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# Gravitational lensing by wormholes supported by electromagnetic, scalar, and quantum effects

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**Abstract.** Wormholes are one of the most interesting topological features in spacetime, offering a rat run between two vastly separated regions of the universe. In this paper, we study the deflection angle of light by wormholes, which are supported by electric charge, magnetic charge, and scalar fields in the weak field limit approximation. To this end, we apply new geometric methods —the Gauss-Bonnet theorem and the optical geometry— to compute the deflection angles. We also verify our findings by using the well-known geodesics method. There exists a similarity between the charge and the quantum corrections on a black hole solution, which has been recently discussed in the context of the relativistic Bohmian quantum mechanics. By replacing classical geodesics with Bohmian trajectories, we introduce a new wormhole solution, whose having matter sources and anisotropic pressure supported by Bohmian quantum effects. The problem of fulfillment of the energy conditions of the Morris-Thorne traversable wormhole is also discussed.

## 1 Introduction

During the last few decades, wormholes have become one of the most popular and intensively studied topics. They are tunnel-like connections made out of spacetime, offering a shorter distance between two vastly separated regions of spacetime (of the different universes or widely separated areas of our own Universe). In 1988, the seminal works on traversable wormholes, as hypothetical shortcuts in spacetime, was proposed by Morris and Thorne [1]. The wormhole solution culminated in the Visser's book with title *Lorentzian Wormholes: From Einstein to Hawking* [2], which reviews the subject up to the year 1995, as well as new notions are presented and hinted at. However, wormhole physics dates back to 1935, when Einstein and Rosen [3] constructed an elementary particle model represented by a bridge linking two identical sheets: Einstein-Rosen bridge (ERB). But, ERB was an unsuccessful particle model. In 1962, Fuller and Wheeler [4] also showed ERB is unstable if it connects two parts of the same universe. At this juncture, it is worth noting that terminology of "wormhole" was first coined by Wheeler [5]. In fact, Wheeler considered wormholes as quantum foams (at the Planck scale), which link separate regions of spacetime [6, 7]. In fact, ERB is a non-traversable wormhole, even by a photon, since the throat pinches off in a finite time. These entities were further explored by Ellis in 1973 [8, 9], by producing a non-singular wormhole solutions using a wrong-sign scalar field. Later, Thorne together with his students came up with a time-machine model using wormholes [10].

Morris and Thorne studied the "traversable wormholes" for human travel [1]. They modeled the wormholes as the objects possessing two mouths and a throat. It was understood that for fabricating such a wormhole solutions, two

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spacetimes that do not posses any event horizon or physical singularities should be invoked. But the wormhole throat is threaded by exotic matter, which violates the null energy condition (NEC) [2]. The existence of the so-called exotic matter sounds to be unusual in general relativity (GR), but such a matter appears in quantum field theory as well as on scalar-tensor theories.

A first example of rotating axially symmetric solution describing a traversable wormhole was given by Teo [11]. However, Teo only showed the generic form (together with some physical properties) of such a rotating wormhole without representing the solution to the associated Einstein-field equations. In a similar way Khatsymovsky [12] discussed some general features of a slowly rotating wormhole. Meanwhile, traversable Lorentzian wormhole in the presence of a generic cosmological constant $\Lambda$ was studied by Lemos *et al.* [13], and further wormhole in anti-de Sitter background was found by [14]. On the other hand, wormhole solutions were studied in the framework of modified and alternative theories of gravity such as the braneworld scenario [15, 16], Einstein Gauss-Bonnet Gravity [17–31], $f(R)$ gravity [32–34], $f(R,\phi)$ gravity [35], $f(R,T)$ gravity [36–39], non-commutative gravity [40–43]. Moreover the gravitational lensing by the wormholes are studied in many papers [44–67].

A decade ago, new global topology method was suggested by Gibbons and Werner [68,69] to calculate the deflection angle for static black holes, in the weak limits, by using the Gauss-Bonnet theorem (GBT). Then, Werner [70] also extended the latter result to the stationary black holes. Afterwards, Gibbons and Werner method (GWM) has become popular and is still used in many studies [71–90]. Different than the ordinary methods, in GWM the deflection angle is computed by taking the integral over a domain $A_\infty$ outside the light ray:

$$\hat{\alpha} = -\iint_{\mathcal{A}_\infty} \mathcal{K}\,\mathrm{d}\sigma. \tag{1}$$

Note that $\mathcal{K}$ stands for the Gaussian optical curvature, $\mathrm{d}\sigma$ is the elementary surface area of the optical geometry, and $\hat{\alpha}$ is the deflection angle.

Despite the success of GR, quantum gravity has become dramatically urgent necessity in order to resolve several problems in cosmos like the missing matter problem or singularities of spacetime. In an recent approach, Das [91] showed that by replacing the classical geodesics with quantal trajectories arising from Bohmian mechanics, one obtains the quantum corrections to the Raychaudhuri equation, which is the so-called quantum Raychaudhuri equation (QRE). QRE has recently been employed for resolving the singularities in cosmology [92] and in black holes [93]. In particular, in the study of [93] modified Schwarzschild metric has been derived and, in sequel, the singularity structure and thermodynamical features of the obtained black hole have been studied in detail. Remarkably, it was shown that quantum corrections change the picture of Hawking radiation when the size of BH approaches the Planck scale.

Therefore our preliminary aim of this study is to see differences between the deflection angle of wormhole with electric charge, magnetic charge and next, scalar fields as well as the quantum effects in the weak field limit approximation using the GWM to do so we use the GBT. The main importance of the paper is to find the difference between the deflection angles which may shed light from an experimental point of view in the near future.

