1851 (2012)
14. S.H. Mazharimousavi, M. Halilsoy, T. Tahamtan, Eur. Phys. J. C **72**, 1958 (2012)
15. G. Darmois, *Mèmorial des Sciences Màthematiques, Fascicule XXV* (Gauthier-Villars, Paris, 1927). Chap V
16. W. Israel, Nuovo Cimento B **44**, 1 (1966)
17. W. Israel, Nuovo Cimento B **48**, 463(E) (1967)
18. J.M.M. Senovilla, Phys. Rev. D **88**, 064015 (2012)
19. N.M. Garcia, F.S.N. Lobo, M. Visser, Phys. Rev. D **86**, 044026 (2012)
20. V. Varela, Phys. Rev. D **92**, 044002 (2015)
21. E.F. Eiroa, G.F. Aguirre, Eur. Phys. J. C **76**, 132 (2016)
22. M.Z. Bhatti, Z. Yosaf, S. Ashraf, Ann. Phys. **383**, 439 (2017)
23. M.Z. Bhatti, A. Anwar, S. Ashraf, Mod. Phys. Lett. A **32**, 1750111 (2017)
24. E.F. Eiroa, G.F. Aguirre, Phys. Rev. D **94**, 044016 (2016)