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Series solutions of nonlinear conformable fractional KdV-Burgers equation with some applications

Ahmad El-Ajou[1], Zeyad Al-Zhour[2], Moa'ath Oqielat[1], Shaher Momani[3,4,5,a], and Tasawar Hayat[4,6]

[1] Department of Mathematics, Faculty of Science, Al Balqa Applied University, Salt 19117, Jordan
[2] Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam 31441, Saudi Arabia
[3] College of Humanities and Sciences, Ajman University, Ajman, United Arab Emirates
[4] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[5] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
[6] Department of Mathematics, Faculty of Science, Quaid-I-Azam University, Islamabad 45320, Pakistan

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Abstract. In this paper, the non-linear fractional KdV-Burgers equation (KdVBE) in terms of conformable fractional derivative (CFD) is reconstituted instead of the Caputo fractional derivative and the series solution of this case is also presented by using the residual power series (RPS) method. Moreover, five important and interesting applications related to the fractional KdVBE are given and discussed in order to show the behavior of the surface graphs of the solutions. More clarifications: Firstly, we compare the solutions of the conformable fractional KdVBE and the Caputo fractional KdVBE. Secondly, in order to demonstrate the generality, potential and superiority of the RPS method, we discuss the simplicity of this method compared with other methods. Thirdly, we present the approximate solutions with graphical results of a time-CFD, space-CFD and time-space-CFD non-linear fractional KdVBEs. Finally, the results indicate that the CFD is very suitable for modeling the KdVBE and computations show that our proposed method for solving the conformable fractional KdVBE does not have mathematical requirements which implies that it is very effective as well as for providing the numerical solutions and more flexible in choosing the initial guesses approximations.

1 Introduction

The importance of fractional calculus and its effectiveness in modeling many of the natural phenomena that we need to understand in the past is no longer hidden for mathematicians. In recent decades, many mathematical formulas that have practical applications in all different sciences such as in engineering, physics, chemistry and biology have been reformulated through fractional calculus. There are many partial differential equations (PDEs) that have been reformulated using the fractional derivatives concept, such as the Whitham-Broer-Kaup equations [1], reaction-diffusion equations [2], Black-Scholes European option pricing equations [3].

The KdVBE developed by Korteweg and de Vries [4] and derived by Su and Gardner [5] to describe nonlinear waves and several physical phenomena and also modeled encountered in several areas of applied mathematics such as heat conduction, acoustic waves, gas dynamics and traffic flow [6] are also a sample of PDEs that are reformulated by fractional derivatives. Recently, fractional KdVBE [7], fractional Schrödinger-KdVBE [8] and fractional Burgers equations [9] were presented to describe many important phenomena and dynamic processes in physics. The exact solutions of the KdVBE were obtained by using many different methods such as the homogeneous balance method [10], truncated expansion method [11] and exponential function method [12]. Very recently, many authors have paid attention to study analytical and numerical methods for solving KdVBEs such as the variational iterations method [13], decomposition method [14,15], element-free Galerkin method [16], explicit restrictive Taylor method [17] and residual power series method [18].

[a] e-mail: s.momani@ju.edu.jo (corresponding author)

In fact, the residual power series (RPS) method is an efficient and attractive method for obtaining the analytical solution of several fractional ordinary differential equations (ODEs) and fractional PDEs [18–28] such as: in the solution of the composite linear and nonlinear fractional ODE [19]; in the multiplicity fractional solutions to boundary value problems [20]; in the nonlinear time-fractional dispersive PDEs [21]; in a numerical solution of wave and telegraph equations [22], in the solution of fuzzy differential equations [23], in the exact solutions of fractional-order time-dependent Schrödinger equations [25], in the approximate analytical solutions of time-fractional diffusion equations [26], in the analytic solutions of the generalized Berger-Fisher equation [27], in the numerical solution of the coupled system of time-fractional nonlinear Boussinesq-Burger's equations [28] and other types of ordinary and fractional PDEs.

Most applications on the nonlinear fractional KdVBE that is mentioned in the researches [29–37] are only of space-fractional or time-fractional derivatives type. Indeed, there are no applications on nonlinear fractional KdVBEs that address the space-time-fractional state, and it is not easy to solve it in the known manner as reported in the literature [29–37], because it needs more execution time to accomplish the job of approximation.

There are many definitions for fractional derivatives that have been used in modeling many natural phenomena. The two most important of these definitions are: the Riemann-Liouville and Caputo definitions. In 2014, R. Khalil *et al.* [38] introduced a new definition of the fractional derivative which is called the "conformable fractional derivative (CFD)" and it is very easily computed compared with other previous definitions such as the Riemann-Liouville and Caputo fractional derivatives. The main advantages of the CFD can be summarized as follows [38–42]:

i) It simulates all the concepts and properties of an ordinary derivative such as: quotient, product and chain rules while the other fractional definitions fail to satisfy these rules.
ii) Very recently, it has received a lot of attention from many researchers and many applications have been remodeled using this definition.
iii) A non-differentiable function can be differentiated in the conformable sense.
iv) It solves exactly and numerically fractional differential equations and systems easily and efficiently.
v) It generalizes the well-known transforms such as the Laplace and Sumudu transforms and is used as a tool for solving some singular fractional differential equations.
vi) It paves the way for new comparisons and applications.
vii) It can be extended to solve fractional PDEs exactly and numerically as we will see in the present paper.

