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A construction of new traveling wave solutions for the 2D Ginzburg-Landau equation

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Abstract. In this work, three mathematical methods, namely, the Riccati-Bernoulli sub-ODE method, the $\exp(-\varphi(\xi))$-expansion method and the sine-cosine approach, are applied for constructing many new exact solutions for the 2D Ginzburg-Landau equation. This equation is a prevalent model for the evolution of slowly varying wave packets in nonlinear dissipative media. The three proposed methods are efficient and powerful in solving a wide class of nonlinear evolution equations. In the end, three-dimensional graphs of some solutions have been plotted. Finally, we compare our results with other results in order to show that the proposed methods are robust and adequate.

1 Introduction

This paper is concerned with the 2D Ginzburg-Landau equation, given by

$$iu_t + \frac{1}{2}u_{xx} + \frac{1}{2}(\beta - if)u_{yy} + (1 - i\delta)|u|^2 u = i\gamma u \tag{1}$$

(see, *e.g.*, [1–6]). Here u is a complex valued function defined on $(2+1)$-dimensional space-time R^{2+1}, *i.e.*, the spatial dimension is 2 with additional time dimension, whereas β, f, δ, γ are real constants and $i = \sqrt{-1}$. This equation is an essentially interesting model in this respect because it is a dissipative version of the nonlinear Schrödinger equation with a nonlinear term [7], a Hamiltonian equation which can possess solutions that form localized singularities in finite time. Equation (1) governs the finite amplitude evolution of instability waves in a large variety of dissipative systems, weakly nonlinear which are close to criticality. Indeed, this equations arises in the areas of chemical physics, fluid dynamics, statistical mechanics and biology.

Nonlinear phenomena occur in many branches of science and engineering, like, gas dynamics physics, fluid mechanics, plasma, chemical reactions, relativity, ecology, optical fiber, solid state physics, see [8–19]. According to the important role of the NPDEs, various papers interested in finding solutions of them. These solutions might be essential and important for the exploring some physical phenomena. Therefore investigating an interesting technique to solve so many problems is so interesting topic. Thus, many new methods have been proposed, like as extended tanh-method [20,21], the Riccati-Bernoulli sub-ODE method [18,19,22–24], the generalized Kudryashov method [25,26], the tanh-sech method [27–29], homogeneous balance method [30,31], Jacobi elliptic function method [32,33], exp-function method [34,35], sine-cosine method [36–38], F-expansion method [39–41], $(\frac{G'}{G})$-expansion method [42,43]. However, many other methods can be used to obtain exact solutions of the nonlinear equations arising in applied mathematics and physics, see [44–48].

There are recent development in analytical and numerical techniques for finding solutions for nonlinear partial differential equations (NPDEs). Indeed, theoretical works about solitons and their applications such as the stability analysis, symmetry analysis and conservation laws give several information for modelling the systems of NPDEs, see [49–56].

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The Riccati-Bernoulli sub-ODE method, the $\exp(-\varphi(\xi))$-expansion method and the sine-cosine method are presented to find exact solutions of nonlinear partial differential equations (NPDEs). By using appropriate traveling wave transformation and the three proposed methods, a set of algebraic equations will be generated. Hence by solving these equations, the solutions of NPDEs are obtained. The 2D Ginzburg-Landau equation is selected to clarify the validity of these methods. Consequently, the three proposed methods will be implemented to find the exact solutions of the 2D Ginzburg-Landau equation. In this case the new solutions are given and compared with other methods and show that this method is efficient, robust and adequate for solving other type of NPDEs. To the best of our knowledge, no previous research work has been done using the proposed methods for solving the mUNLSE.

The novelties of this paper are mainly exhibited in two aspects: First, an interesting method will be used, which is not familiar, the so called Riccati-Bernoulli sub-ODE. This method is used in order to solve the 2D Ginzburg-Landau equation. The proposed method also gives infinite sequence of solutions. Moreover, further two methods, namely the exp-function and sine-cosine methods are used in order to solve the proposed equation. Second, the new types of exact analytical solutions will be obtained. Moreover comparing the results in this paper with other results, one can see that the results here are new and most extensive.

The present paper is arranged as follows. Section 2 describes the Riccati-Bernoulli sub-ordinary differential equations (sub-ODE), the exp-function method and sine-cosine method. In sect. 3, some exact solutions for the 2D Ginzburg-Landau model are presented. In sect. 4, some three-dimensional graphs of some solutions is provided. In sect. 5 we compare our results with other results in order to show that the proposed methods in this paper are efficacious, robust and adequate. Namely, we clarify that the Riccati-Bernoulli sub-ODE method superior to other methods. Conclusion will appear in sect. 6.

2 Analytical methods

We assume that the $(2+1)$-dimensional nonlinear evolution equation for $\theta(x,y,t)$ is in the form

$$G(\theta, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt}, \theta_{yy}, \theta_{xy}, \ldots) = 0. \tag{2}$$

Using the wave transformation for a positive constant c,

$$\theta(x,t) = \theta(\xi), \quad \xi = kx + \beta y - ct, \tag{3}$$

eq. (2) will be transformed to the following ODE:

$$D\left(\theta, \frac{\mathrm{d}\theta}{\mathrm{d}\xi}, \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2}, \frac{\mathrm{d}^3\theta}{\mathrm{d}\xi^3}, \ldots\right) = 0. \tag{4}$$

We will present the Riccati-Bernoulli sub-ODE method, the $\exp(-\varphi(\xi))$-expansion method and the sine-cosine method to find exact solutions of (4).