This paper has the following organization. In sect. 1, we consider a charged wormhole and study the deflection angle of light from it. Sections 2 and 3 are devoted to the deflection angle computation for a wormhole spacetime supported by electric charge and nonlinear electromagnetic theory, respectively. In sect. 4, we perform the same analysis in the case of a scalar field wormhole. In sect. 5, we introduce a new wormhole solution based on the analogy of charge and quantum effects in the framework of the relativistic Bohmian quantum mechanics. Furthermore, we investigate the deflection angle as well as the energy conditions. We draw our conclusions in sect. 6.

## 2 Deflection of light by wormhole with electric charge

Recently, Kim and Lee [94] have found an exact solution for a charged Lorentzian wormhole, whose spacetime is described by the following metric:

$$\mathrm{d}s^2 = -\left(1+\frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1-\frac{b(r)}{r}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2. \tag{2}$$

Throughout the paper, we shall consider the special case when $b(r) = b_0^2/r$. Thus, the charged wormhole metric (2) recasts [94]

$$\mathrm{d}s^2 = -\left(1+\frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1-\frac{b_0^2}{r^2}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2. \tag{3}$$

Note that in order to describe a traversable wormhole, the metric (3) should obey some conditions: namely, it should be $b_0^2 > Q^2$, and in this case values of the radial coordinate $r$ lies in the domain $r \geq r_{th}$, where $r_{th}^2 = b_0^2 - Q^2$ is the throat radius.

To find the charged wormhole optical metric, we simply let $\mathrm{d}s^2 = 0$. Furthermore, it is quite convenient to simplify the problem by considering the deflection of light in the equatorial plane; hence we set $\theta = \pi/2$. The optical metric reads

$$\mathrm{d}t^2 = \frac{\mathrm{d}r^2}{(1+\frac{Q^2}{r^2})(1-\frac{b_0^2}{r^2}+\frac{Q^2}{r^2})} + \frac{r^2}{(1+\frac{Q^2}{r^2})}\mathrm{d}\varphi^2. \tag{4}$$

At this point, one may introduce a new radial coordinate $u$, and a new function $\zeta(u)$ as follows:

$$\mathrm{d}u = \frac{\mathrm{d}r}{\sqrt{(1+\frac{Q^2}{r^2})(1-\frac{b_0^2}{r^2}+\frac{Q^2}{r^2})}}\,, \qquad \zeta(u) = \frac{r}{\sqrt{1+\frac{Q^2}{r^2}}}\,. \tag{5}$$

Thus, our metric can be rewritten as

$$\mathrm{d}t^2 = h_{ab}\,\mathrm{d}\lambda^a\mathrm{d}\lambda^b = \mathrm{d}u^2 + \zeta^2(u)\mathrm{d}\varphi^2. \tag{6}$$

Note that $(a, b = r, \varphi)$ and $h = \det h_{ab}$. The Gaussian optical curvature $\mathcal{K}$, is defined by

$$\mathcal{K} = -\frac{1}{\zeta(u)}\left[\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)\frac{\mathrm{d}\zeta}{\mathrm{d}r} + \left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2\frac{\mathrm{d}^2\zeta}{\mathrm{d}r^2}\right]. \tag{7}$$

Applying eq. (7) to our metric (3), we find the charged wormhole Gaussian optical curvature, which is given by

$$\mathcal{K} = \frac{(3\,Q^2 - b_0^2)r^4 + (5\,Q^4 - 5\,Q^2b_0^2)r^2 + 2\,Q^6 - 2\,Q^4b_0^2}{(Q^2+r^2)r^6}\,. \tag{8}$$

We can approximate the above result in leading orders at the limit $r- > \infty$. Thus, one gets

$$\mathcal{K} \simeq -\frac{b_0^2}{r^4} + \frac{3Q^2}{r^4} + \mathcal{O}(b_0^2, Q^4). \tag{9}$$

This result is going to be used to evaluate the deflection angle. In addition we wish to pint out that terms proportional to $b_0^2Q^2$ and $Q^4$ are actually not important in our computations, however this term will be important if we go beyond the straight line approximation which is not the case in the present article. In other words, the agreement between the GWM and geodesics is exact only in leading-order terms. Going beyond the first order by modifying the integration domain this term certainly should be important. The key point behind the GWM is the GBT, which involves the wormhole optical geometry in our case. More precisely, we can choose a non-singular domain outside the light ray, say $\mathcal{A}_R$, with boundary $\partial\mathcal{A}_R = \gamma_h \cup C_R$, where circle segment $C_R$ of coordinate-radius $R$ centered at the lens from where the GBT can be expressed as

$$\iint_{\mathcal{A}_R}\mathcal{K}\,\mathrm{d}\sigma + \oint_{\partial\mathcal{A}_R}\kappa\,\mathrm{d}t + \sum_i \theta_i = 2\pi\chi(\mathcal{A}_R). \tag{10}$$