For the above reasons, in this paper, we reconstitute the KdVBE using the CFD definition rather than the Caputo fractional derivative that is used in equation modeling by the authors in references [15–18] and then we apply the RPS method to construct an approximate solution of this equation. In addition, we compare the resulting solution with the previous solutions provided in the case of using the Caputo fractional derivative to find out whether there is a significant difference in the graph of the solutions in the two cases and therefore determining the range suitability of using the CFD in modeling KdVBE. To achieve our aims, we consider the following form of the non-linear fractional KdVBE with time-space-CFDs:

$$\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} + \epsilon \psi^r(x,t) \frac{\partial^\beta \psi(x,t)}{\partial x^\beta} + \eta \frac{\partial^2 \psi(x,t)}{\partial x^2} + \mu \frac{\partial^3 \psi(x,t)}{\partial x^3} = 0, \quad 0 < \alpha,\ \beta \leq 1,\ t,\ x > 0, \tag{1}$$

subject to

$$\psi(x,0) = h(x), \tag{2}$$

where ε, η, μ are constants, $r = 0, 1, 2$, α and β refer to the order of time-CFD and space-CFD, respectively, $h(x)$ is an analytic function for all $x > 0$ and $\psi(x,t)$ is a function of two variables space x and time t which is vanishing for $t < 0$.

In particular,

i) if $\alpha = \beta = 1$, then eq. (1) is reduced to the classical KdV-Burgers equation;
ii) if $\beta = 1$ and $0 < \alpha \leq 1$, then eq. (1) is reduced to the time-fractional equation;
iii) if $\alpha = 1$ and $0 < \beta \leq 1$, then eq. (1) is reduced to the space-fractional equation.

This paper is arranged as follows: In the next section, we review some definitions and properties of the conformable fractional calculus and introduce a new form of Taylor's series in the CFD sense. In the third section, we construct the RPS solution of KdVBE. In the fourth section, we present the RPS solutions of five important problems and compare with the results as in [18,35,37]. Finally, we summarize the outcomes of this paper in the conclusion section.

2 Conformable fractional calculus

In this Section, we study the definition and some basic concepts of the α-th–order conformable fractional derivative (CFD) and conformable fractional integral (CFI) [38–42] and then we extend this definition to the partial case with some partial expansions. In addition, we study the multiple fractional power series (MFPS) with some related results.

Definition 1. *The CFD starting from s of a function $f : [s, \infty) \to \mathcal{R}$ of order $\alpha \in (n-1, n]$ is defined by* [31,32]*:*

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} = f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(n-1)}(t + \varepsilon(t-s)^{n-\alpha}) - f^{(n-1)}(t)}{\varepsilon}, \quad t > s, \tag{3}$$

and $f^{(\alpha)}(s) = \lim_{t \to s^+} f^{(\alpha)}(t)$ provided $f(t)$ is $(n-1)$-differentiable and $\lim_{t\to s^+} f^{(\alpha)}(t)$ exists.

Note that the basic rules and properties for some known functions for $\alpha \in (0,1]$ such as the linearity property, power rule, addition rule, product rule, quotient rule and relationship between α-differentiation and ordinary differentiation of $f(t)$, $t > s$ can be found in the literature [38,39].

Remark 1. For $\alpha \in (n-1, n]$, $n \in N$ and f is an n-differentiable at $t > s$, we have:

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t) = (t-s)^{n-\alpha} \frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t).$$

Definition 2. *The CFI starting from s of order $\alpha \in (n-1, n]$, $n \in N$ of f is defined as*

$$I_s^\alpha f(t) = \frac{1}{(n-1)!} \int_s^t \frac{(t-x)^{n-1} f(x)}{(x-s)^{n-\alpha}} \mathrm{d}x, \quad t > s \geq 0. \tag{4}$$

For $\alpha = n$, then the formula as in eq. (3) *is reduced to the following Cauchy formula:*

$$I_s^n f(t) = \frac{1}{(n-1)!} \int_s^t (t-x)^{n-1} f(x) \mathrm{d}x, \quad t > s \geq 0.$$

Theorem 1. *Let $\alpha \in (n-1, n]$, $n \in N$ and f be any n times differentiable function. Then*

i) $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(I_s^\alpha f(t)) = f(t)$;

ii) $I_s^\alpha(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(s)(t-s)^k}{k!}$.

Proof. By applying remark 1, then we have

i)

$$\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(I_s^\alpha f(t)) &= (t-s)^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I_s^\alpha f(t)) \\
&= (t-s)^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\frac{1}{(n-1)!} \int_s^t \frac{(t-x)^{n-1} f(x)}{(x-s)^{n-\alpha}} \mathrm{d}x \right) \\
&= (t-s)^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_s^n \left(\frac{f(t)}{(t-s)^{n-\alpha}} \right) \\
&= (t-s)^{n-\alpha} \frac{f(t)}{(t-s)^{n-\alpha}} = f(t);
\end{aligned}$$

ii)

$$\begin{aligned}
I_s^\alpha \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) \right) &= I_s^\alpha \left((t-s)^{n-\alpha} \frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t) \right) \\
&= \frac{1}{(n-1)!} \int_s^t \frac{(t-x)^{n-1}((x-s)^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x))}{(x-s)^{n-\alpha}} \mathrm{d}x \\
&= \frac{1}{(n-1)!} \int_s^t (t-x)^{n-1} f^{(n)}(x) \mathrm{d}x = I_s^n f^{(n)}(t) \\
&= f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(s)(t-s)^k}{k!}.
\end{aligned}$$

