



Stabilization of the stochastic point reactor kinetic equation through power feedback

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Abstract Most basic models for the power (or equivalently, the neutron population) in a nuclear core consider the power as a function of time (with an energetic and spatial distribution) and lead to deterministic description of the reactor kinetics. While these models are of common use and are undoubtedly the main analytic tool in understanding the reactor kinetics, the true nature of the power in a reactor core is stochastic and should be considered as a stochastic process in time. The stochastic fluctuations of the power around the mean field (which is given by the deterministic models) are referred to as “reactor noise”, and understanding them is a basic topic in nuclear science and engineering. Traditionally, most models for reactor noise consider a sub-critical core, reaching steady state after exposure to an external source. The focus on a sub-critical setting is driven by two main factors. First, from a practical point of view, measuring the power fluctuations in a sub-critical core (known as “noise experiments”) has proven to be a very efficient tool for estimating the static and kinetic parameters of the core. Second, once we assume a critical setting, the current models become statistically unstable, while the mean field solution has a stationary solution, the variance tends to ∞ linearly in time. The instability of the stochastic models is a known problem, and it has been conjectured in the past that this (some what strange) increase in the variance—that is not observed in physical systems—can be restrained by power feedback. However, this conjecture was never proven. The outline of the present study is to present a stochastic analysis to the point reactor kinetics model, proving that once the reactivity has a negative feedback, it not only forces a specific steady-state solution (in terms of the mean field equation), but also prevents the variance to “explode”, and the variance is bounded in time.

1 Introduction

Most basic models for the power (or equivalently, the neutron population) in a nuclear core consider the power as a function of time (with an energetic and spatial distribution) and lead to deterministic description of the reactor kinetics, either in terms of the well-known Boltzmann equation, or the point reactor kinetics [1]. While these models are of common use and are undoubtedly the main analytic tool in understanding the reactor kinetics, the true

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nature of the power in a reactor core is stochastic and should be considered as a stochastic process in time [2]. The stochastic fluctuations of the power around the mean field (which is given by the deterministic models) are referred to in the literature as “reactor noise” and form a basic topic in nuclear science and engineering.

The analysis of reactor noise originated in the seminal work of Feynman [3], and is traditionally done via the Probability Generating Function (PGF) formalism, where the so-called Master Equation (or the Chapman–Kolmogorov equation) is utilized to describe the dynamics of the PGF [4]. In recent years, originating in the work of Hayes and Allen [5], there is an ongoing effort to analyze reactor noise via stochastic differential equation (SDE) and Ito calculus. The basic SDE model is obtained by applying the central limit theorem on the master equation, an approximation referred to in the literature as the *diffusion scale approximation* (see also [6]). We consider [5] to be the first paper to use SDE to model reactor noise (although the Langevin equation was introduced in earlier studies, see [7] and the references within), for two reasons: first, [5] was the first to construct the amplitude of the noise term from first principles and analytically justify the use of the Brownian motion increment. Second, it was the first to use the celebrated Ito calculus to analyze the process.

Looking at the literature on reactor noise, the vast majority deals with a sub-critical core, reaching steady state in the presence of an external source. The focus on a sub-critical setting is driven by two factors. First, from a practical point of view, measuring the power fluctuations in a sub-critical core (known as “noise experiments”) has proven to be a very efficient tool for estimating the static and kinetic parameters of the core and is standard practice in reactor measurement and control [8]. Second, once we assume a critical setting (obtained by taking the limit $k \rightarrow 1$ on the multiplication factor, while nullifying the external source), the model becomes statistically unstable: while the mean field solution has a stationary solution, the variance tends to ∞ linearly in time [9].

This instability, which is obviously not a physical phenomena (otherwise we would encounter random meltdowns and shutdowns), has been a “*constant source of irritation*” (Williams, 1977, [2]). The following is quoted from [2]: “The divergence of the auto-correlation function, the variance to mean ratio and the Rossi- α formula, when the reactor is critical, is a constant source of irritation in the interpretation of noise experiments. . . . Several explanations have been advanced for the absence of this “critical catastrophe”, the most popular being that a reactor is never operated exactly at a critical state because there is always a background source of some kind. This explanation, however, is not overly convincing and it is more likely that either inherent and/or an operator-induced feedback mechanism exists which is of low frequency and is sufficient to prevent any divergences at criticality”.

Although [2] was written some 50 years ago, and since then there were numerous publications on reactor noise, it is still safe to state that the stability issue has not been settled. The above quotation not only surfaces the problem, but it also offers a possible solution: the fluctuations are “restrained” by feedback mechanism—either inherent (such as thermal feedback) or operator induced (such as the regulation system).