Note that $\kappa$ gives the geodesic curvature, $\mathcal{K}$ stands for the Gaussian optical curvature, while $\theta_i$ is the exterior angle at the $i$-th vertex. We can choose a non-singular domain outside of the light ray with the Euler characteristic number $\chi(\mathcal{A}_R) = 1$. The geodesic curvature is defined as

$$\kappa = h\,\left(\nabla_{\dot\gamma}\dot\gamma, \ddot\gamma\right), \tag{11}$$

where the unit speed condition $h(\dot\gamma, \dot\gamma) = 1$, holds. If we let $R \to \infty$, our two jump angles $(\theta_{\mathcal{O}}, \theta_{\mathcal{S}})$ become $\pi/2$, or in other words, the sum of jump angles to the source $\mathcal{S}$, and observer $\mathcal{O}$, satisfies $\theta_O + \theta_S \to \pi$ [68]. Hence, we can rewrite the GBT as

$$\iint_{\mathcal{A}_R}\mathcal{K}\,\mathrm{d}\sigma + \oint_{C_R}\kappa\,\mathrm{d}t \overset{R\to\infty}{=} \iint_{\mathcal{D}_\infty}\mathcal{K}\,\mathrm{d}\sigma + \int_0^{\pi+\hat\alpha}\mathrm{d}\varphi = \pi. \tag{12}$$

Let us now compute the geodesic curvature $\kappa$. To this end, we first point out that $\kappa(\gamma_h) = 0$, since $\gamma_h$ is a geodesic. Thus, we end up with

$$\kappa(C_R) = |\nabla_{\dot C_R}\dot C_R|, \tag{13}$$

in which one can choose $C_R := r(\varphi) = R = \text{const}$. The radial part is evaluated as follows:

$$\left(\nabla_{\dot C_R}\dot C_R\right)^r = \dot C_R^{\varphi}\left(\partial_\varphi \dot C_R^r\right) + \mathit{\Gamma}_{\varphi\varphi}^{r(op)}\left(\dot C_R^{\varphi}\right)^2. \tag{14}$$

In the last equation $\Gamma^{r(op)}_{\varphi\varphi}$ gives the Christoffel symbol which is associated to the optical geometry. From the above equation, it is obvious that the first term vanishes, while the second term is calculated using eq. (13) and the unit speed condition. As $R \to \infty$, the geodesic curvature and $\mathrm{d}t$ approximates to

$$\begin{aligned}\lim_{R\to\infty} \kappa(C_R) &= \lim_{R\to\infty} \left|\nabla_{\dot{C}_R} \dot{C}_R\right|, \\ &\to \frac{1}{R}\end{aligned} \tag{15}$$

and

$$\begin{aligned}\lim_{R\to\infty} \mathrm{d}t &= \lim_{R\to\infty} \left(\frac{R}{\sqrt{1+\frac{Q^2}{R^2}}}\right) \mathrm{d}\varphi \\ &\to R\,\mathrm{d}\varphi.\end{aligned} \tag{16}$$

If we combine eqs. (15) and (16), we find $\kappa(C_R)\mathrm{d}t = \mathrm{d}\varphi$. For getting the deflection angle, it is convenient to approximate the boundary curve of $\mathcal{A}_\infty$ to a notional undeflected ray, which is the line of $r(\varphi) = b/\sin\varphi$. Then, the deflection angle (1) reads

$$\hat{\alpha} = -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \mathcal{K}\,\mathrm{d}\sigma. \tag{17}$$

If we substitute eq. (9) into (17), we find out the following integral:

$$\hat{\alpha} \simeq -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \left(-\frac{b_0^2}{r^4} + \frac{3Q^2}{r^4}\right) \sqrt{\det h_{ab}}\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{18}$$

It is worth noting that using the relation: $\mathrm{d}u \approx \mathrm{d}r$, which is valid in the limit $R \to \infty$, one can easily solve (18) in the leading-order terms. Thus, we have

$$\hat{\alpha} \simeq \frac{\pi\, b_0^2}{4b^2} - \frac{3\pi\, Q^2}{4b^2} + \mathcal{O}(b_0^2, Q^2). \tag{19}$$

Thus, the total deflection angle consists by two leading terms, the first terms is a direct result of the geometry of the spacetime, while the second term a consequence of the electric field.

### 2.1 Geodesics of a wormhole with electric charge

The Lagrangian of metric (2) is expressed as follows:

$$2\,\mathcal{L} = -\left(1+\frac{Q^2}{r(s)^2}\right)\dot{t}^2(s) + \frac{\dot{r}(s)^2}{1-\frac{b_0^2}{r(s)^2}+\frac{Q^2}{r(s)^2}} + r(s)^2\dot{\theta}^2(s) + r(s)^2\sin^2\theta\dot{\varphi}^2(s). \tag{20}$$

As is well known, the geodesic equation has to be supplemented by the normalization (or the so-called metric) condition $2\mathcal{L} = \epsilon$ where for massive particles $\epsilon = 1$ and for photon $\epsilon = 0$. Here, we consider the deflection of planar photons, *i.e.*, $\theta = \pi/2$. After using the spacetime symmetries, one obtains two constants of motion $l$ and $\gamma$, which are given by [95]:

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = 2r^2(s)\dot{\varphi}(s) = \mathcal{G}, \tag{21}$$

$$p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -2\left(1+\frac{Q^2}{r(s)^2}\right)\dot{t}(s) = -\mathcal{E}. \tag{22}$$

Let us now introduce a new variable $u(\varphi)$, which is related to our old radial coordinate as $r = 1/u(\varphi)$. Thus, we obtain the following identity:

$$\frac{\dot{r}}{\dot{\varphi}} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,. \tag{23}$$

Without loss of generality, we can normalize the affine parameter along the light ($\epsilon = 0$) rays by taking $\mathcal{E} = 1$ [95] and approximate the distance of closest approach with the impact parameter, *i.e.*, $u_{\max} = 1/r_{\min} = 1/b$, since we shall consider only leading-order terms [96]. In this case, one can choose the second constant of motion: $\mathcal{G} = b$.