Definition 3. *The time-CFD operator of order $\alpha \in (n-1, n]$ of $\psi(x,t)$ is defined as*

$$\psi_t^{(\alpha)}(x,t) = \frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} = \lim_{\varepsilon \to 0} \frac{\psi_t^{(n-1)}(x, t + \varepsilon(t-s)^{n-\alpha}) - \psi_t^{(n-1)}(x,t)}{\varepsilon}, \quad t > s \geq 0, \tag{5}$$

and the space-CFD operator of order $\beta \in (n-1, n]$ of $\psi(x,t)$ is defined as

$$\psi_x^{(\beta)}(x,t) = \frac{\partial^\beta \psi(x,t)}{\partial x^\beta} = \lim_{\varepsilon \to 0} \frac{\psi_x^{(n-1)}(x + \varepsilon(x-s)^{n-\beta}, t) - \psi_x^{(n-1)}(x,t)}{\varepsilon}, \quad x > s \geq 0. \tag{6}$$

Definition 4. *The power series of the form*

$$\sum_{m=0}^{\infty} h_m(x)(t-s)^{m\alpha} = h_0(x) + h_1(x)(t-s)^{\alpha} + h_2(x)(t-s)^{2\alpha} + \dots, \quad t \geq s,\ 0 \leq n-1 < \alpha \leq n, \tag{7}$$

is called the MFPS about $t = s > 0$, where h_m's are functions of x.

As a special case, if h_m's are constant functions, then we get the fractional power series (FPS) expansion about $t = s$ [19].

Theorem 2. *Assume that $\psi(x,t)$ has a MFPS expansion at $t = s$ of the form*

$$\psi(x,t) = \sum_{m=0}^{\infty} h_m(x)(t-s)^{m\alpha}, \quad \alpha \in (n-1, n],\ x \in I,\ s \leq t < s + R. \tag{8}$$

If $\frac{\partial^{m\alpha}}{\partial t^{m\alpha}}\psi(x,t) = \psi_t^{(m\alpha)}(x,t)$ are continuous on $I \times (s, s+R)$, $m = 0, 1, 2, \dots$, then coefficients $h_m(x)$ of eq. (8) are given as

$$h_m(x) = \psi_t^{(m\alpha)}(x,s)\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha-n+1)}{\Gamma(2\alpha+1)}\dots\frac{\Gamma(m\alpha-n+1)}{\Gamma(m\alpha+1)}, \quad m = 0, 1, 2, \dots, \tag{9}$$

where $\frac{\partial^{m\alpha}}{\partial t^{m\alpha}} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \cdot \frac{\partial^{\alpha}}{\partial t^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ (m-times) and $R = \min_{c \in I} R_c$, in which R_c is the radius of convergence of the FPS $\sum_{m=0}^{\infty} h_m(c)(t-s)^{m\alpha}$.

As a special case of eq. (9), if $n = 1$, then we have:

$$h_m(x) = \frac{\psi_t^{(m\alpha)}\psi(x,s)}{\alpha^m(m!)}, \quad m = 0, 1, 2, \dots, \tag{10}$$

Proof. Assume that $\psi(x,t)$ is a function of two variables that can be represented by a MFPS of eq. (7). Note that if we put $t = s$ into eq. (8), then all terms after the first are vanished, and we obtain

$$h_0(x) = \psi(x,s). \tag{11}$$

Applying the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ one time on eq. (8), which implies

$$\psi_t^{(\alpha)}\psi(x,t) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}h_1(x) + \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-n+1)}h_2(x)(t-s)^{\alpha} + \frac{\Gamma(3\alpha+1)}{\Gamma(3\alpha-n+1)}h_3(x)(t-s)^{2\alpha}. \tag{12}$$

By substituting $t = t_0$ into eq. (12), we get

$$h_1(x) = \psi_t^{(\alpha)}(x,s)\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)}. \tag{13}$$

Again, applying the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ two times on eq. (8), we get

$$\begin{aligned}\psi_t^{(2\alpha)}\psi(x,t) = {} & h_2(x)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-n+1)} + h_3(x)\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-n+1)}\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}(t-s)^{\alpha} \\ & + h_4(x)\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha-n+1)}(t-s)^{2\alpha} + \dots.\end{aligned} \tag{14}$$

By substituting $t = s$ into eq. (14) we get

$$h_2(x) = \psi_t^{(2\alpha)}(x,t_0)\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha-n+1)}{\Gamma(2\alpha+1)}. \tag{15}$$

If we go on to apply the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ repeatedly m times on eq. (8) and then putting $t = s$ in the resulting formula, then we obtain $h_m(x)$ as in eq. (9).

Remark 2. By substituting formula of $h_m(x)$ as in eq. (10) and using eq. (8), we can see that the MFPS of $\psi(x,t)$ at s must be of the following expansion:

$$\psi(x,t) = \sum_{m=0}^{\infty} \frac{\psi_t^{(m\alpha)}\psi(x,s)}{\phi(\alpha,m)}(t-s)^{m\alpha}\,, \quad x \in I,\ s \le t < s+R,\ 0 \le n-1 < \alpha \le n, \tag{16}$$

where

$$\phi(\alpha,m) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-n+1)}\cdots\frac{\Gamma(m\alpha+1)}{\Gamma(m\alpha-n+1)}\,, \quad m = 0,1,2,\ldots, \tag{17}$$

which is a new form of Taylor's series of $\psi(x,t)$.

Note that the classical Taylor's series formula can be easily obtained by setting $\alpha = 1$ in eq. (16).

3 Approximation RPS solutions of the conformable fractional KdVBE

In this Section, we present an analytic approximate solution of the conformable fractional KdVBE as in eqs. (1) based on the RPS method.