A reasonable consideration as to why a feedback analysis was never presented so far is that most literature uses the PGF formalism, which typically involves a partial differential equation. Since a power feedback is bound to involve a nonlinear term, it is much harder to incorporate in the equations, and cannot be analyzed in a complete fashion. Using the SDE formalism, on the other hand, allows easier treatment, both in terms of modeling and in terms of analysis, of nonlinear terms (this, for instance, was demonstrated in [10] in the context of dead time effect).

In recent years, there is a growing interest in understanding the noise phenomenon in critical reactors in full working power (see, for instance [11–13] and the references there

within), due to observations that aging reactors seem to admit a growing noise term (this growth is **not** associated with the instability mentioned earlier, but rather with mechanical aging of the facility). Thus, a stable and well-posed mathematical formalism might prove very beneficial.

The outline of the present study is to show, using classic theory of SDE, that indeed once a power feedback is inserted into the equations, regardless of the strength of the feedback, the solution is stabilized, and has a bounded variance.

The paper is arranged as follows: In the next section, we provide the reader with the necessary mathematical and physical background of the problem, and further discuss (through numeric simulations) the motivation of the study. In Sect. 3, we prove the main theorem of the study, and in Sect. 4 we conclude.

2 Preliminaries

2.1 Stochastic differential equations

Dynamic evolution that involves randomness, viewed at diffusion scale, gives rise to SDE in a large variety of settings. This occurs in application fields such as population genetics, queuing networks, finance, communication systems, and theoretical physics. When the fluctuations of the dynamics are diffusive, working with SDE often simplifies their description considerably while keeping the essence. The method by which evolution dynamics are approximated by SDE is referred to in the literature as *diffusion approximation*. Diffusion approximations have been successfully applied in various application fields in all the areas alluded to above.

The two most basic processes underlying the neutron population dynamics are the Poisson process that provides a natural model for the particle injection, absorption and detection, and branching processes, that model fission. Both processes lie in the classical realm of probability theory, and specifically, their scaling limits, including *law of large numbers* (LLN) and CLT, are well understood. It seems that [14] was first to provide rigorous derivation of LLN and CLT results for the total progeny of sub-critical branching processes with immigration. For nearly critical branching processes, the limiting behavior is given by continuous state branching process with immigration when the initial condition is getting large, a direction that started from [15]. The state of the art is described in the recent book [16] (see e.g., Theorem 3.43 there). In this work, we are interested in the formulation and study of SDE that arise as diffusion approximations of nuclear dynamics.

An SDE is an equation of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x, \quad (2.1)$$

where the unknown is a stochastic process X , that has continuous sample paths taking values in \mathbb{R}^d , for some positive integer d ; b and σ are given coefficients; and W is a d -dimensional Brownian motion (BM). We will refer to b as the *drift* of the equation, and σ will be referred to as the *noise amplitude* or the *noise term*. A process X is regarded a solution if it satisfies, for every t , $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$, where the last term in this integral equation is an *Ito integral*. The special case where $\sigma = 0$ corresponds to an ordinary differential equation.

Questions of existence and uniqueness of solutions to SDE such as (2.1), their Markov structure, properties of the solutions, as well as solution methods have been of great interest and enjoyed a remarkable success ever since the 1960's, although the pioneering work goes back to Ito [17] and Gihman [18]. The rich literature includes qualitative theory such as

boundedness and stability of solutions, solution methods, representation of solutions, most notably via the Girsanov transformation, the study of fine properties of solutions, as well as the description of time evolution and steady-state distribution by means of Kolmogorov’s forward and backward equations; a small sample of books addressing the subject is [19–23].

Perhaps the most basic tool in analysis of SDE’d is Ito’s formula, stating that:

$$X_t Y_t = \int_0^t X dY_t + \int_0^t Y dX_t + \int_0^t dX_t dY_t \tag{2.2}$$

where in the last we may use the “multiplication table” of the Ito calculus, according to which if $dW_t, d\tilde{W}_t$ are two independent BM increments, then $(dt)^2 = 0, dt dW_t = 0, dt d\tilde{W}_t = 0, dW_t d\tilde{W}_t = 0$, while $(dW_t)^2 = (d\tilde{W}_t)^2 = dt$ (see Theorems 4.1.2 and 4.2.1, especially equation (4.1.8), in the book [20]).

