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Arbitrary-order Darboux transformations for two-dimensional Dirac equations with position-dependent mass

Axel Schulze-Halberg[a]

Department of Mathematics and Actuarial Science and Department of Physics, Indiana University Northwest, 3400 Broadway, Gary, IN 46408, USA

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Abstract We construct higher-order Darboux transformations for Dirac equations in two dimensions that feature a position-dependent mass. Our method allows to generate closed-form expressions for both a transformed potential and a transformed position-dependent mass function.

1 Introduction

Dirac materials [3] are lattice systems in which low-energy charge carriers behave like Dirac fermions. The development of these materials started with the isolation of graphene [7,9], a two-dimensional monolayer of carbon atoms forming a hexagonal honeycomb lattice. Further examples for Dirac materials include d-wave superconductors [2,12], superfluids [14,21], and topological insulators [20,23], just to name a few. Dirac materials exhibit many unusual properties, one of which is Klein tunneling. This phenomenon, referring to perfect transmission of Dirac fermions impinging perpendicularly to a potential barrier, was theoretically predicted some time ago [10] and experimentally observed in graphene [24]. The presence of Klein tunneling inhibits the existence of bound states within a Dirac material due to high mobility of the charge carriers. A variety of techniques have been proposed to achieve confinement of Dirac fermions, see [5,6] for an overview. One of the techniques proposed in [6] is the introduction of a position-dependent mass function into the governing Dirac equation. It turned out that charge carriers associated with a spatially varying mass can undergo confinement, provided the mass function is chosen suitably. This was shown by an example featuring closed-form solutions in terms of Bessel functions, see [6] for details. In general, closed-form solutions of Dirac equations are rare and therefore hard to find. One of the most effective techniques to find such solutions is the Darboux transformation. The first version of this transformation [4] that applied to linear second-order equations, meanwhile has been generalized to be compatible with a variety of linear and nonlinear models [8,13], including the Dirac equation. Particularly in the two-dimensional case governing Dirac materials, a method was devised [17] to adapt the first-order Darboux transformation, based on a result on Schrödinger models for quadratically energy-dependent potentials [11]. Besides the Dirac

[a] e-mail: axgeschu@iun.edu (corresponding author)

equation, these models can be linked to Korteweg-de Vries [22] and Klein–Gordon systems [18].

The purpose of the present work is to construct higher-order Darboux transformations for two-dimensional Dirac equations featuring a position-dependent mass in combination with a diagonal potential matrix. Our method uses a recent generalization [15,16] of the results from [11], as well as a connection between two-dimensional Dirac equations and their Schrödinger counterparts for quadratically energy-dependent potentials. The remainder of this note is organized as follows: in Sect. 2, we summarize main results from [15]. Section 3 is devoted to the construction of the Darboux transformation, while we give an example in Sect. 4. Finally, "Appendix A" is devoted to stating a few excessively long expressions that result from our example.

2 Preliminaries

For the sake of completeness, we briefly review the principal results from [15]. Our goal is to establish a Darboux transformation between the following two Schrödinger equations with quadratically energy-dependent potentials:

$$\psi_0''(x) - [E^2 + E\ V_0(x) + U_0(x)]\ \psi_0(x) = 0 \tag{1}$$

$$\psi_n''(x) - [E^2 + E\ V_n(x) + U_n(x)]\ \psi_n(x) = 0. \tag{2}$$

The energy E is a real constant, the functions V_j, U_j, $j = 0, 2$ are the potential terms that do not depend on E, and ψ_0, ψ_n stand for the respective solutions, where the index n is a natural number. Assume that h_j, $j = 0, \ldots, n-1$, are auxiliary solutions to Eq. (1) at energies λ_j, $j = 0, \ldots, n-1$, respectively, such that the constants $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, E are pairwise different. Define functions v_j, $j = 0, \ldots, n-1$, by means of

$$v_j(x) = \exp\left[(E - \lambda_j)\ x\right]\ h_j(x), \quad j = 0, \ldots, n-1. \tag{3}$$

Next, we introduce the n-th order Darboux transformation of the solution ϕ_0 to (1) as

$$D_{h_0,\ldots h_{n-1}}(\psi_0)(x) = \frac{W_{v_0,\ldots,v_{n-1},\psi_0}(x)}{\sqrt{\hat{W}_{n-1}(x)\ W_{v_0,\ldots,v_{n-1}}(x)}}, \tag{4}$$

where $W_{v_0,\ldots,v_{n-1}}$ and $W_{v_0,\ldots,v_{n-1},\psi_0}$ denote the Wronskians of $v_0, \ldots, v_{n-1}$ and of $v_0, \ldots, v_{n-1}, \psi_0$, respectively. Furthermore, the quantities $\hat{W}_j$, $j = 0, \ldots, n-1$, are defined recursively by the rules

$$\hat{W}_0(x) = 2\ v_0'(x) - v_0(x)\ [V_0(x) + 2E] \tag{5}$$

$$\hat{W}_j(x) = 2\ \frac{\hat{W}_{j-1}'(x)\ W_{v_0,\ldots,v_j}(x)}{W_{j-1}(x)} - 2\ \frac{\hat{W}_{j-1}(x)\ W_{v_0,\ldots,v_j}'(x)}{W_{j-1}(x)} + \frac{\hat{W}_{j-1}(x)\ W_{v_0,\ldots,v_j}(x)}{W_{j-1}(x)}\ [V_0(x) + 2E], \quad j = 1, 2, \ldots, n-1. \tag{6}$$

Under these conditions, the function $\psi_n = D_{h_0,\ldots,h_{n-1}}(\psi_0)$ solves Eq. (2), provided the potential terms comply with the following constraints

$$V_n(x) = V_0(x) + \frac{\mathrm{d}}{\mathrm{d}x}\ \log\left[\frac{\hat{W}_{n-1}(x)}{W_{v_0,\ldots,v_{n-1}}(x)}\right] \tag{7}$$

$$U_n(x) = U_0(x) - \frac{n}{2}\, V_0'(x) + \frac{V_0(x)}{2} \left\{ \frac{\mathrm{d}}{\mathrm{d}x}\, \log \left[\frac{\hat{W}_{n-1}(x)}{W_{v_0,\ldots,v_{n-1}}(x)} \right] \right\} + \frac{3\, (\hat{W}_{n-1}'(x))^2}{4\, \hat{W}_{n-1}(x)^2}$$
$$+ \frac{3\, (W_{v_0,\ldots,v_{n-1}}'(x))^2}{4\, W_{v_0,\ldots,v_{n-1}}(x)^2} - \frac{\hat{W}_{n-1}'(x)\, W_{v_0,\ldots,v_{n-1}}'(x)}{2\, \hat{W}_{n-1}(x)\, W_{v_0,\ldots,v_{n-1}}(x)} - \frac{\hat{W}_{n-1}''(x)}{2\, \hat{W}_{n-1}(x)}$$
$$- \frac{W_{v_0,\ldots,v_{n-1}}''(x)}{2\, W_{v_0,\ldots,v_{n-1}}(x)}. \tag{8}$$

The proof of these results and a separate consideration of the second-order case can be found in [15,16], respectively.