To attain our aim, we assume that the power series solution of eq. (1) as in the following form:

$$\psi(x,t) = \sum_{n=0}^{\infty} h_n(x)\frac{t^{n\alpha}}{\alpha^n n!}\,, \quad x \in I,\ 0 \le t < R,\ 0 < \alpha \le 1. \tag{18}$$

Now, let $\psi_k(x,t)$ be the k-th truncated series of $\psi(x,t)$ as follows:

$$\psi_k(x,t) = \sum_{n=0}^{k} h_n(x)\frac{t^{n\alpha}}{\alpha^n n!}\,, \quad 0 < \alpha \le 1,\ x \in I,\ 0 \le t,\ k = 0,1,2,\ldots \tag{19}$$

Since $u(x,t)$ satisfies eq. (2), so by using eq. (18), we get $\psi(x,0) = h_0(x) = h(x)$. Therefore, the k-th truncated series of $\psi(x,t)$ becomes of the form:

$$\psi_k(x,t) = h(x) + \sum_{n=1}^{k} h_n(x)\frac{t^{n\alpha}}{\alpha^n n!}\,, \quad x \in I,\ 0 \le t,\ 0 < \alpha \le 1,\ k = 1,2,3,\ldots. \tag{20}$$

On the other hand, from eq. (19) the zero-th approximate of the RPS solution of $\psi(x,t)$ is

$$\psi_0(x,t) = h_0(x) = h(x).$$

Now, we need to determine the values of coefficients $h_n(x)$, $n = 1,2,3,\ldots$, k in eq. (20) as follows.

Define the so-called residual function of eq. (1) in case $r = 2$ as

$$\mathrm{Res}(x,t) = \frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} + \epsilon\psi^2(x,t)\frac{\partial^\beta \psi(x,t)}{\partial x^\beta} + \eta\frac{\partial^2 \psi(x,t)}{\partial x^2} + \mu\frac{\partial^3 \psi(x,t)}{\partial x^3}\,, \tag{21}$$

and the k-th residual function as follows:

$$\mathrm{Res}_k(x,t) = \frac{\partial^\alpha \psi_k(x,t)}{\partial t^\alpha} + \epsilon\psi_k^2(x,t)\frac{\partial^\beta \psi_k(x,t)}{\partial x^\beta} + \eta\frac{\partial^2 \psi_k(x,t)}{\partial x^2} + \mu\frac{\partial^3 \psi_k(x,t)}{\partial x^3}\,, \quad k = 1,2,3,\ldots. \tag{22}$$

It is clear to verify the following facts:

i) $\lim_{k\to\infty} \mathrm{Res}_k(x,t) = \mathrm{Res}(x,t) = 0$, $x \in I$, $t \ge 0$;
ii) $\frac{\mathrm{d}^{i\alpha}}{\mathrm{d}t^{i\alpha}} \mathrm{Res}(x,t) = 0$, $i = 0,1,2,\ldots$;
iii) $\frac{\mathrm{d}^{i\alpha}}{\mathrm{d}t^{i\alpha}} \mathrm{Res}(x,0) = \frac{\mathrm{d}^{i\alpha}}{\mathrm{d}t^{i\alpha}} \mathrm{Res}_k(x,0)$, $i = 0,1,2,\ldots$.

Now, substitute the k-th truncated series of $\psi(x,t)$ in eq. (22) then we have

$$\frac{\mathrm{d}^{(k-1)\alpha}}{\mathrm{d}t^{(k-1)\alpha}} \mathrm{Res}_k(x,0) = 0, \quad k = 1,2,3,\ldots. \tag{23}$$

Thus, the required coefficients $h(x)$ can be easily determined by solving the algebraic equation as in eq. (23).

Now, $h_1(x)$ can be determined by substituting the 1st-truncated series $\psi_1(x,t)$ into the 1st residual function $\mathrm{Res}_1(x,t)$ to get

$$\mathrm{Res}_1(x,t) = \frac{\partial^\alpha \psi_1(x,t)}{\partial t^\alpha} + \varepsilon\psi_1^2(x,t)\frac{\partial^\beta \psi_1(x,t)}{\partial x^\beta} + \eta\frac{\partial^2 \psi_1(x,t)}{\partial x^2} + \mu\frac{\partial^3 \psi_1(x,t)}{\partial x^3}. \tag{24}$$

Since $\psi_1(x,t) = h(x) + h_1(x)\frac{t^\alpha}{\alpha}$, then eq. (24) implies:

$$\mathrm{Res}_1(x,t) = h_1(x) + \varepsilon\left(\frac{\partial^\beta h(x)}{\partial x^\beta} + \frac{t^\alpha}{\alpha}\frac{\partial^\beta h_1(x)}{\partial x^\beta}\right)\left(h(x) + h_1(x)\frac{t^\alpha}{\alpha}\right)^2 + \eta h''(x) + \eta h_1''(x)\frac{t^\alpha}{\alpha} + \mu h'''(x) + \mu h_1'''(x)\frac{t^\alpha}{\alpha}. \tag{25}$$

Now, based on eq. (23) for $k = 1$ and eq. (25) for $t = 0$ yields

$$h_1(x) = -\left(\varepsilon h^2(x)h^{(\beta)}(x) + \eta h''(x) + \mu h'''(x)\right). \tag{26}$$

Hence, after zero-th approximate of the RPS solution $\psi_0(x,t) = h(x)$, then the 1st approximate of the RPS solution of eqs. (1) and (2) can be expressed as:

$$\psi_1(x,t) = h(x) - \left(\varepsilon h^2(x)h^{(\beta)}h(x) + \eta h''(x) + \mu h'''(x)\right)\frac{t^\alpha}{\alpha}. \tag{27}$$