Finally using the metric condition ($\epsilon = 0$) with eqs. (20)–(23) in the associated Lagrange equation of $u(\varphi)$, we obtain the following equation:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 \frac{1}{(Q^2u^2 - b_0^2u^2 + 1)u^4} - \frac{Q^2}{u^6u^2(Q^2 + \frac{1}{u^2})^2} - \frac{1}{u^8b^2(Q^2 + \frac{1}{u^2})^2} + \frac{1}{u^2} = 0. \tag{24}$$

From here, it follows that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \pm\frac{b\sqrt{Q^2u^2+1}}{\sqrt{b^2\Phi(Q,u,b_0) + Q^2u^2 - b_0^2u^2 + 1}}\,, \tag{25}$$

where

$$\Phi(Q,u,b_0) = -Q^4u^6 + Q^2b_0^2u^6 - 2Q^2u^4 + b_0^2u^4 - u^2.$$

It is well known that the solution of the differential equation (45) is given by the following relation [95,97]:

$$\Delta\varphi = \pi + \hat{\alpha}, \tag{26}$$

where $\hat{\alpha}$ is the deflection angle to be derived. Following the same arguments given in ref. [97], the deflection angle can be calculated as

$$\hat{\alpha} = 2\int_{1/b}^{0}\left|\frac{\mathrm{d}\varphi}{\mathrm{d}u}\right|\mathrm{d}u - \pi. \tag{27}$$

As expected, the deflection angle in the weak limit approximation is found to be the same result found by GBT. Namely, we have

$$\hat{\alpha} \simeq \frac{\pi\, b_0^2}{4b^2} - \frac{3\pi\, Q^2}{4b^2} + \mathcal{O}(b_0^2, Q^2). \tag{28}$$

## 3 Deflection of light by a non-gravitational wormhole

Recently, using a non-linear electromagnetic theory in the framework of Born-Infeld electromagnetism, it was shown that a specific field configuration in flat metric can be viewed as a spherically symmetric wormhole. In particular, the following effective spacetime metric was found by [98]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{29}$$

where the shape function $b(r)$ is given by

$$b(r) = \frac{2r_{th}^4 - r^4 + r^2\sqrt{r^4 - r_{th}^4}}{r^3 + r\sqrt{r^4 - r_{th}^4}}\,, \tag{30}$$

by which $r_{th}$ is the throat location. The corresponding optical metric of eq. (29) can be found to be

$$\mathrm{d}t^2 = \frac{\mathrm{d}r^2}{1 - \frac{2r_{th}^4 - r^4 + r^2\sqrt{r^4 - r_{th}^4}}{r^4 + r^2\sqrt{r^4 - r_{th}^4}}} + r^2\mathrm{d}\varphi^2. \tag{31}$$

Using eq. (7) the Gaussian optical curvature for the wormhole (31) becomes

$$\mathcal{K} = -2\,\frac{r_{th}^4(r^4 - r_{th}^4 + 2\,r^2\sqrt{r^4 - r_{th}^4})}{r^4(r^2 + \sqrt{r^4 - r_{th}^4})^2\sqrt{r^4 - r_{th}^4}}\,. \tag{32}$$

In the following section, we shall use result (32) within the concept of GBT to find the deflection angle. Next, we can approximate this solution at the limit $r- > \infty$ as follows:

$$\mathcal{K} \simeq -\frac{3\,r_{th}^4}{2\,r^6}\,. \tag{33}$$

Substituting eq. (33) into eq. (17), we get

$$\hat{\alpha} = -\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\left(-\frac{3r_{th}^4}{2r^6}\right)\sqrt{\det h_{ab}}\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{34}$$

One can easily solve this integral in the leading-order terms as

$$\hat{\alpha} \simeq \frac{9\,\pi\,r_0^4}{64\,b^4}\,. \tag{35}$$

In the case of magnetic wormhole $\alpha = 0$ and $\beta \neq 0$, we have $r_{th} = \sqrt{2|\beta|/b_0}$. On the other hand, in the case of an electric wormhole ($\beta = 0$ and $\alpha \neq 0$): $r_{th} = 2\sqrt{2|\alpha|/b_0}$, with $\alpha$ and $\beta$ arbitrary constants. Thus, we remark that in leading-order terms the deflection angle is affected by the non-linear electromagnetic fields. For magnetic field case,

$$\hat{\alpha} \simeq \frac{9\pi\beta^2}{b^4\,b_0^2} + \mathcal{O}(\beta^4, b^8) \tag{36}$$

and, for electric field case,

$$\hat{\alpha} \simeq \frac{9\pi\alpha^2}{b^4\,b_0^2} + \mathcal{O}(\alpha^4, b^8). \tag{37}$$

### 3.1 Geodesics of a non-gravitational wormhole

For the geodesics analysis, one can write down the Lagrangian of metric (29) as follows:

$$2\,\mathcal{L} = -\dot{t}^2(s) + \frac{\dot{r}(s)^2}{1 - \frac{2r_{th}^4 - r(s)^4 + r(s)^2\sqrt{r(s)^4 - r_{th}^4}}{r(s)^4 + r(s)^2\sqrt{r(s)^4 - r_{th}^4}}} + r(s)^2\dot{\theta}^2(s) + r(s)^2\sin^2\theta\dot{\varphi}^2(s). \tag{38}$$