3 Construction of the Darboux transformation

As mentioned above, we want to establish a Darboux transformation between two-dimensional Dirac equations with a position-dependent mass. To this end, we start out from our initial equation that we write in the form

$$-\, i\, \sigma_1\, \frac{\partial}{\partial x}\, \Psi(x,y) - i\, \sigma_2\, \frac{\partial}{\partial y}\, \Psi(x,y) + [m(x)\, \sigma_3 + V(x)\,]\, \Psi(x,y) = 0, \tag{9}$$

where σ_j, $j = 1, 2, 3$, are the Pauli matrices, m and V represent the position-dependent mass function and the potential, respectively, and Ψ denotes the two-component solution. The principal idea of our construction is to convert our Dirac equation (9) into Schrödinger form (1), apply the Darboux transformation described in Sect. 2 , and afterward reinstate Dirac form of the resulting transformed equation. Hence, the first step consists in decoupling (9) and afterward convert it to a second-order equation that matches the form (1).

3.1 Decoupling the Dirac equation

In order to decouple the Dirac equation (9), we make use of the fact that both potential and position-dependent mass do not depend on the variable y. We set

$$\Psi(x,y) = \exp(i\, k_y\, y)\, [\Psi_1(x), \Psi_2(x)]\,. \tag{10}$$

Here, the constant k_y stands for free motion in y-direction. We now relate the component functions Ψ_1 and Ψ_2 to each other as follows

$$\Psi_2(x) = i\, \frac{k_y\, \Psi_1(x) - \Psi_1'(x)}{m(x) - V(x)}. \tag{11}$$

Upon substituting this setting along with (10), the second component of the Dirac equation (9) is satisfied, while the first component takes the form

$$\Psi_1''(x) + \frac{V'(x) - m'(x)}{m(x) - V(x)}\, \Psi_1'(x) + [m(x) - V(x)]^{-1} \Big\{ - m(x)^3 + k_y^2\, V(x)$$
$$+ m(x)^2\, V(x) - V(x)^3 + m(x)\left[V(x)^2 - k_y^2\right] + k_y\left[m'(x) - V'(x)\right] \Big\}\, \Psi_1(x) = 0.$$

In the next step, we gauge away the first-derivative term by defining

$$\Psi_1(x) = \sqrt{m(x) - V(x)}\, \psi_0(x), \tag{12}$$

where the function ψ_0 is a solution of

$$\psi_0''(x) - \left\{k_y^2 + \frac{V'(x) - m'(x)}{m(x) - V(x)}\, k_y + \frac{3}{4}\left[\frac{m'(x) - V'(x)}{m(x) - V(x)}\right]^2 + m(x)^2 - V(x)^2 \right.$$
$$\left. + \frac{V''(x) - m''(x)}{m(x) - V(x)}\right\} \psi_0(x) = 0. \tag{13}$$

We observe that this linear second-order equation matches the form (1), provided we make the following definitions

$$E = k_y \tag{14}$$
$$V_0(x) = \frac{V'(x) - m'(x)}{m(x) - V(x)} \tag{15}$$
$$u_0(x) = \frac{3}{4}\left[\frac{m'(x) - V'(x)}{m(x) - V(x)}\right]^2 + m(x)^2 - V(x)^2 + \frac{V''(x) - m''(x)}{m(x) - V(x)}. \tag{16}$$

Due to Eqs. (1) and (13) matching, we are now in position to perform a Darboux transformation.

3.2 The Darboux transformation

We will now apply our Darboux transformation (4) to the solution ψ_0 of Eq. (13). While the transformed solution ψ_n is given by (4), the transformed potential terms V_n, U_n can be found by inserting the above settings (15), (16) into (7) and (8), respectively. This yields

$$V_n(x) = \frac{V'(x) - m'(x)}{m(x) - V(x)} + \frac{d}{dx} \log\left[\frac{\hat{W}_{n-1}(x)}{W_{v_0,\ldots,v_{n-1}}(x)}\right] \tag{17}$$

$$U_n(x) = \frac{3}{4}\left[\frac{m'(x) - V'(x)}{m(x) - V(x)}\right]^2 + m(x)^2 - V(x)^2 + \frac{V''(x) - m''(x)}{m(x) - V(x)}$$
$$-\frac{n}{2}\frac{d}{dx}\left[\frac{V'(x) - m'(x)}{m(x) - V(x)}\right] + \frac{V'(x) - m'(x)}{2\,[m(x) - V(x)]}\frac{d}{dx}\log\left[\frac{\hat{W}_{n-1}(x)}{W_{v_0,\ldots,v_{n-1}}(x)}\right]$$
$$+\frac{3}{4}\left[\frac{\hat{W}_{n-1}'(x)}{\hat{W}_{n-1}(x)}\right]^2 + \frac{3}{4}\left[\frac{W_{v_0,\ldots,v_{n-1}}'(x)}{W_{v_0,\ldots,v_{n-1}}(x)}\right]^2$$
$$-\frac{\hat{W}_{n-1}'(x)\, W_{v_0,\ldots,v_{n-1}}'(x)}{2\,\hat{W}_{n-1}(x)\, W_{v_0,\ldots,v_{n-1}}(x)} - \frac{\hat{W}_{n-1}''(x)}{2\,\hat{W}_{n-1}(x)}$$
$$-\frac{W_{v_0,\ldots,v_{n-1}}''(x)}{2\,W_{v_0,\ldots,v_{n-1}}(x)}. \tag{18}$$

Thus, the function (4) with ψ_0 from (13) is a solution of the transformed equation (2) for the settings (14)–(16). Now that we have generated the transformed linear second-order equation (2), the remaining task is to cast the latter equation in Dirac form.