2.1 Riccati-Bernoulli sub-ODE method

According to the Riccati-Bernoulli sub-ODE technique [22], the solution of eq. (4) is assumed to satisfy the following equation:

$$\theta' = a\theta^{2-n} + b\theta + c\theta^n, \tag{5}$$

where a, b, c and n are constants calculated later. Hence,

$$\theta'' = ab(3-n)\theta^{2-n} + a^2(2-n)\theta^{3-2n} + nc^2\theta^{2n-1} + bc(n+1)\theta^n + (2ac+b^2)\theta, \tag{6}$$

$$\begin{aligned}\theta''' = \big(&ab(3-n)(2-n)\theta^{1-n} + a^2(2-n)(3-2n)\theta^{2-2n} \\ &+ n(2n-1)c^2\theta^{2n-2} + bcn(n+1)\theta^{n-1} + (2ac+b^2)\big)\theta'.\end{aligned} \tag{7}$$

The exact solutions of eq. (5), for an arbitrary constant λ are given as follows:

I) For $n = 1$, the solution $\theta(\xi)$

$$\theta(\xi) = \lambda e^{(a+b+c)\xi}. \tag{8}$$

II) For $n \neq 1$, $\alpha = \frac{1}{1-n}$:

1) $c = 0$

i) $b=0$; ($a \neq 0$), the solution is

$$\theta(\xi) = \left(\frac{a}{\alpha}(\xi+\lambda)\right)^{\alpha}; \tag{9}$$

ii) $b \neq 0$ the solution is

$$\theta(\xi) = \left(\frac{-a}{b} + \lambda e^{\frac{b}{\nu}\xi}\right)^{\alpha}. \tag{10}$$

2) $a \neq 0$, $B = \frac{b}{2a}$, $C = \frac{c}{a}$

i) for $C > B^2$ the solutions are

$$\theta(\xi) = \left(-B + \sqrt{C-B^2}\tan\left(\frac{a}{\alpha}\sqrt{C-B^2}(\xi+\lambda)\right)\right)^{\alpha} \tag{11}$$

and

$$\theta(\xi) = \left(-B - \sqrt{C-B^2}\cot\left(\frac{a}{\alpha}\sqrt{C-B^2}(\xi+\lambda)\right)\right)^{\alpha}; \tag{12}$$

ii) for $C < B^2$ the solutions are

$$\theta(\xi) = \left(-B - \sqrt{B^2-C}\coth\left(\frac{a}{\alpha}\sqrt{B^2-C}(\xi+\lambda)\right)\right)^{\alpha} \tag{13}$$

and

$$\theta(\xi) = \left(-B + \sqrt{B^2-C}\tanh\left(\frac{a}{\alpha}\sqrt{B^2-C}(\xi+\lambda)\right)\right)^{\alpha}; \tag{14}$$

iii) for $C = B^2$ the solution is

$$\theta(\xi) = \left(\frac{\alpha}{a(\xi+\mu)} - B\right)^{\alpha}. \tag{15}$$

2.1.1 Bäcklund transformation

When $\theta_{m-1}(\xi)$ and $\theta_m(\xi)(\theta_m(\xi) = \theta_m(\theta_{m-1}(\xi)))$ are the solutions of eq. (5), then

$$\frac{\mathrm{d}\theta_m(\xi)}{\mathrm{d}\xi} = \frac{\mathrm{d}\theta_m(\xi)}{\mathrm{d}\theta_{m-1}(\xi)}\frac{\mathrm{d}\theta_{m-1}(\xi)}{\mathrm{d}\xi} = \frac{\mathrm{d}\theta_m(\xi)}{\mathrm{d}\theta_{m-1}(\xi)}(a\theta_{m-1}^{2-n} + b\theta_{m-1} + c\theta_{m-1}^{n}),$$

otherwise

$$\frac{\mathrm{d}\theta_m(\xi)}{a\theta_m^{2-n} + b\theta_m + c\theta_m^n} = \frac{\mathrm{d}\theta_{m-1}(\xi)}{a\theta_{m-1}^{2-n} + b\theta_{m-1} + c\theta_{m-1}^n}. \tag{16}$$

Integrating eq. (16) once with respect to ξ, the Bäcklund transformation of eq. (5) is given as follows:

$$\theta_m(\xi) = \left(\frac{-C + A_3\left(\theta_{m-1}(\xi)\right)^{\frac{1}{\alpha}}}{B + A_3 + \left(\theta_{m-1}(\xi)\right)^{\frac{1}{\alpha}}}\right)^{\alpha}. \tag{17}$$

For $c = 0$:

i) $b \neq 0$ we find that

$$\theta_m(\xi) = \frac{K_2^{\alpha}\theta_{m-1}(\xi)}{[BK_1 + K_2 + K_1(\theta_{m-1}(\xi))^{\frac{1}{\alpha}}]^{\alpha}}; \tag{18}$$

ii) $b = 0$

$$\theta_m(\xi) = \frac{K_2^{\alpha}\theta_{m-1}(\xi)}{[K_2 + K_1(\theta_{m-1}(\xi))^{\frac{1}{\alpha}}]^{\alpha}}, \tag{19}$$

where K_1 and K_2 are arbitrary constants. Equation (17) is used to find infinite sequence of solutions of eq. (5), as well of eq. (2).