2.2 The point reactor equation

The most general description of the power in a nuclear reactor is given by the well-known Boltzmann equation, which is an integro-differential equation, governing the dynamics of the energetic and spatial distribution of the neutron flux through time. Due to its complexity, it is often not practical to solve the Boltzmann equation in flux transients, we assume that the flux has a shape function that does not vary in time [24] and the dynamics through time are analyzed in much simpler setting, referred to as the point reactor kinetics (PRK). This allows us to consider the following ODE for the dynamics:

$$\begin{cases} \frac{dP}{dt} = \frac{\rho(t)-\beta}{\Lambda} P + \lambda C + S(t) \\ \frac{dC}{dt} = \frac{\beta}{\Lambda} P - \lambda C \end{cases} \tag{2.3}$$

Here P is the power, C is the power generated by neutrons from the decay of the delayed neutron precursor (which is clearly proportional to the precursor concentration), S is the external source amplitude, ρ is the reactivity, Λ is the generation time, λ is the decay constant of the delayed neutrons and β the reactivity fraction of the delayed neutrons. The PRK is a fundamental tool in reactor kinetics, as it correlates between two very different time scales: the power multiplication rate $\frac{\rho(t)-\beta}{\Lambda}$ and the delayed neutron decay rate λ . In steady-state analysis the different time scale have a small effect, but in modeling power transients, it is crucial. The equation was formulated to provide a theoretical basis for transient measurements such as rod drop experiments [24,25], but applications of the PRK equation can be found in control and feedback analysis [26], reactivity measurements through the transfer function [27,28] and solution techniques for arbitrary reactivity functions [29,30].

In a critical setting, both $\rho = S = 0$, and the solution converges (exponentially fast) to a steady state. Equation (2.3) assumes that the reactivity may be a function of time, but in real physical systems, the reactivity is a function of the power: the macroscopic cross sections of both the fuel and the moderator are strongly effected by core temperature, which is in an obvious correspondence with the power and the neutron flux. This feedback is usually modeled via thermal feedback, which adds two dimensions to the state space: the fuel temperature and the moderator temperature (see [26] and the references there within). To obtain a more tractable system from a mathematical view point, we consider a direct power feedback of the form:

$$\begin{cases} \frac{dP}{dt} = \frac{\delta\rho(t)-\beta}{\Lambda} P + \lambda C \\ \frac{dC}{dt} = \frac{\beta}{\Lambda} P - \lambda C \\ \delta\rho(t) = -\epsilon^2 (P(t) - P_0) \end{cases} \tag{2.4}$$

or:

$$\begin{cases} \frac{dP}{dt} = \frac{-\beta}{\Lambda} P + \lambda C - \frac{\epsilon^2}{\Lambda} (P(t) - P_0) P \\ \frac{dC}{dt} = \frac{\beta}{\Lambda} P - \lambda C \end{cases} \tag{2.5}$$

(in both, P_0 is the working power of the reactor).

Traditionally, the point reactor model has been used to analyze the power in a system out of the steady state. Such transient calculations include rod drops, reactivity oscillations and accident analysis. In the past decade, starting in the work of Hayes and Allen [5], the PRK is modulated into a stochastic differential equation (SDE), by adding a Brownian motion (BM) increment, resulting with a system of the form [5]:

$$\begin{cases} dP = \frac{\rho(t)-\beta}{\Lambda} P dt + \lambda C dt + S + \sigma_{1,1}dW^{(1)} + \sigma_{1,2}dW^{(2)} \\ dC = \frac{\beta}{\Lambda} P dt - \lambda C dt + \sigma_{2,1}dW^{(1)} + \sigma_{2,2}dW^{(2)} \end{cases} \tag{2.6}$$

where $dW^{(1)}, dW^{(2)}$ are two independent BM increments, and $\sigma_{i,j}$ $i, j = 1, 2$ are the amplitudes of the BM increments. In most applications, it is assumed the both $\sigma_{i,j}$ $i, j = 1, 2$ are constant [5,6]. This is often due to a first-order approximation, and in their true nature, $\sigma_{i,j}$ $i, j = 1, 2$ may be functions of P and C [5,6]. To allow a general as possible theory, we will only assume that they are bounded.

Looking at (2.6) in a critical setting, meaning $S = \rho = 0$, we can easily observe the upper mentioned instability: simply by summing both equations, we have that:

$$d(P + C) = (\sigma_{1,1} + \sigma_{2,1})dW^{(1)} + (\sigma_{1,2} + \sigma_{2,2})dW^{(2)} \tag{2.7}$$

which means that $P + C$ have properties of a BM: while the mean value is fixed, the variance grows linearly in time. It should be clear that this instability is not a property of the modeling scheme, and the exact same phenomenon is observed when using the more classic PGF formalism [4]. So, is this instability a true reflection of the nature of things? clearly not, and for two reasons. First, from a pragmatic point of view, random shutdowns or meltdowns are not observed. Second, a critical system is always subjected to feedback response. Either an engineering feedback, through the regulation system, or inherent physical feedback, such as temperature feedback (it is worth mentioning that according to all licensing protocols, any reactivity feedback must be negative). As mentioned earlier, it has been suggested in [2] that the feedback mechanism is responsible for the stability, but this was never validated. Motivated by these observations, the present study addresses the following problem: consider an SDE of the form:

$$\begin{cases} dP = \frac{-\beta}{\Lambda} P dt + \lambda C dt - \frac{\epsilon^2}{\Lambda} (P(t) - P_0) P dt + \sigma_{1,1}dW^{(1)} + \sigma_{1,2}dW^{(2)} \\ dC = \frac{\beta}{\Lambda} P dt - \lambda C dt + \sigma_{2,1}dW^{(1)} + \sigma_{2,2}dW^{(2)} \end{cases} \tag{2.8}$$

It is a clear observation that for the mean field equation, obtained by taking $\sigma_{i,j} = 0$ $i, j = 1, 2$, the solution is stable (as is (2.3) in critical conditions). Is the variance of the solution to Eq. (2.8) bounded uniformly for all values of t ?