Similarly, $f_2(x)$ can be obtained also by substituting the 2nd truncated series; $\psi_2(x,t) = h(x) + h_1(x)\frac{t^\alpha}{\alpha} + h_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ of eqs. (1) and (2) into eq. (22) when $k = 2$ to get:

$$\begin{aligned}\mathrm{Res}_2(x,t) = {}& h_1(x) + \eta h''(x) + \mu h'''(x) + \varepsilon\left(h^{(\beta)}(x) + h_1^{(\beta)}(x)\frac{t^\alpha}{\alpha} + h_2^{(\beta)}(x)\frac{t^{2\alpha}}{2\alpha^2}\right)\\ &\times\left(h(x) + h_1(x)\frac{t^\alpha}{\alpha} + h_2(x)\frac{t^{2\alpha}}{2\alpha^2}\right)^2 + \left(h_2(x) + \eta h_1''(x) + \mu h_1'''(x)\right)\frac{t^\alpha}{\alpha} + \left(\eta h_2''(x) + \mu h_2'''(x)\right)\frac{t^{2\alpha}}{2\alpha^2}.\end{aligned} \tag{28}$$

Now, operating $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ on eq. (28) gives the α-th–order time CFD of $\mathrm{Res}_2(x,t)$ as

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\mathrm{Res}_2(x,t) = {}& h_2(x) + 2\varepsilon h(x)h_1(x)h^{(\beta)}(x) + \varepsilon h^2(x)h_1^{(\beta)}(x) + \eta h_1''(x) + \mu h_1'''(x)\\ &+ \frac{\varepsilon}{\alpha}\left(2h_1^2(x)h^{(\beta)}(x) + 2h(x)h_2(x)h^{(\beta)} + 4h(x)h_1(x)h_1^{(\beta)}(x) + h^2(x)h_2^{(\beta)}(x) + \eta h_2''(x) + \mu h_2'''(x)\right)t^\alpha\\ &+ \frac{3\varepsilon}{\alpha^2}\left(h_1^2(x)h_1^{(\beta)}(x) + h_1(x)h_2(x)h^{(\beta)}(x) + h(x)h_2(x)h_1^{(\beta)}(x) + h(x)h_1(x)h_2^{(\beta)}(x)\right)t^{2\alpha}\\ &+ \frac{\varepsilon}{\alpha^3}\left(h_2^2(x)h^{(\beta)}(x) + 2h(x)h_2(x)h_2^{(\beta)}(x) + 4h_1(x)h_2(x)h_1^{(\beta)}(x) + 2h_1^2(x)h_2^{(\beta)}(x)\right)t^{3\alpha}\\ &+ \frac{\varepsilon}{4\alpha^4}\left(5h_2^2(x)h_1^{(\beta)}(x) + 10h_1(x)h_2(x)h_2^{(\beta)}(x)\right)t^{4\alpha} + \frac{3\varepsilon}{4\alpha^5}h_2^2(x)h_2^{(\beta)}(x)t^{5\alpha}.\end{aligned} \tag{29}$$

By considering $k = 2$ in eq. (23) and solving $h_2(x)$ in the resulting algebraic equation, it is easy to access

$$h_2(x) = -\left(\varepsilon\left(2h(x)h_1(x)h^{(\beta)}(x) + h^2(x)h_1^{(\beta)}(x)\right) + \eta h_1''(x) + \mu h_1'''(x)\right). \tag{30}$$

Thus, the 2nd approximate solution of the IVP (1) and (2) is given by

$$\begin{aligned}\psi_2(x,t) = {}& h(x) - \left(\varepsilon h^2(x)h^{(\beta)}(x) + \eta h''(x) + \mu h'''(x)\right)\frac{t^\alpha}{\alpha}\\ &- \left(\varepsilon\left(2h(x)h_1(x)h^{(\beta)}(x) + h^2(x)h_1^{(\beta)}(x)\right) + \eta h_1''(x) + \mu h_1'''(x)\right)\frac{t^{2\alpha}}{2\alpha^2}.\end{aligned} \tag{31}$$

Similarly, by applying the same process above for $k = 3$, then one can easily obtain:

$$h_3(x) = -\left(\varepsilon\left(2h_1^2(x)h^{(\beta)}(x) + 4h(x)h_1(x)h_1^{(\beta)} + 2h(x)h_2(x)h^{(\beta)}(x) + h^2(x)h_2^{(\beta)}(x)\right) + \eta h_2''(x) + \mu h_2'''(x)\right). \tag{32}$$

Indeed, based on the previous results of $h_0(x)$, $h_1(x)$, and $h_2(x)$ and the result of eq. (32), the 3rd approximate solution of KdVBE (1), in case $r=2$, is given by

$$\begin{aligned}\psi_3(x,t)&=h(x)-\left(\varepsilon h^2(x)h^{(\beta)}(x)+\eta h''(x)+\mu h'''(x)\right)\frac{t^\alpha}{\alpha}-\left(\varepsilon\left(2f(x)h_1(x)h^{(\beta)}(x)+h^2(x)h_1^{(\beta)}(x)\right)+\eta h_1''(x)+\mu h_1'''(x)\right)\frac{t^{2\alpha}}{2\alpha^2}\\&-\left(\varepsilon\left(2h_1^2(x)h^{(\beta)}(x)+4h(x)h_1(x)h_1^{(\beta)}(x)+2h(x)h_2(x)h^{(\beta)}(x)+h^2(x)h_2^{(\beta)}(x)\right)+\eta h_2''(x)+\mu h_2'''(x)\right)\frac{t^{3\alpha}}{6\alpha^3}.\end{aligned} \tag{33}$$

Finally, following the previous procedure for finding an approximate solution for the IVP (1) and (2) in the case of $r=2$, we can find an approximate solution to the same IVP in the cases of $r=0$ and $r=1$. The reader can verify that the approximate solution for IVP (1) and (2) for cases $r=0$ and $r=1$, respectively, is as follows.