2.2 The $\exp(-\varphi(\xi))$-expansion method

Suppose that the solution of eq. (4) can be written in the polynomial form of $\exp(-\varphi(\xi))$

$$\theta(\xi) = a_m e^{-m\varphi(\xi)} + a_{m-1}e^{-(m-1)\varphi(\xi)} + \ldots = \sum_{i=0}^{m} a_i e^{-i\varphi(\xi)}, \quad a_m \neq 0, \tag{20}$$

where $\varphi(\xi)$ obeys the following ODE:

$$\varphi'(\xi) = e^{-\varphi(\xi)} + \mu\, e^{\varphi(\xi)} + \beta. \tag{21}$$

The solutions of eq. (21) are the following.

1) For $\mu \neq 0$ we get:
   i) $\mu < \frac{\beta^2}{4}$

$$\varphi(\xi) = \ln\left(-\sqrt{\frac{\beta^2}{4} - \mu}\tanh\left(\sqrt{\frac{\beta^2}{4} - \mu}(\xi+\gamma)\right) - \frac{\beta}{2}\right) - \ln\mu, \tag{22}$$

   ii) $\mu > \frac{\beta^2}{4}$

$$\varphi(\xi) = \ln\left(\sqrt{\mu - \frac{\beta^2}{4}}\tan\left(\sqrt{\frac{\beta^2}{4} - \mu}(\xi+\gamma)\right) - \frac{\beta}{2}\right) - \ln\mu, \tag{23}$$

   iii) $\mu = \frac{\beta^2}{4}$

$$\varphi(\xi) = \ln\left(-\frac{2(\beta(\xi+\gamma)+2)}{\beta^2(\xi+\delta)}\right). \tag{24}$$

2) At $\mu = 0$:
   i) $\beta \neq 0$

$$\varphi(\xi) = \ln\left(e^{\beta(\xi+\gamma)} - 1\right) - \ln\beta, \tag{25}$$

   ii) $\beta = 0$

$$\varphi(\xi) = \ln(\xi+\gamma). \tag{26}$$

   Here γ is an arbitrary constant.

Finally, superseding eqs. (20) and (21) into eq. (4) and aggregating all terms of the same power $\exp(-m\varphi(\xi))$, $m = 0, 1, 2, 3, \ldots$. After equating them to zero, the algebraic equations are obtained, which can be solved by Mathematica or Maple to obtain the values of a_i. Hence, the solutions (20) are obtained, which give the exact solutions of eq. (4).

2.3 The sine-cosine method

The solution of eq. (4) is [57,58]

$$\theta(x,t) = \begin{cases} \lambda\sin^r(\mu\xi), & |\xi| \leq \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{27}$$

or

$$\theta(x,t) = \begin{cases} \lambda\cos^r(\mu\xi), & |\xi| \leq \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where λ, μ and $r \neq 0$, are parameters determined later. Equation (27) gives

$$\begin{aligned} \theta(\xi) &= \lambda\sin^r(\mu\xi), \\ \theta^n(\xi) &= \lambda^n\sin^{nr}(\mu\xi), \\ (\theta^n)_\xi &= n\mu r\lambda^n\cos(\mu\xi)\sin^{nr-1}(\mu\xi), \\ (\theta^n)_{\xi\xi} &= -n^2\mu^2r^2\lambda^n\sin^{nr}(\mu\xi) + n\mu^2\lambda^n r(nr-1)\sin^{nr-2}(\mu\xi) \end{aligned} \tag{29}$$

and eq. (28) gives

$$\begin{aligned}
\theta(\xi) &= \lambda \cos^r(\mu\xi),\\
\theta^n(\xi) &= \lambda^n \cos^{nr}(\mu\xi),\\
(\theta^n)' &= -n\mu r\lambda^n \sin(\mu\xi)\cos^{nr-1}(\mu\xi),\\
(\theta^n)'' &= -n^2\mu^2 r^2\lambda^n \cos^{nr}(\mu\xi) + n\mu^2\lambda^n r(nr-1)\cos^{nr-2}(\mu\xi),
\end{aligned} \tag{30}$$

and so on.

Finally, eqs. (29) or (30) are superseded into eq. (4), balance the terms of the cosine functions or the sine functions, when (30) or (29) is used, respectively. Then all terms of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ are summing and equating their coefficients with zero to obtain the algebraic equations in the unknowns λ, μ and r. Hence all possible values of λ, μ and r are determined.