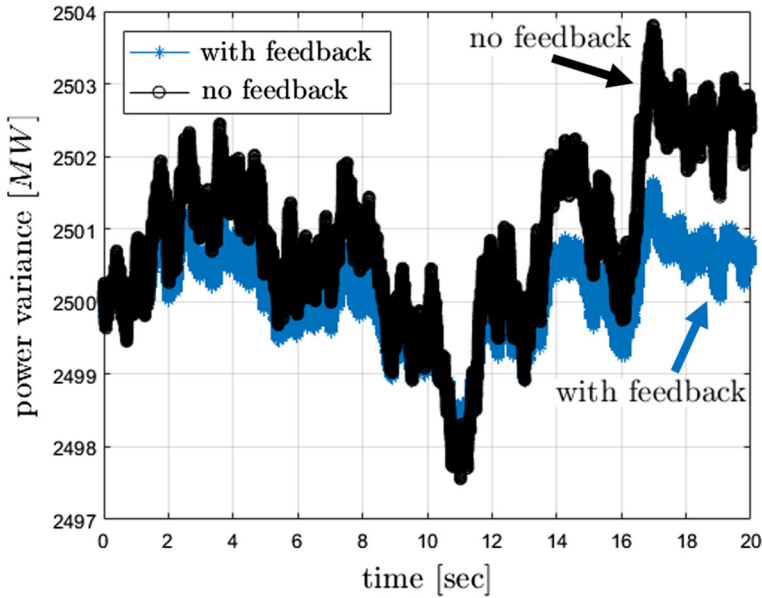


Fig. 1 Sample mean

2.3 On the stochastic instability

In the present section, we discuss a very basic question regarding the above-mentioned stability issue: First, what does “instability” mean, and second what is the motivation for proving the stability of a feedback system. For the first question, we will address the issue through some numeric simulation. A computer-based simulation is used to sample paths of the solutions to Eqs. (2.6) and (2.8). The parameters used were typical to a Three Mile Island type reactor, taken from [26]. Simulations were done using an For Euler–Maruyama [31] numeric scheme. For each equation, the solution was sampled for 20 s, and 1000 paths (histories) were sampled. Figure 1 shows the sample mean for both equations. For both equations, we see a very similar mean field solution: the mean value is very close to the steady-state solution P_0 . It should be mentioned that the properties of the steady state are different: Eq. (2.8) has exactly two steady-state solutions: at $P = 0$ and $P = P_0$. Equation (2.6), on the other hand, has infinitely many steady-state solution (the entire kernel of the state matrix), and the asymptotic solution is determined by the initial conditions. In this simulation, the initial conditions were chose such that both equations have the exact same asymptotic solution. As we can see, on the right-hand side there is a certain deviation between the two simulations, but the results are very similar (the fact that the two sample means also have similar trends is not accidental: to avoid any statistical artifacts, we have used the exact same sampled values for $dW^{(1)}$, $dW^{(2)}$ in simulating the paths for both equations).

However, looking at the sample variance, as shown in Fig. 2, we see a totally different behavior: while the variance of the system without feedback shows a variance increasing linearly in time, the standard deviation (on the power) in the system with feedback is bounded at about 2%. Figures 3 and 4 present randomly chosen 150 trajectories for the system with and without feedback. Again, we see a similar situation: both simulations show the same mean value, equal to the steady state, but in the system with feedback, the feedback “restrains” the