i) In case $r=0$, the first few coefficients in the series (20) are recorded as:

$$\begin{aligned}h_0(x)&=h(x),\\h_1(x)&=-\left(\varepsilon h^{(\beta)}(x)+\eta h''(x)+\mu h'''(x)\right),\\h_2(x)&=-\left(\varepsilon h_1^{(\beta)}(x)+\eta h_1''(x)+\mu h_1'''(x)\right),\\h_3(x)&=-\left(\varepsilon h_2^{(\beta)}(x)+\eta h_2''(x)+\mu h_2'''(x)\right).\end{aligned} \tag{34}$$

ii) In case $r=1$, the first few coefficients in the series (20) are listed as:

$$\begin{aligned}h_0(x)&=h(x),\\h_1(x)&=-\left(\varepsilon h(x)h^{(\beta)}(x)+\eta h''(x)+\mu h'''(x)\right),\\h_2(x)&=-\left(\varepsilon\left(h_1(x)h^{(\beta)}(x)+h(x)h_1^{(\beta)}(x)\right)+\eta h_1''(x)+\mu h_1'''(x)\right),\\h_3(x)&=-\left(\varepsilon\left(h_2(x)h^{(\beta)}(x)+2h_1(x)h_1^{(\beta)}(x)+h(x)h_2^{(\beta)}(x)\right)+\eta h_2''(x)+\mu h_2'''(x)\right).\end{aligned} \tag{35}$$

Thus, we completed constructing the RPS approximate solution for the non-linear conformable fractional KdVBE with time-CFD. The construction of the RPS approximate solutions for the non-linear conformable fractional KdVBE with space-CFD and time-space-CFDs is similar to the case of time-CFD and will be illustrated through the tested applications in the next section.

4 Some applications and graphical results

In this section, five applications are considered and discussed to demonstrate the potentiality, generality and superiority of the RPS method for solving the conformable fractional KdVBE as in eq. (1) as well as compared with other methods. These applications are:

- The time-conformable fractional nonlinear KdVBEs (first and second applications).
- The space-conformable fractional nonlinear KdVBE (third and fourth applications).
- The time-space-conformable fractional nonlinear KdVBE (last application).

The MATHEMATICA7 Software Package is used for getting our computations.

Application 1. Given the following time-conformable fractional nonlinear KdVBE:

$$\frac{\partial^\alpha\psi(x,t)}{\partial t^\alpha}+6\psi^2(x,t)\frac{\partial\psi(x,t)}{\partial x}+\frac{\partial^3\psi(x,t)}{\partial x^3}=0,\quad x\in I,\ t>0,\ 0<\alpha\le 1, \tag{36}$$

subject to

$$\psi(x,0)=\sqrt{c}\,\mathrm{sech}\left(\omega+\sqrt{c}x\right). \tag{37}$$

Now by using the RPS method, taking $r=\beta=1$, $\varepsilon=6$, $\eta=0$, $\mu=1$ and starting with the initial function $h_0(x)=\sqrt{c}\,\mathrm{sech}(\omega+\sqrt{c}x)$, where c and ω are constants with the following k-th residual function:

$$\mathrm{Res}_k(x,t)=\frac{\partial^\alpha\psi_k(x,t)}{\partial t^\alpha}+6\psi_k^2(x,t)\frac{\partial\psi_k(x,t)}{\partial x}+\frac{\partial^3\psi_k(x,t)}{\partial x^3},\quad k=1,2,3. \tag{38}$$