3 Application

This section is concerned with eq. (1). The Riccati-Bernoulli sub-ODE, the exp-function and sine-cosine methods are applied to get some new exact solutions of model (1). Assume that eq. (1) has the following exact solution:

$$u(x,y,t) = \nu(\xi)e^{i\vartheta}, \quad \vartheta = lx + kt, \tag{31}$$

where $\nu(x,y,t)$ is a real function and l, k are constants to be calculated. Superseding (34) into (1) gives

$$i\nu_t + \frac{1}{2}(\nu_{xx} + \beta\nu_{yy}) - \frac{1}{2}if\nu_{yy} + i(l\nu_x + \beta\nu_y) - i\gamma\nu - \left[k + \frac{1}{2}l^2\right]\nu + \nu^3 - i\delta\nu^3 = 0. \tag{32}$$

Diving eq. (32) into real part and imaginary part, yields

$$\begin{aligned}
&\frac{1}{2}(\nu_{xx} + \beta\nu_{yy}) + \nu^3 - \left[k + \frac{1}{2}l^2\right]\nu = 0,\\
&\nu_t - \frac{1}{2}f\nu_{yy} + (l\nu_x + \beta\nu_y)\delta\nu^3 - \gamma\nu = 0.
\end{aligned} \tag{33}$$

The traveling wave solutions are posed as follows:

$$\nu(x,y,t) = U(\xi), \quad \xi = c_1x + c_2y + c_3t, \tag{34}$$

where c_1, c_2, c_3 are constants. Substituting (34) into eq. (33) and after tedious computation, see [4], the NODEs for $U(\xi)$ are obtained

$$hU'' + gU^3 + eU = 0, \tag{35}$$

where

$$h = -\frac{1}{2}fc_2^2, \qquad g = -\frac{fc_2^2}{c_1^2 + \beta c_2^2}, \qquad e = -fc_2^2\frac{k + \frac{1}{2}l^2}{c_1^2 + \beta c_2^2}. \tag{36}$$

3.1 On solving eq. (1) using the Riccati-Bernoulli sub-ODE

Superseding eqs. (6) into eq. (35), the following equation is obtained:

$$h\left(ab(3-n)U^{2-n} + a^2(2-n)U^{3-2n} + nc^2U^{2n-1} + bc(n+1)U^n + (2ac+b^2)U\right) + gU^3 + eU = 0. \tag{37}$$

Setting $n = 0$, eq. (37) is reduced to

$$h(3abU^2 + 2a^2U^3 + bc + (2ac+b^2)U) + gU^3 + eU = 0. \tag{38}$$

Equating each coefficient of U^i $(i = 0,1,2,3)$ to zero, yield

$$hbc = 0, \quad h \neq 0, \tag{39}$$

$$h(2ac + b^2) + e = 0, \tag{40}$$

$$hab = 0, \tag{41}$$

$$2ha^2 + g = 0. \tag{42}$$

Solving eqs. (39)–(42) gives

$$b = 0, \tag{43}$$

$$c = \mp \frac{e}{\sqrt{-2gh}}, \tag{44}$$

$$a = \pm\sqrt{\frac{-g}{2h}}. \tag{45}$$

Hence, the cases of solutions for eqs. (35) and (1) are given, respectively:

1) When $b = 0$ and $c = 0$, the solution of eq. (35) is

$$U_1(x, y, t) = (-a(c_1x + c_2y + c_3t + \lambda))^{-1}. \tag{46}$$

Using eqs. (46), and (34) the solution of eq. (1) is

$$u_1(x, y, t) = e^{i(lx+kt)} (-a(c_1x + c_2y + c_3t + \lambda))^{-1}, \tag{47}$$

where l, k, c_1, c_2, c_3, λ are arbitrary constants.

2) When $\frac{e}{h} < 0$, substituting eqs. (43)–(45) and (34) into eqs. (11) and (12), the following exact solutions of eq. (1) are obtained:

$$U_{2,3}(x, y, t) = \pm\sqrt{\frac{e}{g}} \tan\left(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right) \tag{48}$$

and

$$U_{4,5}(x, y, t) = \pm\sqrt{\frac{e}{g}} \cot\left(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right). \tag{49}$$

Using eqs. (48), (49) and (34) the solutions of eq. (1) are

$$u_{2,3}(x, y, t) = \pm\sqrt{\frac{e}{g}} e^{i(lx+kt)} \tan\left(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right) \tag{50}$$

and

$$u_{4,5}(x, y, t) = \pm\sqrt{\frac{e}{g}} e^{i(lx+kt)} \cot\left(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right), \tag{51}$$

where l, k, c_1, c_2, c_3, e, g, h, λ are arbitrary constants.

3) When $\frac{e}{h} > 0$, substituting eqs. (43)–(45) and (34) into eqs. (13) and (14), the following exact solutions of eq. (1) are given:

$$U_{6,7}(x, y, t) = \pm\sqrt{\frac{-e}{g}} \tanh\left(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right) \tag{52}$$

and

$$U_{8,9}(x, y, t) = \pm\sqrt{\frac{-e}{g}} \coth\left(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right). \tag{53}$$

Using eqs. (52), (53) and (34) the solutions of eq. (1) are

$$u_{6,7}(x, y, t) = \pm\sqrt{\frac{-e}{g}} e^{i(lx+kt)} \tanh\left(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right) \tag{54}$$

and

$$u_{8,9}(x, y, t) = \pm\sqrt{\frac{-e}{g}} e^{i(lx+kt)} \tanh\left(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda)\right), \tag{55}$$

where l, k, c_1, c_2, c_3, e, g, h, λ are arbitrary constants.