3.3 Reinstating Dirac form: matching conditions

In order to revert the decoupling procedure for the Dirac equation, we need to match the transformed equation (2) for (17) and (18) with the general shape of (13). More precisely, we require the transformed equation (2) to read

$$\psi_n''(x) - \left\{ k_y^2 + \frac{U'(x) - M'(x)}{M(x) - U(x)}\, k_y + \frac{3}{4}\left[\frac{M'(x) - U'(x)}{M(x) - U(x)}\right]^2 + M(x)^2 - U(x)^2 \right.$$
$$\left. + \frac{U''(x) - M''(x)}{M(x) - U(x)} \right\} \psi_n(x) = 0, \tag{19}$$

introducing a transformed Dirac potential U and a transformed position-dependent mass function M. We will now proceed by matching the coefficients pertaining to the k_y-powers. Since the term k_y^2 is already matching, we continue with the coefficient of k_y in (19), using its explicit form (17). This results in the equation

$$\frac{U'(x) - M'(x)}{M(x) - U(x)} = \frac{V'(x) - m'(x)}{m(x) - V(x)} + \frac{\mathrm{d}}{\mathrm{d}x} \log\left[\frac{\hat{W}_{n-1}(x)}{W_{v_0,\ldots,v_{n-1}}(x)}\right].$$

We can solve this constraint for the transformed mass function M by means of logarithmic integration. The result can be simplified as to remove all integrations in the following way:

$$M(x) = U(x) - \exp\left\{\int^x -\frac{V'(t) - m'(t)}{m(t) - V(t)} - \frac{\mathrm{d}}{\mathrm{d}t} \log\left[\frac{\hat{W}_{n-1}(t)}{W_{v_0,\ldots,v_{n-1}}(t)}\right] \mathrm{d}t\right\}$$
$$= U(x) - \exp\left[\int^x -\frac{V'(t) - m'(t)}{m(t) - V(t)}\, \mathrm{d}t\right] \exp\left\{-\int^x \frac{\mathrm{d}}{\mathrm{d}t} \log\left[\frac{\hat{W}_{n-1}(t)}{W_{v_0,\ldots,v_{n-1}}(t)}\right] \mathrm{d}t\right\}$$
$$= U(x) - \left[V(x) - m(x)\right] \frac{W_{v_0,\ldots,v_{n-1}}(x)}{\hat{W}_{n-1}(x)}. \tag{20}$$

It remains to match the terms that do not depend on k_y. This gives a condition on the transformed Dirac potential U in the form

$$\frac{3}{4}\left[\frac{M'(x) - U'(x)}{M(x) - U(x)}\right]^2 + M(x)^2 - U(x)^2 + \frac{U''(x) - M''(x)}{M(x) - U(x)}$$
$$= \frac{3}{4}\left[\frac{m'(x) - V'(x)}{m(x) - V(x)}\right]^2 + m(x)^2 - V(x)^2 + \frac{V''(x) - m''(x)}{m(x) - V(x)} + U_n(x) - U_0(x), \tag{21}$$

where the function U_n and the transformed mass M are displayed in (18) and (20), respectively. We omit to include the explicit form of these functions, as the resulting expressions would be very large. We can solve Eq. (21) with respect to the potential U, which gives the lengthy result

$$U(x) = \sqrt{V(x) - m(x)} \left\{ \frac{W_{v_0,\ldots,v_{n-1}}(x)}{2\, \hat{W}_{n-1}(x)} + \frac{\hat{W}_{n-1}(x)}{2\, W_{v_0,\ldots,v_{n-1}}(x)} \left[V(x) + m(x)\right]\right\}$$
$$+ \sqrt{\frac{1}{V(x) - m(x)}} \left\{ -\frac{[U_2(x) - U_0(x)]\, \hat{W}_{n-1}}{2\, W_{v_0,\ldots,v_{n-1}}(x)} + \frac{3\, \hat{W}_{n-1}(x)\, [W'_{v_0,\ldots,v_{n-1}}(x)]^2}{8\, W_{v_0,\ldots,v_{n-1}}(x)^3} \right.$$
$$- \frac{W'_{v_0,\ldots,v_{n-1}}(x)\, \hat{W}'_{n-1}(x)}{4\, W_{v_0,\ldots,v_{n-1}}(x)^2} - \frac{[\hat{W}'_{n-1}(x)]^2}{8\, W_{v_0,\ldots,v_{n-1}}(x)\, \hat{W}_{n-1}(x)}$$
$$\left. - \frac{\hat{W}_{n-1}(x)\, W''_{v_0,\ldots,v_{n-1}}(x)}{4\, W_{v_0,\ldots,v_{n-1}}(x)^2} + \frac{\hat{W}''_{n-1}(x)}{4\, W_{v_0,\ldots,v_{n-1}}(x)} \right\}$$

$$
\begin{aligned}
&+\left[V(x)-m(x)\right]^{-\frac{3}{2}}\left\{\left[V'(x)-m'(x)\right]\frac{\hat{W}_{n-1}(x)\ W'_{v_0,\ldots,v_{n-1}}(x)}{8\ W_{v_0,\ldots,v_{n-1}}(x)^2}\right.\\
&-\left[V'(x)-m'(x)\right]\frac{\hat{W}'_{n-1}(x)}{8\ W_{v_0,\ldots,v_{n-1}}(x)}\\
&\left.+\left[V''(x)-m''(x)\right]\frac{\hat{W}_{n-1}(x)}{8\ W_{v_0,\ldots,v_{n-1}}(x)}\right\}\\
&+\left[V(x)-m(x)\right]^{-\frac{5}{2}}\left\{\frac{7\ \hat{W}_{n-1}(x)}{32\ W_{v_0,\ldots,v_{n-1}}(x)}\left[-[m'(x)]^2+2\ m'(x)\ V'(x)\right.\right.\\
&\left.\left.-\hat{W}_{n-1}(x)\ [V'(x)]^2\right]\right\}. \qquad (22)
\end{aligned}
$$

In summary, if we choose the transformed mass M and the transformed Dirac potential U as in (20) and (22), then equation (19) matches the form (2), where the potential terms V_n and U_n are given by (17) and (18), respectively. We are now ready to convert our transformed linear second-order equation (2) for (17) and (18) to Dirac form.