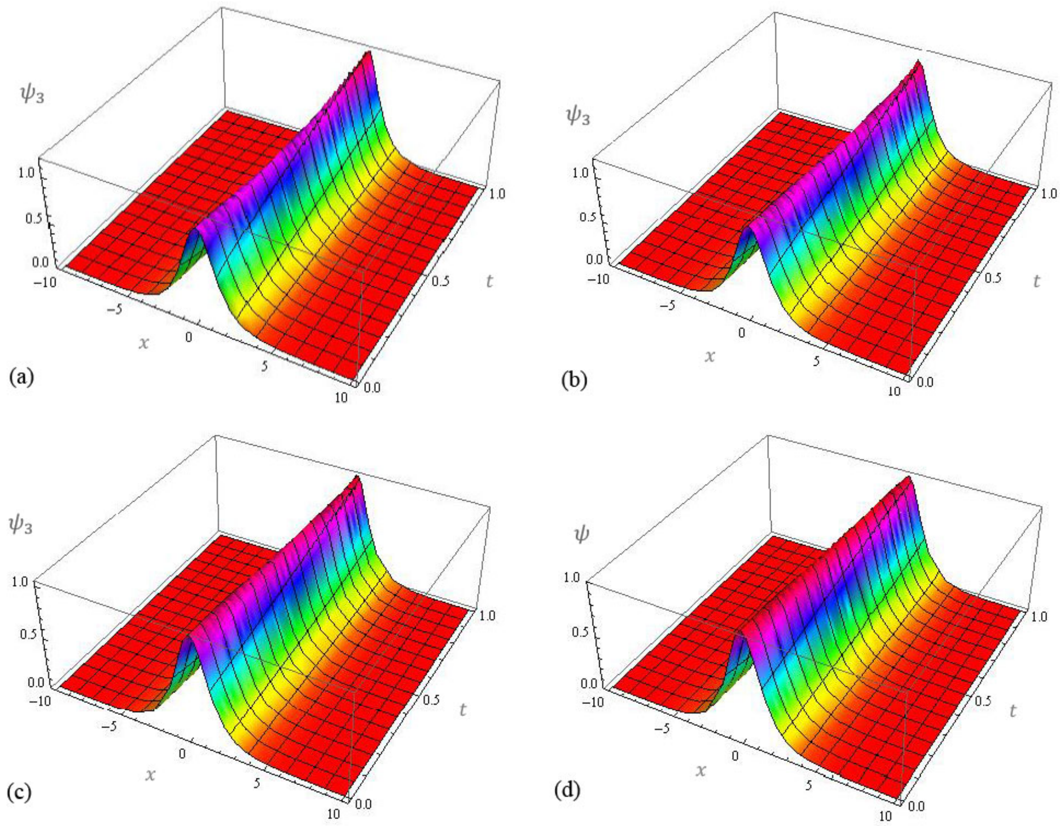

Fig. 1. The graph of the exact solution $\psi(x,t)$ and the approximate solution $\psi_3(x,t)$ of eq. (36) in the CFD sense: (a) $\psi_3(x,t)$ when $\alpha = 0.75$, (b) $\psi_3(x,t)$ when $\alpha = 0.90$, (c) $\psi_3(x,t)$ when $\alpha = 1$, (d) $\psi(x,t)$ when $\alpha = 1$.

Then the next forms for $h_n(x)$, $n = 1,2,3$ are obtained by applying eqs. (26), (30) and (32)

$$\begin{aligned}
h_0(x) &= \sqrt{c}\,\mathrm{sech}\left(\omega + \sqrt{c}x\right),\\
h_1(x) &= c^2 \tanh\left(\omega + \sqrt{c}x\right)\mathrm{sech}\left(\omega + \sqrt{c}x\right),\\
h_2(x) &= -\frac{1}{2}c^{7/2}\,\mathrm{sech}^3\left(\omega + \sqrt{c}x\right)\left(3 - \cosh\left(2\left(\omega + \sqrt{c}x\right)\right)\right),\\
h_3(x) &= \frac{1}{2}c^5\left(-11 + \cosh\left(2\left(\omega + \sqrt{c}x\right)\right)\right)\mathrm{sech}^3\left(\omega + \sqrt{c}x\right)\tanh\left(\omega + \sqrt{c}x\right),\\
&\ \ \vdots
\end{aligned} \tag{39}$$

Here, we just consider the first three terms in eq. (19) in the solutions of eqs. (36) and (37) by letting the approximation of the initial guess $\psi_0(x,t) = h_0(x)$. As a result, the 3rd approximate of the RPS solution of eq. (36) coincide precisely with the following expansion:

$$\psi_3(x,t) = h_0(x) + h_1(x)\frac{t^\alpha}{\alpha} + h_2(x)\frac{t^{2\alpha}}{(2!)\alpha^2} + h_3(x)\frac{t^{3\alpha}}{(3!)\alpha^3}. \tag{40}$$

Remark 3. If we track the pattern in the coefficients in eq. (39) when $\alpha = 1$, we find the general form of the solution of eq. (36) is coinciding with the next exact solution as a special case:

$$\psi(x,t) = \sqrt{c}\,\mathrm{sech}\left(\omega + \sqrt{c}\,(x - ct)\right). \tag{41}$$

To demonstrate the geometric behaviors of the RPS approximate solution of eq. (36), a comparison between the approximate $\psi_3(x,t)$ and exact solution $\psi(x,t)$ is given in 3D for a different value of α and $c = 1$, $\omega = 0$, see fig. 1.

From fig. 1, it is obvious that the behaviors of the subfigures are almost similar and coinciding specially for panels (c) and (d) where $\alpha = 1$, they appear identical and in perfect agreement in terms of accuracy to each other. Consequently,