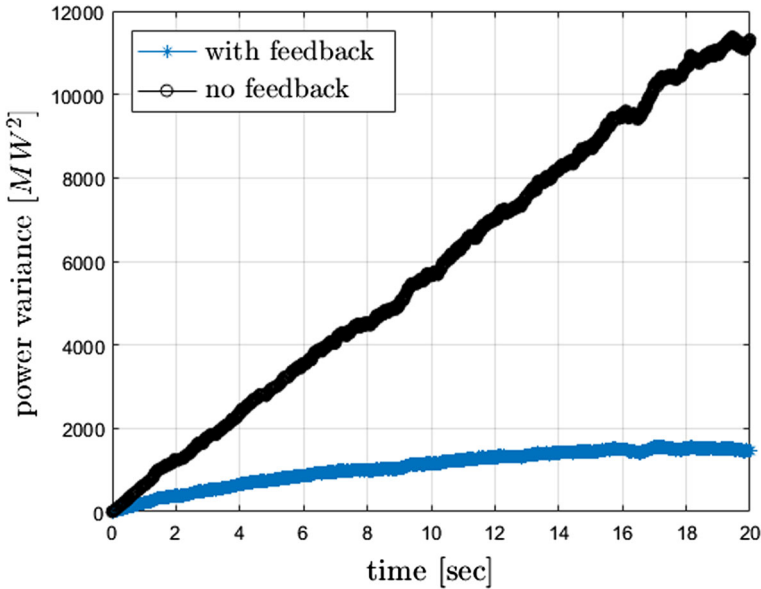


Fig. 2 Sample variance

trajectories from deviating from the steady state, and thus dramatically reduces the spread of the trajectories around the steady state (notice that the two figures have a very different Y-axis scale).

From a strict mathematical point of view, the purpose of the study is very simple: to prove that the situation observed in the simulations is the general case: for any values of β , Λ , P_0 and ϵ , and regardless of noise amplitudes σ_1 and σ_2 , the solution to the SDE if (2.8) has a bounded variance.

But the results are more than a mathematical statement and carry practical significance. The importance of the feedback to the analysis and optimization of critical cores has been well recognized (see, for instance [26] and the references there within). Yet, it is clear that to obtain any practical results, the system must be linearized which is commonly done by taking the approximation

$$\epsilon^2 P(P - P_0) \approx \epsilon^2 P_0(P - P_0) \tag{2.9}$$

The justification for the linear approximation in (2.9) is that if $(P - P_0) \ll P_0$, then the difference between the left-hand side and the right-hand side is proportional to $(P - P_0)^2$, and once the equations are normalized to describe the relative power $\frac{P}{P_0}$, the term $\frac{(P - P_0)^2}{P_0^2}$ becomes a second-order term and can be neglected. However, this can only be possible if the variance of P is bounded, otherwise we expect that as time increases, $(P - P_0)$ will be scalable with P_0 , and the linear approximation would collapse. Therefore, from a practical point of view, the outline of the study is to “close the gap” allowing us to consider the linear approximation, which would lead to a more complete analysis of the feedback effect in stochastic models.

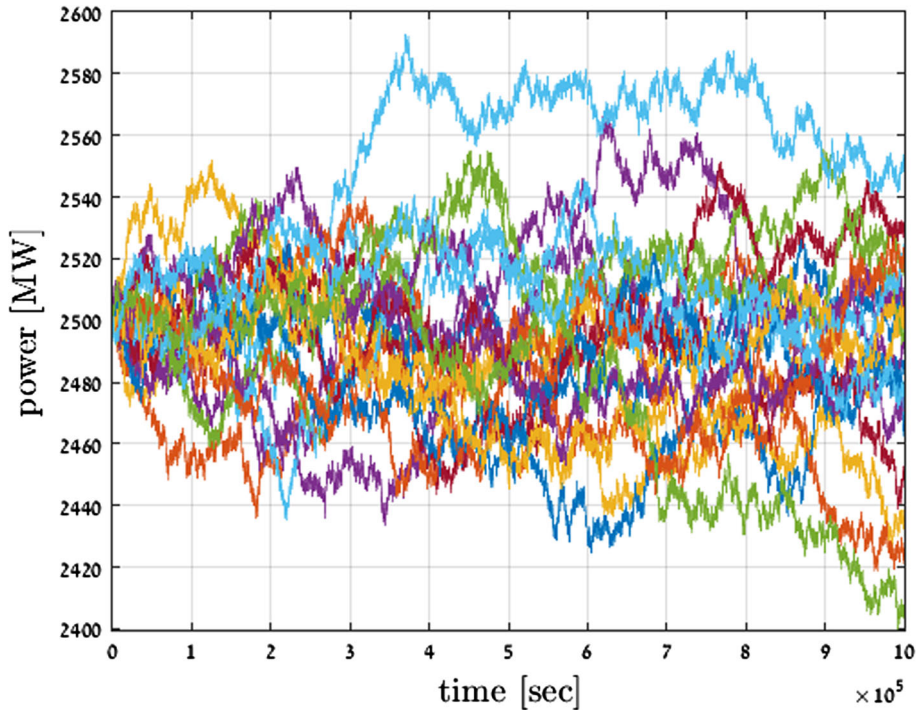


Fig. 3 Sampled trajectories with feedback

2.4 On the positivity of solutions to SDE's

Although SDE's have been recognized as a powerful tool for analyzing stochastic dynamics, there is often a repeating issue, causing a constant debate: what happens if and when the solution becomes negative, while the physical properties of the solution demands that it stays non-negative?