Remark 1. *Using eq.* (17) *for* $U_i(x, y, t)$, $i = 1, 2, \ldots, 9$, *an infinite sequence of solutions of eq.* (35), *otherwise for eq.* (1) *are obtained. For illustration, by applying eq.* (17) *to* $U_i(x, y, t)$, $i = 1, 2, \ldots, 9$, *once, the following new solutions of eq.* (35) *are obtained*

$$U_1^\star(x,y,t) = \frac{A_3}{-aA_3(c_1x + c_2y + c_3t + \lambda) \pm 1}, \tag{56}$$

$$U_{2,3}^\star(x,y,t) = \frac{\pm\frac{e}{\sqrt{-2gh}} \pm A_3\sqrt{\frac{e}{g}}\tan(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda))}{A_3 \pm \sqrt{\frac{e}{g}}\tan(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda))}, \tag{57}$$

$$U_{4,5}^\star(x,y,t) = \frac{\pm\frac{e}{\sqrt{-2gh}} \pm A_3\sqrt{\frac{e}{g}}\cot(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda))}{A_3 \pm \sqrt{\frac{e}{g}}\cot(\sqrt{\frac{-e}{2h}}(c_1x + c_2y + c_3t + \lambda))}, \tag{58}$$

$$U_{6,7}^\star(x,y,t) = \frac{\pm\frac{e}{\sqrt{-2gh}} \pm A_3\sqrt{\frac{-e}{g}}\tanh(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda))}{A_3 \pm \sqrt{\frac{-e}{g}}\tanh(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda))}, \tag{59}$$

$$U_{8,9}^\star(x,y,t) = \frac{\pm\frac{e}{\sqrt{-2gh}} \pm A_3\sqrt{\frac{-e}{g}}\coth(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda))}{A_3 \pm \sqrt{\frac{-e}{g}}\coth(\sqrt{\frac{e}{2h}}(c_1x + c_2y + c_3t + \lambda))}, \tag{60}$$

where A_3, c_1, c_2, c_3, e, g, h, λ *are arbitrary constants.*

3.2 On solving eq. (1) using the $\exp(-\varphi(\xi))$-expansion method

Balancing the highest order derivative U'' and non-linear term U^3, gives $n = 1$. Consequently, eq. (35) has the following solution:

$$U = a_0 + a_1 \exp(-\varphi), \tag{61}$$

where A_0 and A_1 are constants to be calculated, with $a_1 \neq 0$. Using (61) gives

$$U'' = a_1\mu\beta + (\beta^2 + 2\mu)\exp(-\varphi) + 3\beta\exp(-2\varphi) + 2\exp(-3\varphi), \tag{62}$$

$$U^3 = a_0^3 + 3a_0^2a_1\exp(-\varphi) + 3a_0a_1^2\exp(-2\varphi) + a_1^3\exp(-3\varphi). \tag{63}$$

Superseding U, U'', U^3 into eq. (75) and equating the coefficients of $\exp(-\varphi)$ to zero, give

$$ha_1\beta\mu + ga_0^3 + ea_0 = 0, \tag{64}$$

$$ha_1(\beta^2 + 2\mu) + 3ga_0^2a_1 + ea_1 = 0, \tag{65}$$

$$ha_1\beta + ga_0a_1^2 = 0, \tag{66}$$

$$2ha_1 + ga_1^3 = 0. \tag{67}$$

Solving eqs. (64)–(67), gives

$$h = \frac{2e}{\beta^2 - 4\mu}, \qquad a_0 = \pm\beta\sqrt{\frac{e}{-g(\beta^2 - 4\mu)}}, \qquad a_1 = \pm 2\sqrt{\frac{e}{-g(\beta^2 - 4\mu)}}.$$

Actually only case from the above cases is considered, the other cases follow in the same way. Now substituting the values of a_0, a_1 into eq. (61) yields

$$U(\xi) = \pm\sqrt{\frac{e}{g(4\mu - \beta^2)}}\left(\beta + 2\exp(-\varphi(\xi))\right), \tag{68}$$

where

$$\xi = c_1x + c_2y + c_3t.$$

Now substituting eqs. (22)–(26) into eq. (68), respectively, the following solutions are obtained.

Case 1) At $\mu \neq 0$:

i) For $\beta^2-4\mu>0$ the solutions are

$$\hat{U}_{1,2}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g(\beta^2-4\mu)}}\left(\beta-\frac{4\mu}{\sqrt{\beta^2-4\mu}\tanh(\frac{\sqrt{\beta^2-4\mu}}{2}(c_1x+c_2y+c_3t+\gamma))+\beta}\right). \tag{69}$$

Using eqs. (69) and (34) the solutions of eq. (1) are

$$\hat{u}_{1,2}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g(\beta^2-4\mu)}}e^{i(lx+kt)}\left(\beta-\frac{4\mu}{\sqrt{\beta^2-4\mu}\tanh(\frac{\sqrt{\beta^2-4\mu}}{2}(c_1x+c_2y+c_3t+\gamma))+\beta}\right), \tag{70}$$

where l, k, c_1, c_2, c_3, γ are arbitrary constants and h, g given by (36).

ii) For $\beta^2-4\mu<0$ the solutions are:

$$\hat{U}_{3,4}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g(\beta^2-4\mu)}}\left(\beta+\frac{4\mu}{\sqrt{4\mu-\beta^2}\tan(\frac{\sqrt{4\mu-\beta^2}}{2}(c_1x+c_2y+c_3t+\gamma))-\beta}\right). \tag{71}$$

Using eqs. (71) and (34) the solutions of eq. (1) are

$$\hat{u}_{3,4}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g(\beta^2-4\mu)}}e^{i(lx+kt)}\left(\beta+\frac{4\mu}{\sqrt{4\mu-\beta^2}\tan(\frac{\sqrt{4\mu-\beta^2}}{2}(c_1x+c_2y+c_3t+\gamma))-\beta}\right), \tag{72}$$

where l, k, c_1, c_2, c_3, γ are arbitrary constants and h, g given by (36).