3.4 Reinstating Dirac form: employing initial quantities

We obtain our desired Dirac form by reverting the decoupling process that was performed in Sect. 3.1 . To this end, let us first state the transformed Dirac equation. It reads

$$
-\,i\ \sigma_1\ \frac{\partial}{\partial x}\ \Phi(x,y)-i\ \sigma_2\ \frac{\partial}{\partial y}\ \Phi(x,y)+[M(x)\ \sigma_3+U(x)\ ]\ \Phi(x,y)=0, \qquad (23)
$$

where the mass M and the potential U are given in (20) and (22), respectively. The two-component solution of (23) can be constructed in a similar way as its counterpart Ψ. We just need to rewrite our results (10), (11), (12), where we replace initial quantities by their transformed partners. We find

$$
\Phi(x,y)=\exp(i\ k_y\ y)\ \left[\Phi_1(x),\Phi_2(x)\right]. \qquad (24)
$$

The component functions Φ_1 and Φ_2 are interrelated by means of

$$
\Phi_2(x)=i\ \frac{k_y\ \Phi_1(x)-\Phi_1'(x)}{M(x)-U(x)}, \qquad (25)
$$

and the first component Φ_1 of (24) is given by

$$
\begin{aligned}
\Phi_1(x)&=\sqrt{M(x)-U(x)}\ \psi_n(x)\\
&=\sqrt{M(x)-U(x)}\ \frac{W_{v_0,\ldots,v_{n-1},\psi_0}(x)}{\sqrt{\hat{W}_{n-1}(x)\ W_{v_0,\ldots,v_{n-1}}(x)}}. \qquad (26)
\end{aligned}
$$

At this point, there is one more task remaining: we observe that the transformed quantities (20), (22), (25), (26) are expressed in terms of solutions ψ_0 and h_j, $j=0,\ldots,n-1$, to the Schrödinger-type equation (1), see Sect. 2 for their definition. This is not desirable, since we are aiming at constructing a Darboux transformation between Dirac equations without the use of any intermediate result or equation. For this reason, we must now establish a connection between solutions of (13) and corresponding solutions of the initial Dirac equation (9). The first of these connections is given by (12). Inversion gives

$$\psi_0(x) = \sqrt{\frac{1}{m(x) - V(x)}} \; \Psi_1(x). \tag{27}$$

Next, we need to consider the functions v_j, $j = 0, \ldots, n-1$, that are defined in (3). We will now relate these functions to auxiliary solutions u_j, $j = 0, \ldots, n-1$, of the Dirac equation (9) that pertain to k_y-values λ_j, $j = 0, \ldots, n-1$, respectively. Upon taking into account (3) and (12), we obtain

$$v_j = \sqrt{\frac{1}{m(x) - V(x)}} \; \exp\left[(E - \lambda_j)\, x\right] u_j(x), \quad j = 0, \ldots, n-1. \tag{28}$$

For the sake of brevity, let us now introduce the abbreviation

$$w_{v_0,\ldots,v_j}(x) = \exp\left[(E - \lambda_j)\, x\right] u_j(x), \quad j = 0, \ldots, n-1. \tag{29}$$

We can now combine (27)–(29) in order to rewrite the Wronskians that appear in the transformed quantities (20), (22), (25), (26). Starting out with the Wronskian $W_{v_0,\ldots,v_{n-1}}$ of the functions v_j, $j = 0, \ldots, n-1$, we obtain

$$W_{v_0,\ldots,v_{n-1}}(x) = [m(x) - V(x)]^{-\frac{n}{2}} \; W_{w_0,\ldots,w_{n-1}}(x). \tag{30}$$

Here, we made use of the fact that a common factor in each entry of the matrix associated with the Wronskian can be pulled out like a constant [19]. A similar argumentation leads to

$$W_{v_0,\ldots,v_{n-1},\psi_0}(x) = [m(x) - V(x)]^{-\frac{n+1}{2}} \; W_{w_0,\ldots,w_{n-1},\Psi_1}(x). \tag{31}$$

In the final step, we proceed to rewrite the quantity $\hat{W}_j$, as defined in (6). To this end, we note that the latter quantity can be stated in explicit form as an actual Wronskian [15]. To summarize, we have

$$\hat{W}_{n-1}(x) = \frac{(-2)^n}{F(x)} \; W_{v_0,\ldots,v_{n-1},F}(x), \tag{32}$$

where the function F is given by

$$F(x) = \exp\left(\frac{1}{2} \int^x V_0(t) + 2\, k_y \; \mathrm{d}t\right). \tag{33}$$

Upon implementation of (28) and (29), we can cast (32) in the form

$$\hat{W}_{n-1}(x) = \frac{(-2)^n}{F(x)} \; [m(x) - V(x)]^{-\frac{n+1}{2}} \; W_{w_0,\ldots,w_{n-1},\sqrt{m-V}F}(x). \tag{34}$$

We are now ready to state the Darboux transformation that connects the initial Dirac equation (9) with its transformed counterpart (23). Substitution of (30), (31), (34) into (26) gives the result

$$\Phi_1(x) = \left(-\frac{1}{2}\right)^{\frac{n}{2}} \frac{\sqrt{F(x)\,[M(x) - U(x)]}}{[m(x) - V(x)]^{\frac{1}{4}}} \; \frac{W_{w_0,\ldots,w_{n-1},\Psi_1}(x)}{W_{w_0,\ldots,w_{n-1},\sqrt{m-V}F}(x) \; W_{w_0,\ldots,w_{n-1}}(x)}. \tag{35}$$

This is the first component of the solution to the transformed Dirac equation (23). The second component can be obtained by plugging (35) into (25). We do not show the explicit form of the resulting expression due to its length. Next, let us state the mass (20) that enters in the transformed equation (23). Since this mass is expressed in terms of a Wronskian ratio, it is convenient to determine the latter ratio first. Combination of (30) and (34) gives

$$\frac{W_{v_0,\ldots,v_{n-1}}(x)}{\hat{W}_{n-1}(x)} = \left(-\frac{1}{2}\right)^n F(x)\sqrt{m(x)-V(x)}\,\frac{W_{w_0,\ldots,w_{n-1}}(x)}{W_{w_0,\ldots,w_{n-1},\sqrt{m-V}F}(x)}. \tag{36}$$

Upon plugging this expression into the mass function (20), we arrive at

$$M(x) = U(x) + \left(-\frac{1}{2}\right)^n F(x)\left[m(x)-V(x)\right]^{\frac{3}{2}}\,\frac{W_{w_0,\ldots,w_{n-1}}(x)}{W_{w_0,\ldots,w_{n-1},\sqrt{m-V}F}(x)}. \tag{37}$$

Now that we have rewritten the mass function in terms of quantities related to the initial Dirac equation, we can perform the same process with the transformed Dirac potential U by substituting the Wronskian ratio (36) into (22). Since the resulting form of the latter potential will become very long and involved, we omit to show it here.