Before discussing the issue, we mention that this question is certainly not unique to SDE's. Typically, to construct an SDE, we must approximate the noise term using a central limit theorem. Doing so, we often take the chance that the random variable will become negative; this is true for both the distribution of neutrons, and the shoe-size in central America, and it is a "risk" we always take when using any central limit theorem. Still, using central limit approximations is perhaps one of the most widely used methodologies in all possible disciplines. The reason is simple: once the standard deviation is sufficiently smaller than the mean, the theoretical "threat" of a negative value will never be realized: if we understand the term "probability" correct, then nothing above 6 standard deviations will ever be manifested.

On the other hand, when the analysis itself demand that the solution is non-negative, such a consideration is no longer acceptable, and we must either prove that the solution is always positive, or "force" the solution to be positive by restrictions or manipulations on the noise term. The problem of determining if an equation has only positive paths is a very well-studied problem [22,32], and although (to the best of our knowledge) the equation at hand is not of any form known to be positive, this is not a unique situation, and conditions are often "forced" on the equation to assure a positive solution. In [33], conditions are forced

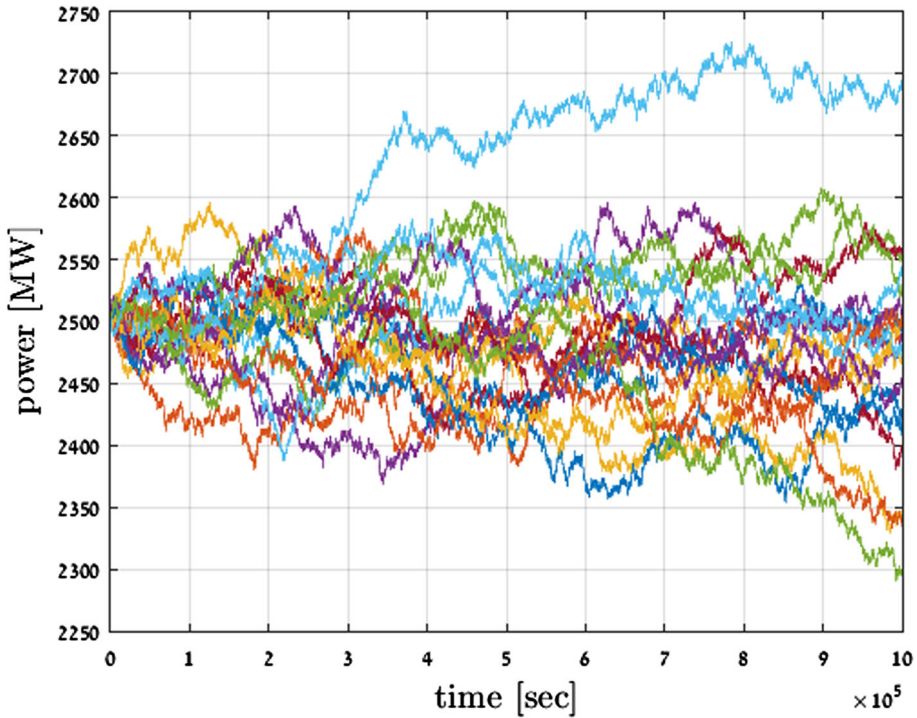


Fig. 4 Sampled trajectories without feedback

on the noise coefficient to assure a positive solution. In [34], the so-called fully implicit stochastic- α (FIS- α) method is applied, where we discretize the time interval, and halt the dynamics if at least one of its states parameters becomes zero. A more general approach is the so-called Skorokhod problem [35], where a reflection term (around 0) is added to the equation, and the list goes on.

In the present study, we suggest a simple modification of the problem, some what similar to the approach in [34] (although we do not discretize the equation): to assure both positivity and the existence of a (unique) solution, we will artificially nullify the noise amplitudes σ_1, σ_2 in a continuous manner in an infinitesimally small domain around the P and C axis. Formally, this is done by two manipulation on σ_1 and σ_2 : first we choose positive value ϵ_1 and nullify σ_1 and σ_2 on the domain $\mathcal{D}_{\epsilon_1} = \{(P, C) | P < \epsilon_1 \text{ or } C < \epsilon_1\}$. Then, we choose a second positive value ϵ_2 , and continuously connect (using a monotonic function) the values of σ_1, σ_2 to 0 in the domain $\mathcal{D}_{\epsilon_1, \epsilon_2} = \{(P, C) | \epsilon_1 \leq P \leq \epsilon_2 \text{ or } \epsilon_1 \leq C \leq \epsilon_2\}$. Since now the coefficients of the SDE are continuous, a unique solution exists. Since in an open domain D_{ϵ_1} surrounding the P and C axis the stochastic terms are nullified, in that domain the system can be analyzed as the simple ODE (2.5), and through basic consideration, it can be shown that the (2.5) cannot obtain negative values of P and C .

However, we must consider the question: how do these modifications of the noise term affect the applicability of our results and the credibility of the physical model? To the best of our understanding, they will not effect them at all, for two main reasons.