Case 2) At $\beta^2-4\mu>0$, $\mu=0$, $\beta\neq 0$:

$$U_{5,6}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g}}\left(1+\frac{2}{\exp(\beta(c_1x+c_2y+c_3t+\gamma))-1}\right). \tag{73}$$

Using eqs. (73) and (34) the solutions of eq. (1) are

$$\hat{u}_{5,6}(x,y,t)=\pm\beta\sqrt{\frac{e}{-g}}e^{i(lx+kt)}\left(1+\frac{2}{\exp(\beta(c_1x+c_2y+c_3t+\gamma))-1}\right), \tag{74}$$

where l, k, c_1, c_2, c_3, γ are arbitrary constants and h, g given by (36).

3.3 On solving eq. (1) using the sine-cosine method

Now the sine-cosine method for solving eq. (35) is applied. Subtitling (29) into (35), gives

$$h\left(-\mu^2r^2\lambda\sin^r(\mu\xi)+\mu^2\lambda r(r-1)\sin^{r-2}(\mu\xi)\right)+g\lambda^3\sin^{3r}(\mu\xi)+e\lambda\sin^r(\mu\xi)=0. \tag{75}$$

Equating the coefficients for the power of sine functions gives the algebraic equations:

$$\begin{aligned}&r-1\neq 0,\qquad r-2=3r,\\&h\mu^2\lambda r(r-1)+g\lambda^3=0,\\&-h\mu^2r^2\lambda+e\lambda=0.\end{aligned} \tag{76}$$

Solving this system gives

$$r=-1,\qquad \lambda=\pm\sqrt{\frac{-2e}{g}},\qquad \mu=\pm\sqrt{\frac{e}{h}}, \tag{77}$$

for $\frac{e}{g}<0$ and $\frac{e}{h}>0$. Similarly, the result (77) can be easily obtained, when the cosine method (30) is used. Thus, the following periodic solutions are given:

$$\tilde{U}_{1,2}(x,y,t)=\pm\sqrt{\frac{-2e}{g}}\sec\left(\sqrt{\frac{e}{h}}(c_1x+c_2y+c_3t)\right),\qquad \left|\sqrt{\frac{e}{h}}(c_1x+c_2y+c_3t)\right|<\frac{\pi}{2} \tag{78}$$

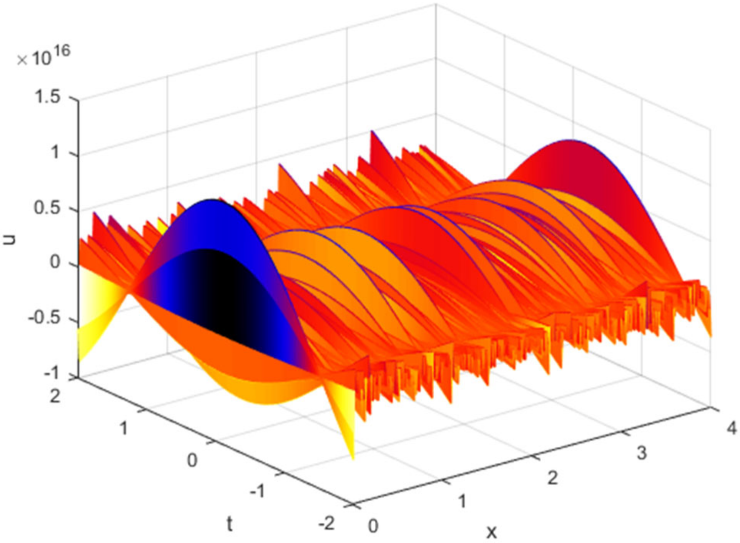

Fig. 1. Graph of $|u_1|$ in (47) with $f = 1$, $c_1 = 0.5$, $c_2 = 0.3$, $c_3 = 0.2$, $\beta = 0.4$, $l = 1.5$, $k = -1.125$, $\lambda = 2$ and $-2 \le x \le 2$ $0 \le t \le 4$.

and

$$\tilde{U}_{3,4}(x,y,t) = \pm\sqrt{\frac{-2e}{g}}\csc\left(\sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t)\right), \qquad 0 < \sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t) < \pi. \tag{79}$$

Using eqs. (78), (79) and (34) the solutions of eq. (1) are

$$\tilde{u}_{1,2}(x,y,t) = \pm\sqrt{\frac{-2e}{g}}e^{i(lx+kt)}\sec\left(\sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t)\right), \qquad \left|\sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t)\right| < \frac{\pi}{2} \tag{80}$$

and

$$\tilde{u}_{3,4}(x,y,t) = \pm\sqrt{\frac{-2e}{g}}e^{i(lx+kt)}\csc\left(\sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t)\right), \qquad 0 < \sqrt{\frac{e}{h}}(c_1x + c_2y + c_3t) < \pi, \tag{81}$$

where l, k, c_1, c_2, c_3 are arbitrary constants and h, g, e given by (36).