4 Application: hyperbolic potential and mass

We will now demonstrate how the Darboux transformation between Dirac equations with position-dependent mass is applied. To this end, let us start by considering a particular case of our initial equation (9). We choose the potential and the mass functions as follows

$$V(x) = 20\ \mathrm{sech}^2(x) + \frac{1}{2} \qquad\qquad m(x) = 20\ \mathrm{sech}^2(x). \tag{38}$$

Let us mention here that the primary reason for choosing these settings is to keep subsequent calculations as simple and transparent as possible. In general, even Darboux transformations of first order result in expressions that are not manageable due to their length. In the following, we will perform a second-order transformation, where excessively long expressions that result from it will be stated in "Appendix A." Now, upon implementing the settings (38), the Dirac equation (9) is taken at zero energy and admits solutions of bound-state type, characterized by particular values of the parameter k_y. We define these values as

$$k_y = \frac{1}{2}\sqrt{1+4\,n_{k_y}^2}, \tag{39}$$

where n_{k_y} stands for a nonnegative integer. A particular solution to the Dirac equation for the settings (38) and (39) is given by (10) with components

$$\Psi_1(x) = \frac{i}{\sqrt{2}}\,P_4^{n_{k_y}}\,[\tanh(x)] \tag{40}$$

$$\Psi_2(x) = -\sqrt{2}\,(n_{k_y}-5)\,P_5^{n_{k_y}}\,[\tanh(x)] + \sqrt{\frac{1}{2}}\left[\sqrt{1+4\,n_{k_y}} \right. \\ \left. -10\,\tanh(x)\right] P_4^{n_{k_y}}\,[\tanh(x)]. \tag{41}$$

Here, P stands for the associated Legendre functions of the first kind [1]. Note that the second component can be obtained from its counterpart by means of (11). Figure 1 shows normalized probability densities $|\Psi(x,0)|^2$ for several values of n_{k_y}.

We will now apply a second-order Darboux transformation to our Dirac equation (9) for the settings (38). To this end, we need to identify two auxiliary solutions of the latter equation. We take the first components of these solutions from(40) as

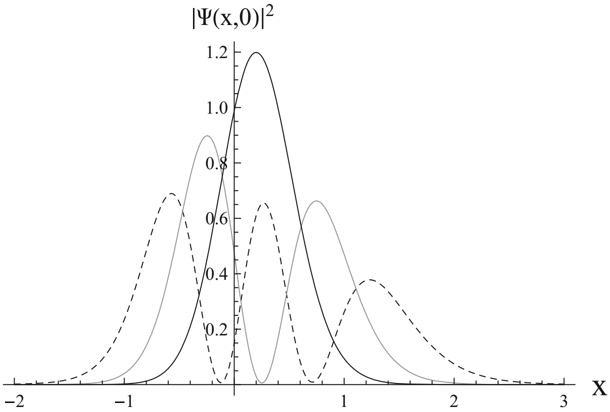

Fig. 1 Graph of the initial probability density $|\Psi(x,0)|^2$ for $n_{k_y} = 4$ (black curve), $n_{k_y} = 3$ (gray curve), and $n_{k_y} = 2$ (dashed curve)

$$u_0(x) = \frac{i}{\sqrt{2}}\, P_4^0\,[\tanh(x)] \qquad\qquad u_1(x) = \frac{i}{\sqrt{2}}\, P_4^1\,[\tanh(x)]\,,$$

while the second components of the auxiliary solutions are irrelevant here. In accordance with (29), we define

$$w_0(x) = \exp\left[\left(E - \frac{1}{2}\right)x\right]u_0(x) \qquad\qquad w_1(x) = \exp\left[\left(E - \frac{\sqrt{5}}{2}\right)x\right]u_1(x), \quad (42)$$

note that the values for λ_0 and λ_1 in (29) are obtained from (39) for $n_{k_y} = 0$ and $n_{k_y} = 1$, respectively. We are now ready to apply our Darboux transformation that is determined by the transformed solution Φ_1, the mass function M, and the associated potential U, as given in (35), (20), and (22), respectively. Let us first consider the difference between the transformed mass M and the potential U. Substitution of (42) into (20) gives

$$\begin{aligned} M(x) - U(x) = &-\Big\{2\,\cosh^2(x)\Big[2\,\cosh(x) + 36\,\cosh(3x) - 4\,\cosh(5x) + 2\,\cosh(7x) \\ &+(\sqrt{5}-1)\Big[-329\,\sinh(x) + 138\,\sinh(3x) - 22\,\sinh(5x) \\ &+\sinh(7x)\Big]\Big]\Big\} \\ &\times\Big\{6\,(2873 + 14\,\sqrt{5})\,\cosh(x) + 10\,(-1159 + 14\,\sqrt{5})\,\cosh(3x) \\ &+30\,(191 + 2\,\sqrt{5})\cosh(5x) + (-2 + 4\,\sqrt{5})\,\cosh(9x) - 2\Big[-4020 \\ &+3860\,\sqrt{5} + 5\,(567 - 575\,\sqrt{5})\,\cosh(2x) - 14\,(-9 + \sqrt{5})\,\cosh(4x) \\ &+(-9 + \sqrt{5})\,\cosh(6x)\Big]\Big[-6\,\sinh(x) + \sinh(3x)\Big]\Big\}^{-1}. \end{aligned} \quad (43)$$

Figure 2 shows two graphs of this expression. We observe that away from zero, both transformed mass and potential differ by a constant amount, where close to zero they are

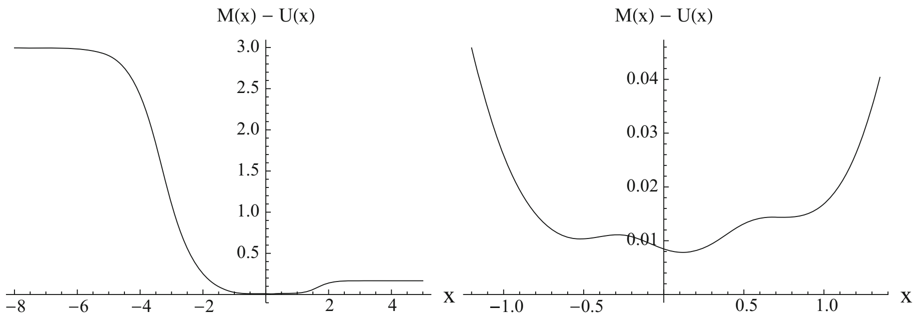

Fig. 2 Graphs of the function $M - U$, as given in (43)