Following sect. 2.1, we first derive the two constants of motion of the geodesics in the non-gravitational wormhole:

$$\mathcal{G} = 2r^2(s)\dot{\varphi}(s), \tag{39}$$

$$\mathcal{E} = 2\dot{t}(s). \tag{40}$$

In the sequel, without loss of generality, we normalize the affine parameter along the light rays by taking $\mathcal{E} = 1$ [95] and approximate the distance to the closest point by an impact parameter, *i.e.*, $u_{\max} = 1/r_{\min} = 1/b$ (recall that we consider only the leading-order terms [96]). In this case, one can choose the second constant of motion as $\mathcal{G} = b$. Finally using the metric condition ($\epsilon = 0$) together with eqs. (38)–(40), one can derive the following Lagrange equation:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 \frac{1}{(1 - \frac{2r_{th}^4}{\Xi} + \frac{1}{\Xi u^4} - \frac{1 - r_{th}^4 u^4}{\Xi u^4})u^4} + \frac{1}{u^2} = 0, \tag{41}$$

where

$$\Xi = \frac{1 + \sqrt{1 - r_{th}^4 u^4}}{u^4}\,. \tag{42}$$

After some manipulation, eq. (41) admits the following differential equation:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \pm\frac{b\,b_0(\sqrt{\Delta} + 1)}{\sqrt{2(4\beta^2u^4 - b_0^2)(b^2u^2 - 1)(\sqrt{\Delta} + 1)}}\,, \tag{43}$$

where

$$\Delta = \frac{b_0^2 - 4\beta^2u^4}{b_0^2}\,. \tag{44}$$

After evaluating the integral, the leading-order terms lead to the following result for the magnetic field case:

$$\hat{\alpha} \simeq \frac{9\pi\beta^2}{b^4\,b_0^2}\,. \tag{45}$$

If we compute the deflection angle of the non-gravitational wormhole possessing pure electric field, we get

$$\hat{\alpha} \simeq \frac{9\pi\alpha^2}{b^4\,b_0^2}\,. \tag{46}$$

One can easily observe that eqs. (45) and (46) are nothing but the deflection angles obtained in the weak limit approximation of GBT.

## 4 Deflection angle of wormhole with scalar field

Here we take the spacetime geometry to be that of a wormhole with scalar field with metric [94]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(1 - \frac{b(r)}{r} + \frac{\alpha}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right). \tag{47}$$

We will restrict our attention to the case $b(r) = b_0^2/r$, which converts the metric (47) to the following form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(1 - \frac{b_0^2}{r^2} + \frac{\alpha}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right). \tag{48}$$

Note that in order to describe a traversable wormhole, the metric (48) should obey some conditions: namely, it should be $b_0^2 > \alpha$, and in this case values of the radial coordinate $r$ lies in the domain $r \geq r_{th}$, where $r_{th}^2 = b_0^2 - \alpha$ is the throat radius.

From here on in, one can immediately derive the optical metric as follows:

$$\mathrm{d}t^2 = \frac{\mathrm{d}r^2}{(1 - \frac{b_0^2}{r^2} + \frac{\alpha}{r^2})} + r^2\mathrm{d}\varphi^2. \tag{49}$$

The Gaussian optical curvature (7) of metric (49) is then found to be

$$\mathcal{K} = \frac{-b_0^2 + \alpha}{r^4}. \tag{50}$$

If we substitute this equation into the deflection angle formula (17), we obtain

$$\hat{\alpha} = -\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\left(\frac{-b_0^2 + \alpha}{r^4}\right)\sqrt{\det h_{ab}}\, \mathrm{d}r\, \mathrm{d}\varphi. \tag{51}$$

After approximating $u \approx r$ and evaluating the integration the leading order terms yield

$$\hat{\alpha} \simeq \frac{\pi\, b_0^2}{4b^2} - \frac{\pi\,\alpha}{4b^2}. \tag{52}$$

Hence we conclude the effect of the scalar field is to bend the light outward the wormhole, in fact similar to the effect of electric charge.

### 4.1 Geodesics of a wormhole with scalar field

With the following Lagrangian, one can begin to perform the geodesics analysis of a wormhole metric with scalar field (48).

$$2\mathcal{L} = -\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{b_0^2}{r^2} + \frac{\alpha}{r^2}} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2. \tag{53}$$

Considering the planar photons ($\theta = \pi/2$) and taking cognizance of the two constants of motion $l$ and $\gamma$:

$$\mathcal{G} = 2r^2(s)\dot{\varphi}(s), \quad \text{and} \quad \mathcal{E} = 2\dot{t}(s), \tag{54}$$

we can normalize the affine parameter along the light rays by taking $\mathcal{E}$ =1 (metric condition) [95]. Again, we consider the nearest approach distance with the impact parameter *i.e.*, $u_{max} \simeq 1/b$ and use the second constant of motion $\mathcal{G} = b$ to focus on the leading order terms [96]. In sequel, we use the metric condition with eqs. (53) and (54) and employ the Lagrange equations. Thus, we obtain the following equation of motion for $u(\varphi)$:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 \frac{1}{(\alpha u^2 - b_0^2u^2 + 1)u^4} - \frac{1}{u^4b^2} + \frac{1}{u^2} = 0. \tag{55}$$

From here, it follows that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \pm\frac{b}{\sqrt{b^2(b_0^2u^4 - \alpha u^4 - u^2) - b_0^2u^2 + \alpha u^2 + 1}}. \tag{56}$$