However, for $\frac{e}{g} > 0$ and $\frac{e}{h} < 0$, the soliton and complex solutions are

$$\tilde{U}_{5,6}(x,y,t) = \pm\sqrt{\frac{2e}{h}}\,\mathrm{sech}\left(\sqrt{\frac{-e}{h}}(c_1x + c_2y + c_3t)\right) \tag{82}$$

and

$$\tilde{U}_{7,8}(x,y,t) = \pm\sqrt{\frac{2e}{h}}\,\mathrm{csch}\left(\sqrt{\frac{-e}{h}}(c_1x + c_2y + c_3t)\right). \tag{83}$$

Using eqs. (82), (83) and (34) the solutions of eq. (1) are

$$\tilde{u}_{5,6}(x,y,t) = \pm\sqrt{\frac{2e}{h}}e^{i(lx+kt)}\,\mathrm{sech}\left(\sqrt{\frac{-e}{h}}(c_1x + c_2y + c_3t)\right) \tag{84}$$

and

$$\tilde{u}_{7,8}(x,y,t) = \pm\sqrt{\frac{2e}{h}}e^{i(lx+kt)}\,\mathrm{csch}\left(\sqrt{\frac{-e}{h}}(c_1x + c_2y + c_3t)\right), \tag{85}$$

where l, k, c_1, c_2, c_3 are arbitrary constants and h, g, e given by (36).

4 Graphs for the solutions

In this section 3D graphics of some solutions have been plotted, namely figs. 1, 2, 3, 4 and 5.

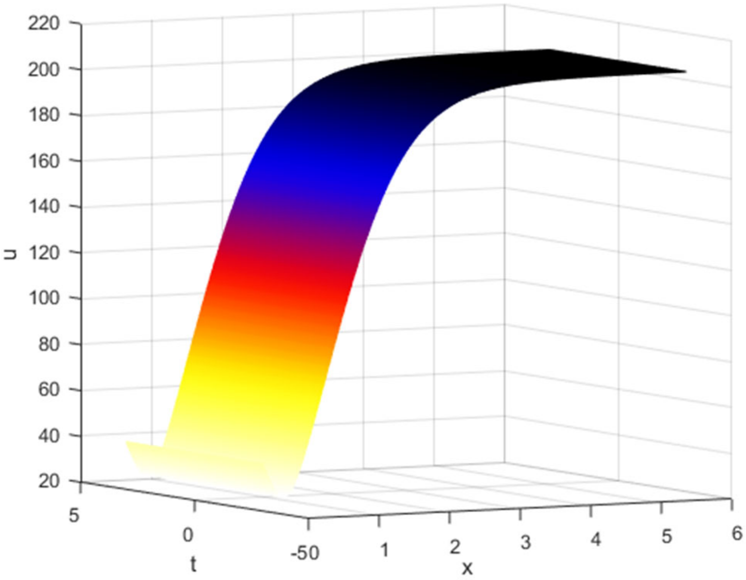

Fig. 2. Graph of $|u_6|$ in (54) with $f = 1.2$, $c_1 = 1.5$, $c_2 = 3$, $c_3 = 1.2$, $\beta = 1.2$, $l = 1.5$, $k = 1.3$, $\lambda = 1$ and $-3 \le x \le 3$ $0 \le t \le 6$.

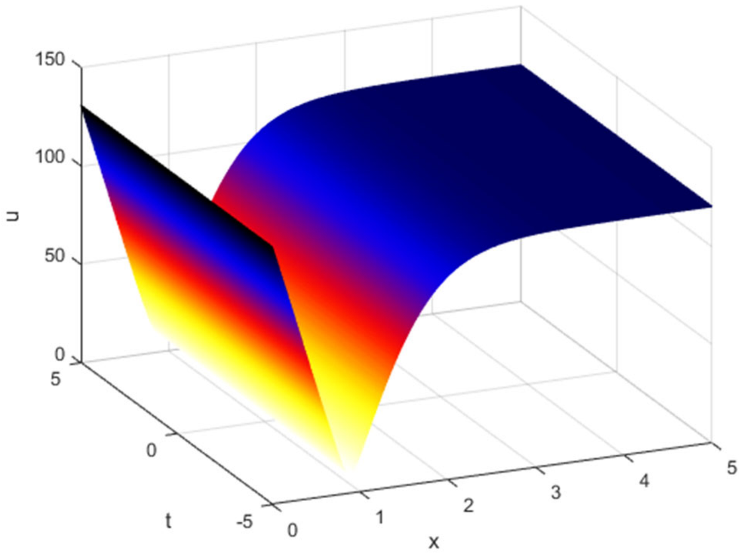

Fig. 3. Graph of $|\hat{u}_1|$ in (70) with $f = 1.2$, $c_1 = 1.5$, $c_2 = 3$, $c_3 = 1.2$, $\beta = 1.2$, $l = 1.3$, $k = 1.5$, $\mu = 1$ and $-5 \le x \le 5$ $0 \le t \le 5$.