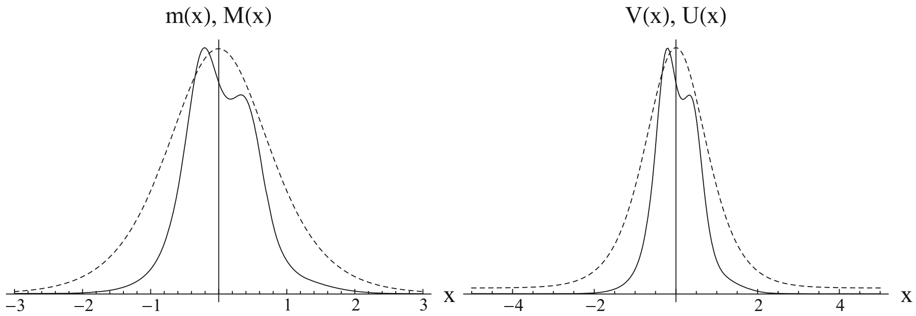

Fig. 3 Left plot: vertically scaled graph of initial mass (38) (dashed curve) and transformed mass (20) (solid curve). Right plot: vertically scaled graph of initial potential (38) (dashed curve) and transformed potential (22) (solid curve)

almost equal. Let us now state the results of our Darboux transformation. Starting out with the transformed mass (20), it turns out that after substitution of (22), (33), (38), and (42), we obtain a very long and involved expression. After having this expression simplified by Mathematica (Wolfram Research Inc.), the result is shown in "Appendix A." Similarly, we use the latter result for finding the explicit form of the transformed Dirac potential (22) by means of (43), see "Appendix A." Figure 3 shows graphs of the initial mass and potential, along with their transformed counterparts. Note that the graphs are scaled in vertical direction, as the transformed quantities take much larger values as compared to their initial partners. It remains to determine the solution (35) for the present case. We follow the same route as for the transformed mass and potential functions by stating the explicit form of the latter solution in "Appendix A." The second component of the solution can be obtained through (25); we omit to present its explicit form. Figure 4 shows normalized probability densities $|\Phi(x, 0)|^2$ for several values of the parameter n_{k_y}.

Inspection of the figure indicates that the probability densities pertain to solutions of bound-state type.

5 Concluding remarks

We have constructed arbitrary-order Darboux transformation for Dirac equations with position-dependent mass. Even though applications typically result in large expressions for the transformed quantities, our method can be easily implemented by means of symbolic calculators. In particular, no symbolic integration is required except for the typically simple integral in (33). Our method has a technical shortcoming that concerns the transformed position-dependent mass function (37). In order to be physically meaningful, this function must be nonnegative. However, due to its complicated form and involved dependence on parameters, we do not have a condition for nonnegativity. A similarly important task is to obtain regularity constraints for the transformed potential terms (7) and (8), as they exist in the conventional Darboux transformation (SUSY formalism). Finally, we point out that our method can be extended in a straightforward manner to work with non-diagonal matrix potentials.

Appendix A: explicit form of the Darboux transformation

The transformed mass function M, as given in (20), takes a long and involved form. For this reason, we do not show it as part of Sect. 4 , but rather display its simplified form here

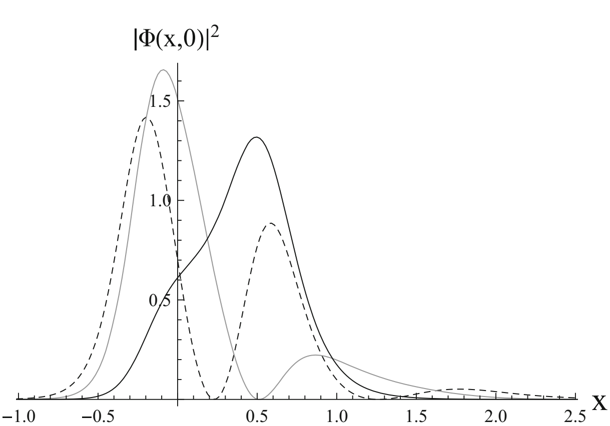

Fig. 4 Graphs of the probability density $|\Phi(x,0)|^2$ for $n_{k_y} = 4$ (black curve), $n_{k_y} = 3$ (gray curve), and $n_{k_y} = 2$ (dashed curve)

```
(Sech[x]^4 (3788769480278793390 - 1629081843496137786 √5 + 4 (-1674851617983534395 + 719636701627230553 √5) Cosh[2 x] -
  560 (-8257260052666931 + 3539903019307978 √5) Cosh[4 x] - 2481967971935486055 Cosh[6 x] + 1059590448708460653 √5 Cosh[6 x] -
  1028625571095233950 Cosh[8 x] - 436173053895059066 √5 Cosh[8 x] - 325432415432312975 Cosh[10 x] + 136575899019118597 √5 Cosh[10 x] +
  77392507159719000 Cosh[12 x] - 31971170648078808 √5 Cosh[12 x] - 13610679942588680 Cosh[14 x] + 5485913786904952 √5 Cosh[14 x] +
  1760307561817660 Cosh[16 x] - 682223764299956 √5 Cosh[16 x] - 173129296567500 Cosh[18 x] + 63932623091940 √5 Cosh[18 x] -
  14980868804920 Cosh[20 x] - 5519365054872 √5 Cosh[20 x] - 1414682669265 Cosh[22 x] - 587329277659 √5 Cosh[22 x] -
  143618715810 Cosh[24 x] - 67486025286 √5 Cosh[24 x] - 12161691705 Cosh[26 x] + 6250448563 √5 Cosh[26 x] + 656986720 Cosh[28 x] -
  354846000 √5 Cosh[28 x] - 15424965 Cosh[30 x] + 9417495 √5 Cosh[30 x] - 30890 Cosh[32 x] + 11310 √5 Cosh[32 x] + 645 Cosh[34 x] -
  215 √5 Cosh[34 x] + 531467084715170880 Sinh[2 x] - 264005503865589184 √5 Sinh[2 x] - 743126394207754080 Sinh[4 x] +
  369653304210588448 √5 Sinh[4 x] + 609015164149942560 Sinh[6 x] - 303725215180351200 √5 Sinh[6 x] - 342997179757203360 Sinh[8 x] +
  171784338954708704 √5 Sinh[8 x] + 138158457887794080 Sinh[10 x] - 69667038076927328 √5 Sinh[10 x] - 40109877026560800 Sinh[12 x] +
  20442941966870112 √5 Sinh[12 x] + 8336301940154880 Sinh[14 x] - 4322681147128576 √5 Sinh[14 x] - 1219767972850560 Sinh[16 x] +
  651281888679040 √5 Sinh[16 x] + 123777706383360 Sinh[18 x] - 69117983398656 √5 Sinh[18 x] - 9693613228320 Sinh[20 x] +
  5554819214432 √5 Sinh[20 x] + 982169064480 Sinh[22 x] - 489289061088 √5 Sinh[22 x] - 136366705440 Sinh[24 x] +
  57047816544 √5 Sinh[24 x] + 13661200800 Sinh[26 x] - 5322323040 √5 Sinh[26 x] - 792624480 Sinh[28 x] + 293079328 √5 Sinh[28 x] +
  21080160 Sinh[30 x] - 6911136 √5 Sinh[30 x] - 24960 Sinh[32 x] + 13696 √5 Sinh[32 x] - 480 Sinh[34 x] + 288 √5 Sinh[34 x])) /
 (64 √2 (2 Cosh[x] + 36 Cosh[3 x] - 4 Cosh[5 x] + 2 Cosh[7 x] + 329 Sinh[x] + 329 √5 Sinh[x] - 138 Sinh[3 x] +
    138 √5 Sinh[3 x] + 22 Sinh[5 x] - 22 √5 Sinh[5 x] - Sinh[7 x] + √5 Sinh[7 x])^3
  (6 (2873 + 14 √5) Cosh[x] + 10 (-1159 + 14 √5) Cosh[3 x] + 5730 Cosh[5 x] + 60 √5 Cosh[5 x] - 2 Cosh[9 x] + 4 √5 Cosh[9 x] -
  67959 Sinh[x] + 66431 √5 Sinh[x] + 24285 Sinh[3 x] - 24885 √5 Sinh[3 x] - 2025 Sinh[5 x] +
  2785 √5 Sinh[5 x] - 180 Sinh[7 x] + 20 √5 Sinh[7 x] + 9 Sinh[9 x] - √5 Sinh[9 x]))
```