First, from a practical point of view, our interest is in systems that in practice, the probability for a random shutdown (or meltdown) is not feasible (assuming the working power is not

arbitrarily small). since both ϵ_1 and ϵ_2 are arbitrarily small, for the modification to have any effect on the system trajectories, either P or C must be arbitrarily close to 0 (which means that they must actually reach 0). If the standard deviation is bounded and sufficiently small, although this is clearly a theoretical risk (a risk that, as we have mentioned, not unique to the present model), from a practical point of view this risk will never be realized in experiment. To explain this point, we return to the numeric simulations shown in Sect. 2.3. In the numeric simulation, we have used fixed values for both σ_1, σ_2 . If we would repeat the simulation, but now nullifying the noise term once P or C are sufficiently small—for a pre-determined criteria of “sufficiently small” which is not met—then the simulation results would be exactly the same. Now, since the sampled standard deviation on the power is about 2%, to reach any value less than 1 MW would require about 50 standard deviations!! Thus, the fact that we have nullified the noise term will never effect the simulation.

Second, the fact the noise term is nullified can be motivated physically. The neutron population, and hence the power, is a discrete random variable, and the assumption that the noise term is “shut down” once the power is reduced under a certain threshold is valid for all practical purposes.

3 The stochastic stability of the SDE with state feedback

The present section forms, from a theoretical point of view, the main contribution of this study. We consider a system of the form (2.8), where $\sigma_{i,j}$ are bounded function, following the conditions described in Sect. 2.4. First, through basic algebraic manipulations, we may re-write Eq. (2.8) in a matricidal form as

$$dX = AXdt - \epsilon^2 \begin{pmatrix} (P - P_0)^2 \\ 0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} P_0 \\ 0 \end{pmatrix} + \Sigma dW \tag{3.1}$$

where

$$X = \begin{pmatrix} P \\ C \end{pmatrix}, \quad W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix} \quad A = \begin{pmatrix} -\left(\frac{\beta}{\Lambda} + \epsilon^2 P_0\right) & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{pmatrix}, \quad \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix}$$

If we denote

$$q = \frac{\sqrt{\left(\frac{\beta}{\Lambda} + \epsilon^2 P_0 - \lambda\right)^2 + 4\lambda \frac{\beta}{\Lambda}} - \left(\frac{\beta}{\Lambda} + \epsilon^2 P_0 - \lambda\right)}{2\lambda}$$

$$\gamma = \frac{1}{2} \left(\left(\frac{\beta}{\Lambda} + \epsilon^2 P_0 - \lambda\right) - \sqrt{\left(\frac{\beta}{\Lambda} + \epsilon^2 P_0 - \lambda\right)^2 - 4\lambda \frac{\beta}{\Lambda}} \right) \tag{3.2}$$

through direct computation we have that $(q \ 1) A = -\gamma (q \ 1)$ (in other word, $v = (q \ 1)$ is a left eigen vector of A , associated with the eigen value $-\gamma$).

Since β, Λ, λ and P_0 are all positive, it is easy to see that both q and γ are positive as well. Defining the auxiliary variable $y = v \times X = qP(t) + C(t)$, $y(t)$ satisfies the SDE

$$dy = -\gamma ydt + q\epsilon^2 P_0 dt - q\epsilon^2 (P - P_0)^2 dt + (q\sigma_{1,1} + \sigma_{2,2})dW^{(1)} + (q\sigma_{1,2} + \sigma_{2,2})dW^{(2)} \tag{3.3}$$

[with q, γ as defined in (3.2)] Since $dW^{(1)}$ and $dW^{(2)}$ are independent, any combination of the two is once again a BM, and we may write :

$$dy = -\gamma y dt + q\epsilon^2 P_0 dt - q\epsilon^2 (P - P_0)^2 dt + \sigma_{tot} dW^{(tot)} \tag{3.4}$$

where $\sigma_{tot} = \sqrt{(q\sigma_{1,1} + \sigma_{2,2})^2 + (q\sigma_{1,2} + \sigma_{2,2})^2}$.
 in a trivial manner, we have:

$$E[y] = qE[P] + E[C], \quad E[y^2] = q^2E[P^2] + 2qE[P \times C] + E[C^2]$$

Since both P and C are positive, if $E[y]$ and $E[y^2]$ are bounded, then $E[P], E[P^2], E[C], E[C^2]$ are all bounded as well. Thus, our strategy is to prove that $E[y]$ and $E[y^2]$ are bounded.

In the coming analysis, we make use the following observation: if $f(t)$ satisfies the inequality

$$f'(t) \leq af(t) + b$$

then $f(t) \leq g(t)$, where g is the solution to the equation

$$g'(t) = ag(t) + b$$

with the initial condition $g(0) = f(0)$.