Solution of the differential eq. (56) can be expressed as follows [95]

$$\Delta\varphi = \pi + \hat{\alpha}, \tag{57}$$

where $\hat{\alpha}$ is the deflection angle to be calculated. The deflection angle results in

$$\hat{\alpha} \simeq \frac{\pi\, b_0^2}{4b^2} - \frac{\pi\,\alpha}{4b^2}\,. \tag{58}$$

## 5 Quantum corrected Morris-Thorne traversable wormhole

As is well known, we can introduce a quantum velocity field $u_\alpha$ with a wave function of a quantum fluid as [93]

$$\psi(x^\alpha) = \mathcal{R}e^{i\,S(x^\alpha)}, \tag{59}$$

where $\psi(x^\alpha)$ is a normalizable wave function, $\mathcal{R}(x^\alpha)$ and $S(x^\alpha)$ are some real continuous functions associated with the four-velocity field $u_\alpha = \frac{\hbar}{m}\,\partial_\alpha S$, in which $(\alpha = 0, 1, 2, 3)$. In a recent work, due to implications of quantum corrections, Ali and Khalil [93] presented an interesting model for a black hole solution known as modified Schwarzschild black hole, when they replaced quantal (Bohmian) geodesics with classical geodesics like the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{\hbar\eta}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(1 - \frac{2M}{r} + \frac{\hbar\eta}{r^2})} + r^2\mathrm{d}\Omega_2^2, \tag{60}$$

where $\eta$ is a dimensionless constant and $\hbar$ has a dimension of $(\text{length})^2$ in geometric units. Note that a similar metric with semiclassical corrections to a regularized Schwarzschild metric was introduced in [99].

The spacetime ansatz for seeking static spherically symmetric traversable (without event horizon) wormhole spacetime can be written in the Schwarzschild coordinates as [1]

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{61}$$

where $(t, r, \theta, \varphi)$ are the standard Schwarzschild coordinates, with $\Phi(r)$ and $b(r)$ are the redshift and shape functions, respectively. Since we seek a wormhole solution, the redshift function $\Phi(r)$ should be finite everywhere, in order to avoid the presence of an event horizon. Another essential characteristics of a wormhole geometry is about the shape function $b(r)$, which should satisfy the condition of $b(r_0) = r_0$, in which $r_0$ is the radius of the wormhole throat where the two spacetimes (upper/lower spacetimes) are joined. Consequently, the shape function must satisfy the flaring-out condition:

$$\frac{b(r) - rb'(r)}{b^2(r)} > 0, \tag{62}$$

in which $b'(r) = \frac{\mathrm{d}b}{\mathrm{d}r} < 1$ must hold at the throat of the wormhole. In addition to this, the shape function also satisfy the asymptotically flat limit, *i.e.*, $b(r)/r \to 0$ as $r \to \infty$.

From now on, we focus on the effect of quantum corrections. Einstein equations remain the same with the effect of quantum corrections in the energy-momentum tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T^{\text{eff}}_{\mu\nu}, \tag{63}$$

where $G_{\mu\nu}$ and $T^{\text{eff}}_{\mu\nu}$ are the Einstein tensor and effective energy-momentum tensor, respectively. The total-derivative of the source term in the field equations is written as

$$T^{\text{eff}}_{\mu\nu} = T_{\mu\nu} + T^{(\text{corr.})}_{\mu\nu}. \tag{64}$$

For an anisotropic fluid $T_{\mu\nu}$, the energy-momentum tensor is defined as follows:

$$T^\mu{}_\nu = \left(-\rho, \mathcal{P}_r, \mathcal{P}_\theta, \mathcal{P}_\varphi\right), \tag{65}$$

where $\rho$ is the energy density with $\mathcal{P}_r$, $\mathcal{P}_\theta$ and $\mathcal{P}_\varphi$ are non-zero components of the diagonal terms, as measured in the orthogonal direction, respectively. The second term of eq. (64) has some modifications when the quantum corrections are taken into consideration:

$$T^\mu{}_\nu{}^{(\text{corr.})} = \left(-\rho^{(\text{corr.})}, P_r^{(\text{corr.})}, P_\theta^{(\text{corr.})}, P_\varphi^{(\text{corr.})}\right). \tag{66}$$

Using the definition of the 4-momentum given as follows,

$$p_\alpha = \hbar\,\partial_\alpha S, \tag{67}$$

using the wave function solution one can show that the geodesic motion is modified due to the relativistic quantum potential given by

$$V_Q = \hbar^2 \frac{\Box \mathcal{R}}{\mathcal{R}} . \tag{68}$$

In ref. [93] it was shown that for a general and static spherically symmetric spacetime metric in Schwarzschild coordinates, the stress-energy tensor has the following non-zero components:

$$\rho^{(\mathrm{corr.})} = P_r^{(\mathrm{corr.})} = P_\theta^{(\mathrm{corr.})} = P_\varphi^{(\mathrm{corr.})} = \frac{\hbar\eta}{8\pi r^4} , \tag{69}$$

in which $\eta$ is a dimensionless constant. It just happened that the effect of this quantum potential to be similar to the effect of the electric field described by a similar energy-momentum tensor components simply by letting $Q^2 \to \hbar\eta$ [94]. Moreover, the geometry encoded in the Einstein equations due to the quantum effect can be obtained by shifting the shape function in the following way:

$$b \to b_{\mathrm{eff}} = b - \frac{\hbar\eta}{r} . \tag{70}$$

More precisely, the quantum effect on the wormhole reveals when shifting the shape function $b$ into $(b - \hbar\eta/r)$. Our obtained solution refurbishes the work of charged wormhole solution inwhich it was assumed a probable charged metric in a combination of MT-type static spherically symmetric wormhole and Reissner-Nordström (RN) spacetime. We now demonstrate that our result, due to the quantum effects, is similar to that procedure. The key point is that we can recast the spacetime metric as

$$\mathrm{d}s^2 = -\left(1 + \frac{\hbar\eta}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r} + \frac{\hbar\eta}{r^2}} + r^2\mathrm{d}\Omega_2^2, \tag{71}$$

which is a combination of a Morris-Thorne–type wormhole with quantum corrections.