Fig. 5 The transformed mass function (20), calculated by Mathematica (Wolfram Research Inc.)

```
(2 Cosh[x]^2 (2 Cosh[x] + 36 Cosh[3 x] - 4 Cosh[5 x] + 2 Cosh[7 x] + (-1 + √5) (-329 Sinh[x] + 138 Sinh[3 x] - 22 Sinh[5 x] + Sinh[7 x])))/
  (6 (2873 - 14 √5) Cosh[x] - 10 (-1159 - 14 √5) Cosh[3 x] - 30 (191 + 2 √5) Cosh[5 x] - (-2 + 4 √5) Cosh[9 x] -
   2 (-4020 + 3860 √5 + 5 (567 - 575 √5) Cosh[2 x] - 14 (-9 + √5) Cosh[4 x] + (-9 + √5) Cosh[6 x]) (-6 Sinh[x] + Sinh[3 x])) +
 (Sech[x]^4 (3 788 769 480 278 793 390 - 1 629 081 843 496 137 786 √5 - 4 (-1 674 851 617 983 534 395 - 719 636 701 627 230 553 √5) Cosh[2 x] -
     560 (-8 257 260 052 666 931 + 3 539 903 019 307 978 √5) Cosh[4 x] - 2 481 967 971 935 486 055 Cosh[6 x] + 1 059 590 448 708 460 653 √5 Cosh[6 x] +
     1 028 625 571 095 233 950 Cosh[8 x] - 436 173 053 895 059 066 √5 Cosh[8 x] - 325 432 415 432 312 975 Cosh[10 x] + 136 575 899 019 118 597 √5 Cosh[10 x] +
     77 392 507 159 719 000 Cosh[12 x] - 31 971 170 648 078 808 √5 Cosh[12 x] - 13 610 679 942 588 680 Cosh[14 x] + 5 485 913 786 904 952 √5 Cosh[14 x] +
     1 760 307 561 817 660 Cosh[16 x] - 682 223 764 299 956 √5 Cosh[16 x] - 173 129 296 567 500 Cosh[18 x] + 63 932 623 091 940 √5 Cosh[18 x] +
     14 980 868 804 920 Cosh[20 x] - 5 519 365 054 872 √5 Cosh[20 x] - 1 414 682 669 265 Cosh[22 x] + 587 329 277 659 √5 Cosh[22 x] +
     143 618 715 810 Cosh[24 x] - 67 486 025 286 √5 Cosh[24 x] - 12 161 691 705 Cosh[26 x] + 6 250 448 563 √5 Cosh[26 x] + 656 986 720 Cosh[28 x] -
     354 846 000 √5 Cosh[28 x] - 15 424 965 Cosh[30 x] + 9 417 495 √5 Cosh[30 x] - 30 890 Cosh[32 x] - 11 310 √5 Cosh[32 x] + 645 Cosh[34 x] -
     215 √5 Cosh[34 x] + 531 467 084 715 170 880 Sinh[2 x] - 264 005 503 865 589 184 √5 Sinh[2 x] - 743 126 394 207 754 080 Sinh[4 x] +
     369 653 304 210 588 448 √5 Sinh[4 x] + 609 015 164 149 942 560 Sinh[6 x] - 303 725 215 180 351 200 √5 Sinh[6 x] - 342 997 179 757 203 360 Sinh[8 x] +
     171 784 338 954 708 704 √5 Sinh[8 x] + 138 158 457 887 794 080 Sinh[10 x] - 69 667 038 076 927 328 √5 Sinh[10 x] - 40 109 877 026 560 800 Sinh[12 x] +
     20 442 941 966 870 112 √5 Sinh[12 x] + 8 336 301 940 154 880 Sinh[14 x] - 4 322 681 147 128 576 √5 Sinh[14 x] - 1 219 767 972 850 560 Sinh[16 x] +
     651 281 888 679 040 √5 Sinh[16 x] + 123 777 706 383 360 Sinh[18 x] - 69 117 983 398 656 √5 Sinh[18 x] - 9 693 613 228 320 Sinh[20 x] +
     5 554 819 214 432 √5 Sinh[20 x] + 982 169 064 480 Sinh[22 x] - 489 289 061 088 √5 Sinh[22 x] - 136 366 705 440 Sinh[24 x] +
     57 047 816 544 √5 Sinh[24 x] + 13 661 200 800 Sinh[26 x] - 5 322 323 040 √5 Sinh[26 x] - 792 624 480 Sinh[28 x] + 293 079 328 √5 Sinh[28 x] -
     21 080 160 Sinh[30 x] - 6 911 136 √5 Sinh[30 x] - 24 960 Sinh[32 x] - 13 696 √5 Sinh[32 x] - 480 Sinh[34 x] - 288 √5 Sinh[34 x]))/
  (64 √2 (2 Cosh[x] + 36 Cosh[3 x] - 4 Cosh[5 x] + 2 Cosh[7 x] + 329 Sinh[x] - 329 √5 Sinh[x] - 138 Sinh[3 x] + 138 √5 Sinh[3 x] +
      22 Sinh[5 x] - 22 √5 Sinh[5 x] - Sinh[7 x] + √5 Sinh[7 x])^3 (6 (2873 + 14 √5) Cosh[x] + 10 (-1159 + 14 √5) Cosh[3 x] +
     5730 Cosh[5 x] + 60 √5 Cosh[5 x] - 2 Cosh[9 x] + 4 √5 Cosh[9 x] - 67 959 Sinh[x] + 66 431 √5 Sinh[x] + 24 285 Sinh[3 x] -
     24 885 √5 Sinh[3 x] - 2025 Sinh[5 x] + 2785 √5 Sinh[5 x] - 180 Sinh[7 x] + 20 √5 Sinh[7 x] + 9 Sinh[9 x] - √5 Sinh[9 x]))
```