5 Comparisons

Here we compare our results with other results in order to show that the our methods are efficacious, robust and adequate. Moreover, we show that the Riccati-Bernoulli sub-ODE method superior to other methods. Namely, we consider the comparison between the solutions given in [3–5] and our solutions. Achab and Bekir [3] have introduced only two solutions for the 2D Ginzburg-Landau equation, using the first integral method. Whereas Achab [4] given five solutions of the 2D Ginzburg-Landau equation, using a uniform algebraic method. Zhon *et al.* [5] given twelve solutions of the 2D Ginzburg-Landau equation, using the homogeneous balance principle and general Jacobi elliptic-function method. Indeed his proposed method is simple, resilient, easy to use and produces very accurate results. His result is much better than the result given in [3, 4]. In our paper, we presented new and so many solutions, using the $\exp(-\varphi(\xi))$-expansion method and the sine-cosine. Moreover, it can be seen that by choosing suitable values for the parameters similar solutions can be verified. Indeed we used an interesting method, the Riccati-Bernoulli sub-ODE method. The main advantages of the Riccati-Bernoulli sub-ODE method over the first integral method, the uniform algebraic method and the homogeneous balance principle and general Jacobi elliptic-function method is that it produce many new exact traveling wave solutions with additional free parameters. If we also compare between these methods

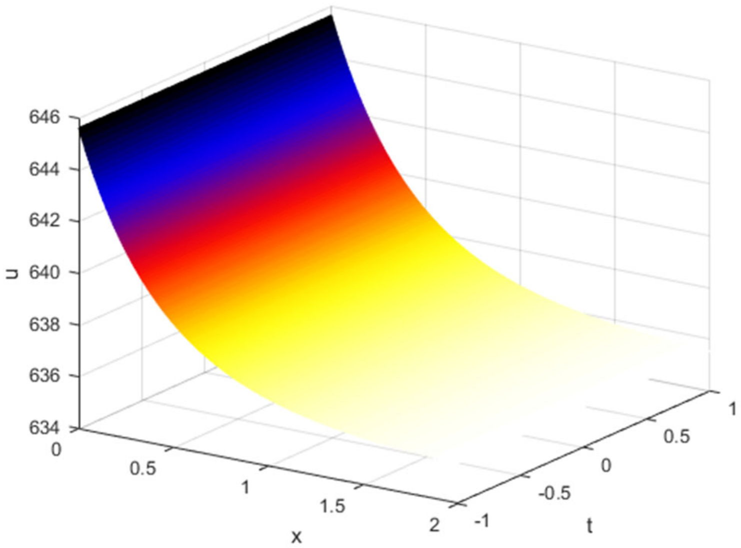

Fig. 4. Graph of $|\hat{u}_3|$ in (74) with $f = -2$, $c_1 = 0.5$, $c_2 = 2$, $c_3 = 1.6$, $\beta = 1$, $l = 1.5$, $k = 0.3$, $\mu = 4$, $\gamma = 2$ and $-1 \le x \le 1$ $0 \le t \le 2$.

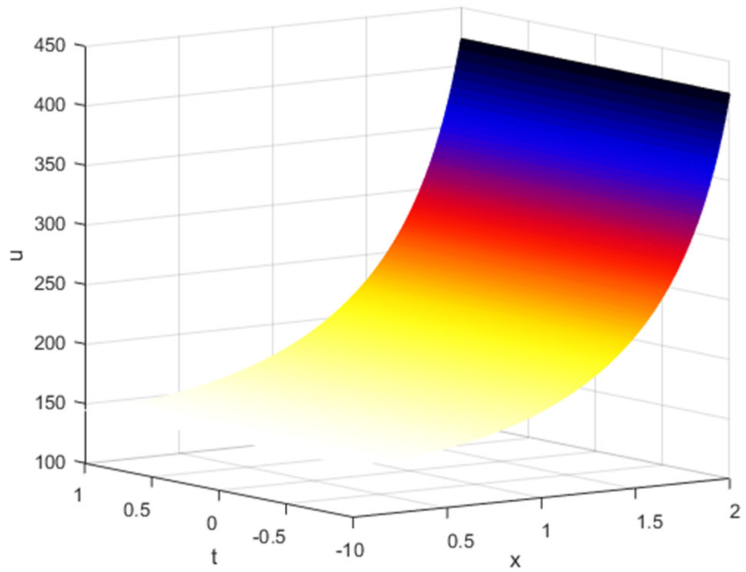

Fig. 5. Graph of $|\tilde{u}_1|$ in (80) with $f = 2$, $c_1 = 0.5$, $c_2 = 1.4$, $c_3 = 0.3$, $\beta = -0.6$, $k = -1.7$, $l = 1.5$ and $-1 \le x \le 1$ $0 \le t \le 2$.

and the proposed methods in this paper, the Riccati-Bernoulli sub-ODE technique is more effective in providing many new solutions than these methods. Indeed the Riccati-Bernoulli sub-ODE method has a very important characteristic, that provides infinite sequence of solutions of equation, which is clarified in sect. 2.1.1. Actually, this feature has never given for any another method. Consequently, this method is efficacious, robust and adequate to solve similar nonlinear problems in mathematical physics and applied mathematics.

6 Conclusions

The Riccati-Bernoulli sub-ODE technique, the $\exp(-\varphi(\xi))$-expansion method and the sine-cosine approach have successfully been applied to find exact solutions for the 2D Ginzburg-Landau equation. As a result, some new exact solutions for them have successfully been obtained. These solutions have so important contribution for the explanation of some physical problems. The graphs of some solutions are depicted for suitable coefficients. Actually the proposed three methods provide a very effective and powerful mathematical tool for solving NPDEs in mathematical physics and natural sciences.

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