### 5.1 Gravitational lensing

Let us now continue our study by investigating the deflection of light in the quantum corrected spacetime geometry given by eq. (71). Choosing $b(r) = b_0^2/r$, metric (71) reduces to

$$\mathrm{d}s^2 = -\left(1 + \frac{\hbar\eta}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{b_0^2}{r^2} + \frac{\hbar\eta}{r^2}} + r^2\mathrm{d}\Omega_2^2. \tag{72}$$

The quantum corrected optical Gaussian curvature at the limit $r \to \infty$ therefore yields

$$\mathcal{K} \simeq \frac{3\hbar\eta - b_0^2}{r^4} - \frac{4b_0^2\hbar\eta}{r^6} . \tag{73}$$

Thus, the asymptotic form of the geodesic curvature becomes

$$\lim_{R\to\infty} \kappa(C_R) \to \frac{1}{R} , \tag{74}$$

which leads the following deflection angle:

$$\hat{\alpha} \simeq \frac{\pi b_0^2}{4\,b^2} - \frac{3\pi\hbar\eta}{4\,b^2} + \frac{3\,\pi\hbar\eta\, b_0^2}{8b^4} . \tag{75}$$

### 5.2 Energy conditions

In this section, we argue the issue of energy conditions and make some regional plots to check the validity of all energy conditions. WEC is defined by $T_{\mu\nu}U^\mu U^\nu \geq 0$, *i.e.*, $\rho \geq 0$ and $\rho(r) + \mathcal{P}_r(r) \geq 0$, where $T_{\mu\nu}$ is the energy-momentum tensor and $U^\mu$ denotes the timelike vector. This means that local energy density is positive and it gives rise to the

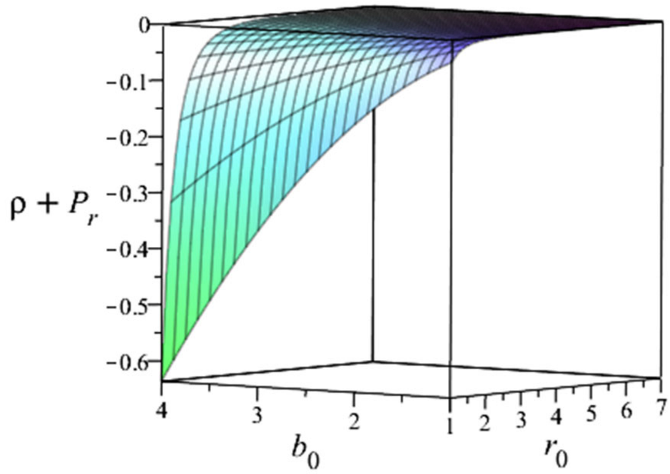


**Fig. 1.** The behavior of the NEC, *i.e.*, $\rho + \mathcal{P}_r$ as a function of $r_0$ and $b_0$, for the chosen $\hbar = 1$ and $\eta = 1$. We see that the NEC is violated.

continuity of NEC, which is defined by $T_{\mu\nu}k^\mu k^\nu \geq 0$, *i.e.*, $\rho(r) + \mathcal{P}_r(r) \geq 0$, where $k^\mu$ is a null vector. From Einstein's field equations written in terms of the effective shape function it follows:

$$\begin{aligned}\rho(r) &= \frac{b'_{\rm eff}}{8\pi r^2}\,,\\ \mathcal{P}_r(r) &= \frac{1}{8\pi}\left[2\left(1-\frac{b_{\rm eff}(r)}{r}\right)\frac{\Phi'}{r} - \frac{b_{\rm eff}(r)}{r^3}\right],\\ \mathcal{P}(r) &= \frac{1}{8\pi}\left(1-\frac{b_{\rm eff}(r)}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'_{\rm eff}r - b_{\rm eff}}{2r(r-b_{\rm eff})}\Phi' - \frac{b'_{\rm eff}r - b_{\rm eff}}{2r^2(r-b_{\rm eff})} + \frac{\Phi'}{r}\right],\end{aligned} \tag{76}$$

where

$$\Phi(r) = \frac{1}{2}\ln\left(1 + \frac{\hbar\eta}{r^2}\right). \tag{77}$$

In this regard, we have the following energy condition at the throat region:

$$\rho(r_0) = \frac{b'_{\rm eff}(r_0)}{8\pi r_0^2}. \tag{78}$$

Now, using the field equations, one finds the following relations:

$$\rho(r) + \mathcal{P}_r(r) = \frac{1}{8\pi}\left[2\left(1-\frac{b_{\rm eff}}{r}\right)\frac{\Phi'}{r} + \frac{rb'_{\rm eff} - b_{\rm eff}}{r^3}\right] \tag{79}$$

and

$$\rho(r) + \mathcal{P}(r) = \frac{1}{8\pi}\left[\left(1-\frac{b_{\rm eff}(r)}{r}\right)\left(\Phi'' + (\Phi')^2 - \frac{b'_{\rm eff}r - b_{\rm eff}}{2r(r-b_{\rm eff})}\Phi' - \frac{b'_{\rm eff}r - b_{\rm eff}}{2r^2(r-b_{\rm eff})} + \frac{b'_{\rm eff}}{r^2} + \frac{\Phi'}{r}\right)\right]. \tag{80}$$