If a and b are constant, $g(t)$ is explicitly given by $g(t) = (1 - e^{at}) \frac{-b}{a}$, and if in addition $a \leq 0, b \geq 0$, then we have that

$$f(t) \leq \frac{-b}{a} \tag{3.5}$$

Upon integration and averaging of (3.4), we have:

$$\begin{aligned} E[y] &= -\gamma \int_0^t E[y] - q\epsilon^2 \int_0^t E[(P - P_0)^2] dt + q\epsilon^2 P_0 t \\ &\leq -\gamma \int_0^t E[y] + q\epsilon^2 P_0 t \end{aligned} \tag{3.6}$$

(The noise term is nullified, since the average value of integration with respect to a BM increment is always 0). By the above-mentioned inequality, we have that $E[y]$ is bounded, and that:

$$E[y] \leq \frac{q\epsilon^2 P_0}{\gamma} \tag{3.7}$$

Using the Ito formula (2.2) on y^2 , taking into account the expression from (3.4) for dy gives:

$dy^2 = 2ydy + (dy)^2 = 2y(-\gamma y dt + q\epsilon^2 P_0 dt - q\epsilon^2 (P - P_0)^2 dt + \sigma_{tot} dW^{(tot)}) + \sigma_{tot}^2 dt$
 (here we have used the fact that $dt^2 = dW dt = 0$). Upon integration and averaging we have:

$$\begin{aligned} E[y^2] &= -2\gamma \int_0^t E[y^2] d\tau + 2q\epsilon^2 P_0 \int_0^t E[y] d\tau \\ &\quad - 2q\epsilon^2 \int_0^t E[y(P - P_0)^2] d\tau + \int_0^t \sigma_{tot}^2 d\tau \end{aligned} \tag{3.8}$$

Since we assume σ_{tot}^2 bounded, say by M_σ , we have that

$$\int_0^t \sigma_{tot}^2 d\tau \leq M_\sigma t$$

(notice that in the last inequality, we have dropped the dependence of the bound on ϵ_1, ϵ_2). Also, since y is positive and $E[y]$ is bounded by $\frac{q\epsilon^2 P_0}{\gamma}$, we have that:

$$\frac{dE[y^2]}{dt} \leq -2\gamma E[y^2] + 2q\epsilon^2 P_0 \frac{q\epsilon^2 P_0}{\gamma} + M_\sigma$$

Using inequality (3.5) once again, we have:

$$E[y^2] \leq \frac{2q\epsilon^2 P_0 \frac{q\epsilon^2 P_0}{\gamma} + M_\sigma}{2\gamma} \tag{3.9}$$

Finally, since both $E[y^2]$ and $E[y]$ are bounded, then $Var[y] = E[y^2] - E[y]^2$ is bounded as well, which completes the proof.

The proof is constructive, and not only proves that the variance is bounded, but also provides an upper bound:

$$Var[y] \leq E[y^2] \leq \frac{2q\epsilon^2 P_0 \frac{q\epsilon^2 P_0}{\gamma} + M_\sigma}{2\gamma}$$

where q, γ are defined in (3.2). This bound is clearly not a tight bound, since we have totally neglected the reduction in the term $E^2[y]$.

We conclude this section with two remarks: First, it is of utmost importance to notice that the bound obtained is not depending on the terms ϵ_1, ϵ_2 (otherwise, since the two are arbitrary, the results would be meaningless). Second, the functional form of $\sigma_{i,j}$ was not discussed, and the only condition is that it is bounded. In fact, the true condition used is that $\sigma_{tot} = \sqrt{(q\sigma_{1,1} + \sigma_{2,2})^2 + (q\sigma_{1,2} + \sigma_{2,2})^2}$ is bounded.

4 Concluding remarks

In the study, we have proven the stochastic stability of a nonlinear SDE with negative feedback, arising from the reactor point kinetics equations with a BM increment noise term. This equation can be interpreted as a diffusion scale approximation of the master equation (notice, while the master equation is commonly given in terms of PGF, the SDE is given directly in terms of the power). In the study, we did assume any specific functional form of the noise term, and only assumed it to be bounded. To prevent the paths of the governing stochastic process to obtain negative values (which is clearly nonphysical), we have nullified the noise term in an arbitrarily small domain surrounding the P and C axis. It is worth stating that there is a vast literature regarding conditions on the functional form of $\sigma_{u,j}$ that will inherently assure the positivity of the paths, and it is very possible that in future models, which will correspond to specific structures of the noise amplitude, the suggested construction will be redundant. The proof presented is constructive, and a bound (in terms of the system parameters) was presented, but we do not consider this as a tight bound.

The true importance of the study is not only in the physical interpretation of the results, but in the fact that it gives a justification (assuming that the measured variance is sufficiently small) to use a linear approximation of the feedback system. This will lead to a better understanding of the noise phenomenon in critical cores (including feedback effects)—a topic that is not understood and did not enjoy a full analytic treatment.

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