Fig. 6 The transformed potential function (20), calculated by Mathematica (Wolfram Research Inc.)

```
-(2 (-5 + nky) Cosh[x] (-2 (-6 + nky) Cosh[x] LegendreP[6, nky, Tanh[x]] (2 Cosh[x] + 36 Cosh[3 x] - 4 Cosh[5 x] + 2 Cosh[7 x] +
         329 Sinh[x] - 329 √5 Sinh[x] - 138 Sinh[3 x] + 138 √5 Sinh[3 x] + 22 Sinh[5 x] - 22 √5 Sinh[5 x] - Sinh[7 x] + √5 Sinh[7 x]) +
      LegendreP[5, nky, Tanh[x]] (6579 - 6581 √5 + 2 √(1 - 4 nky^2) + (-8977 - 8939 √5 + 38 √(1 - 4 nky^2)) Cosh[2 x] - 32 (-85 + 86 √5 - √(1 - 4 nky^2)) Cosh[4 x] -
         333 Cosh[6 x] + 335 √5 Cosh[6 x] - 2 √(1 - 4 nky^2) Cosh[6 x] + 11 Cosh[8 x] - 13 √5 Cosh[8 x] + 2 √(1 - 4 nky^2) Cosh[8 x] + 1138 Sinh[2 x] +
         191 √(1 - 4 nky^2) Sinh[2 x] - 191 √(5 - 20 nky^2) Sinh[2 x] - 904 Sinh[4 x] - 116 √(1 - 4 nky^2) Sinh[4 x] + 116 √(5 - 20 nky^2) Sinh[4 x] +
         150 Sinh[6 x] + 21 √(1 - 4 nky^2) Sinh[6 x] - 21 √(5 - 20 nky^2) Sinh[6 x] - 26 Sinh[8 x] - √(1 - 4 nky^2) Sinh[8 x] + √(5 - 20 nky^2) Sinh[8 x])) +
    LegendreP[4, nky, Tanh[x]] (21 (-430 - √5 - 2 nky^2 - 95 √(1 - 4 nky^2) + 96 √(5 - 20 nky^2)) Cosh[x] -
      35 (-290 + √5 + 2 nky^2 - 85 √(1 - 4 nky^2) + 84 √(5 - 20 nky^2)) Cosh[3 x] - 5250 Cosh[5 x] - 15 √5 Cosh[5 x] - 30 nky^2 Cosh[5 x] -
      1125 √(1 - 4 nky^2) Cosh[5 x] + 1140 √(5 - 20 nky^2) Cosh[5 x] + 600 Cosh[7 x] + 150 √(1 - 4 nky^2) Cosh[7 x] - 150 √(5 - 20 nky^2) Cosh[7 x] -
      70 Cosh[9 x] - √5 Cosh[9 x] - 2 nky^2 Cosh[9 x] - 5 √(1 - 4 nky^2) Cosh[9 x] + 6 √(5 - 20 nky^2) Cosh[9 x] + 87 287 Sinh[x] -
      86 735 √5 Sinh[x] - 191 nky^2 Sinh[x] + 191 √5 nky^2 Sinh[x] - 552 √(1 - 4 nky^2) Sinh[x] - 43 005 Sinh[3 x] + 43 125 √5 Sinh[3 x] -
      75 nky^2 Sinh[3 x] + 75 √5 nky^2 Sinh[3 x] - 120 √(1 - 4 nky^2) Sinh[3 x] + 9865 Sinh[5 x] - 10 225 √5 Sinh[5 x] + 95 nky^2 Sinh[5 x] -
      95 √5 nky^2 Sinh[5 x] + 360 √(1 - 4 nky^2) Sinh[5 x] - 940 Sinh[7 x] + 1000 √5 Sinh[7 x] - 20 nky^2 Sinh[7 x] + 20 √5 nky^2 Sinh[7 x] -
      60 √(1 - 4 nky^2) Sinh[7 x] + 23 Sinh[9 x] - 35 √5 Sinh[9 x] + nky^2 Sinh[9 x] - √5 nky^2 Sinh[9 x] + 12 √(1 - 4 nky^2) Sinh[9 x])))/
 (2^(1/4) (6 (2873 + 14 √5) Cosh[x] + 10 (-1159 + 14 √5) Cosh[3 x] + 5730 Cosh[5 x] + 60 √5 Cosh[5 x] - 2 Cosh[9 x] + 4 √5 Cosh[9 x] -
     67 959 Sinh[x] + 66 431 √5 Sinh[x] + 24 285 Sinh[3 x] - 24 885 √5 Sinh[3 x] - 2025 Sinh[5 x] +
     2785 √5 Sinh[5 x] - 180 Sinh[7 x] + 20 √5 Sinh[7 x] + 9 Sinh[9 x] - √5 Sinh[9 x]))
```

Fig. 7 The first component (35) of the transformed solution, calculated by Mathematica (Wolfram Research Inc.)

in Fig. 5. The latter form was obtained by the symbolic calculator package Mathematica (Wolfram Research Inc.). By substituting this mass function into (43), we can construct the explicit form of the associated Dirac potential U, as defined in (22). The result is shown in Fig. 6. Finally, in Fig. 7, we state the first component Φ_1 of the transformed solution, obtained by substitution of (20), (22), (33), (40), (42), (43) into (35).

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