One can see, from eq. (70), that the solutions are asymptotically flat, *i.e.*, $\frac{b_{\rm eff}(r)}{r} \to 0$ as $r \to \infty$. With the above solution, one also verifies that the form function obeys $b'_{\rm eff}(r_0) = (\hbar\eta - b_0^2)/r^2 < 1$, for the flaring-out condition at the throat. Taking into account the condition we should impose the restriction $\hbar\eta < 2r_0^2$. When we consider the wormhole throat, eq. (79) reduces to

$$(\rho + \mathcal{P}_r)|_{r=r_0} = \frac{1}{8\pi r_0^4}\left[-\frac{2(\hbar\eta)^2}{r_0^2 + \hbar\eta} + 2(\hbar\eta - r_0^2)\right]. \tag{81}$$

Taking into account the condition $\hbar\eta < 2r_0^2$, one verifies that matter configuration violates NEC at the throat, *i.e.* $(\rho + \mathcal{P}_r)|_{r=r_0} < 0$. In fig. 1, we show the behavior of the null energy condition.

### 5.3 Volume integral quantifier

We now consider the "volume integral quantifier", which provides information about measure of the amount of exotic matter required for wormhole maintenance and is related only to $\rho$ and $\mathcal{P}_r$, not to the transverse components, with the following definite integrals, $I_V = \int(\rho(r) + \mathcal{P}_r(r))\mathrm{d}V$, where $\mathrm{d}V = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$. Taking into account eq. (79) and performing integration by parts, which is

$$\oint[\rho + \mathcal{P}_r]\mathrm{d}V = \left[(r - b_{\mathrm{eff}}) \ln\left(\frac{1 + \hbar\eta/r^2}{1 - b_{\mathrm{eff}}/r}\right)\right]_{r_0}^{\infty} - \int_{r_0}^{\infty}\left[(r - b'_{\mathrm{eff}}) \ln\left(\frac{1 + \hbar\eta/r^2}{1 - b_{\mathrm{eff}}/r}\right)\right]\mathrm{d}r. \tag{82}$$

For the construction of wormhole the boundary term at $r_0$ vanishes, and due to asymptotic behavior the boundary term at infinity also vanishes. Thus the above expression reduces to

$$\oint[\rho + \mathcal{P}_r]\,\mathrm{d}V = -\int_{r_0}^{\infty}\left[(r - b'_{\mathrm{eff}}) \ln\left(\frac{1 + \hbar\eta/r^2}{1 - b_{\mathrm{eff}}/r}\right)\right]\mathrm{d}r. \tag{83}$$

The value of this volume-integral provides information about the "total amount" of ANEC violating matter in the spacetime, and we are going to solve the integral for our particular choice of shape function $b(r) = b_0^2/r$. Here, we suppose that the wormhole extends from $r_0$ to a radius situated at $a'$ and then we get the very simple result

$$I_V = \frac{b_0^2}{\sqrt{\hbar\eta}}\left[\arctan\left(\frac{r_0}{\sqrt{\hbar\eta}}\right) - \arctan\left(\frac{a}{\sqrt{\hbar\eta}}\right)\right]. \tag{84}$$

For an interesting observation when $a \to r_0$ then $\int(\rho + \mathcal{P}_r) \to 0$, and thus one may interpret that the wormhole can be contracted for with arbitrarily small quantities of ANEC violating matter.

## 6 Conclusions

In this paper, we have pointed out in detail the effect of electric charge, magnetic charge, scalar fields, and quantum effects on the deflection angle in the various wormhole geometries. To this end, by considering the optical geometries of the wormholes, we have employed the GWM in the context of GBT. Then, the deflection angles are computed via the surface integrals of the associated Gaussian optical curvatures on a domain outside the light ray, which reveals the effect of gravitational lensing as a global effect of the spacetime.

For the wormholes having electric/scalar fields, the deflection angles are found out to be proportional to the throat of the wormhole that bands the light rays towards itself. On the other hand, it is seen that electric/scalar fields play an important role to band the light rays outward the wormhole center. In the case of a non-linear electromagnetic theory in which the wormhole geometry is constructed purely by non-linear magnetic and electric fields, we have shown that deflection angle is proportional to the electromagnetic fields.

We have also constructed a Morris-Thorne wormhole, supported by the quantum effects coming from the Bohmian quantum mechanics. In particular, it has been shown that the exotic matter concentrated on the throat of the wormhole violates the null energy conditions. It is worth noting that the quantum effects on the wormhole geometry are correlated with the shape function: shifting of the shape function is in complete analogy with the charge wormhole solution.

Finally, we want to emphasize that although our results show consistency in the first-order terms in the deflection angles, the harmonized results of the methods disappear with the consideration of the second-order correction terms. This is simply due to the straight line approximation used in the integration domain. In near future, we plan to commit a study that will show fit in the results of the second-order correction terms. The latter aim is going to be possible when we manage to produce a suitable equation for the light ray or to modify the integration domain associated to the optical geometry.

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