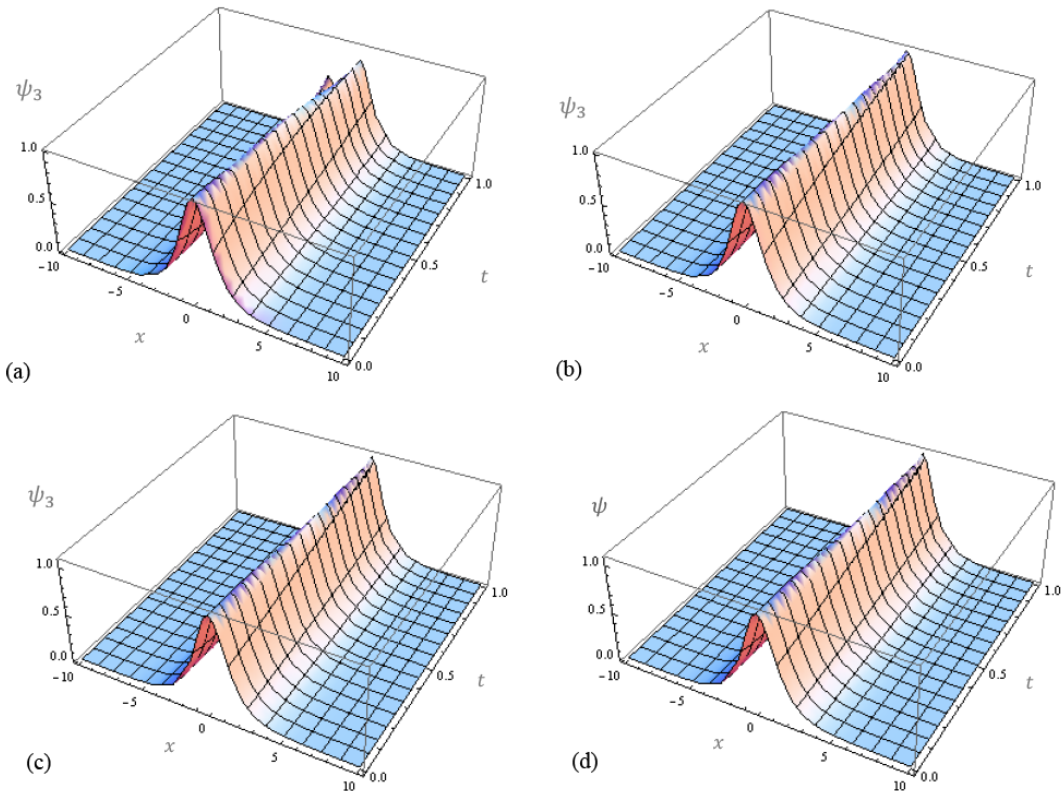

Fig. 2. The graph of the exact solution $\psi(x,t)$ and the approximate solution $\psi_3(x,t)$ of eq. (36) in the Caputo sense: (a) $\psi_3(x,t)$ when $\alpha = 0.75$, (b) $\psi_3(x,t)$ when $\alpha = 0.90$, (c) $\psi_3(x,t)$ when $\alpha = 1$, (d) $\psi(x,t)$ when $\alpha = 1$.

implementing only a few terms can produce an excellent approximation comparing with the exact solution. Therefore, the total error can vanish by finding a larger number of the series terms.

The authors presented in [18] an approximate RPS solution for eqs. (36) and (37) in the Caputo sense, where the 3rd approximate was as follows:

$$\begin{aligned}\psi_3(x,t) &= \sqrt{c}\,\mathrm{sech}\left(\omega+\sqrt{c}x\right) + c^2\,\mathrm{sech}\left(\omega+\sqrt{c}x\right)\tanh\left(\omega+\sqrt{c}x\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\ &\quad -\frac{1}{2}c^{7/2}\left(3-\cosh\left(2\left(\omega+\sqrt{c}x\right)\right)\right)\mathrm{sech}^3\left(\omega+\sqrt{c}x\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &\quad +\left(\frac{c^5}{8}\,\mathrm{sech}^5\left(\omega+\sqrt{c}x\right)\tanh\left(\omega+\sqrt{c}x\right)\frac{24\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\left(-7+3\cosh\left(2\left(\omega+\sqrt{c}x\right)\right)\right)\right.\\ &\quad \left.+\,315-164\cosh\left(2(\omega+\sqrt{c}x)\right)+\cosh\left(4(\omega+\sqrt{c}x)\right)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\,. \end{aligned} \tag{42}$$

Figure 2 illustrates the surface graph of the approximate solution of eqs. (36) and (37) in eq. (42). It is evident that the graph for the approximate solution of eqs. (36) and (37) in the sense of CFD contained in eq. (40) simulates the approximate solution graph in the sense derived from the Caputo fractional derivative contained in eq. (42). It is also noticeable that the surface of the approximate solution in approximate solution given in eq. (42), especially when eq. (40) is smooth on all its domain, while the $\alpha = 0.75$, is not so and this is the advantage of the modeling of the KdVBE by CFD.

Application 2. A time-conformable fractional nonlinear KdVBE is given by:

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}} - 6\psi(x,t)\frac{\partial\psi(x,t)}{\partial x} + \frac{\partial^3\psi(x,t)}{\partial x^3} = 0, \quad x,\ t>0,\ 0<\alpha\le 1, \tag{43}$$

subject to

$$\psi(x,0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}\,. \tag{44}$$

By applying the method of RPS, taking $r = \mu = \beta = 1$, $\varepsilon = -6$, $\eta = 0$, and the initial function $h_0(x) = -2\frac{\omega^2 e^{\omega x}}{(1+e^{\omega x})^2}$, ($\omega$ is a constant) and the next k-th residual function:

$$\mathrm{Res}_k(x,t) = \frac{\partial^\alpha \psi_k(x,t)}{\partial t^\alpha} - 6\psi_k(x,t)\frac{\partial \psi_k(x,t)}{\partial x} + \frac{\partial^3 \psi_k(x,t)}{\partial x^3}, \quad k = 1,2,3. \tag{45}$$

Now, computations by using eq. (53) give the following forms for $h_n(x)$, $n = 1,2,3$:

$$\begin{aligned}
h_0(x) &= -\frac{2e^{\omega x}k^2}{(1+e^{\omega x})^2},\\
h_1(x) &= -\frac{2e^{\omega x}(-1+e^{\omega x})k^5}{(1+e^{\omega x})^3},\\
h_2(x) &= -\frac{2e^{\omega x}(1-4e^{\omega x}+e^{2\omega x})k^8}{(1+e^{\omega x})^4},\\
h_3(x) &= -\frac{2e^{\omega x}(-1+e^{\omega x})(1-10e^{\omega x}+e^{2\omega x})k^{11}}{(1+e^{\omega x})^5},\\
&\ \ \vdots
\end{aligned} \tag{46}$$

Consequently, the 3rd approximate of the RPS solution of eqs. (43) and (44) takes the following expansion:

$$\psi_3(x,t) = h_0(x) + h_1(x)\frac{t^\alpha}{\alpha} + h_2(x)\frac{t^{2\alpha}}{(2!)\alpha^2} + h_3(x)\frac{t^{3\alpha}}{(3!)\alpha^3}. \tag{47}$$

Our new technique offers an approximate solution in a way of rapidly convergent series with simple calculable components which implies that our approach is fast and also efficient by calculating farther more terms of the k-th truncated series of eq. (19).

Remark 4. The solution of eq. (43) has a general form that harmonizes with the following exact solution when $\alpha = 1$, as a special case:

$$\psi(x,t) = -\frac{2e^{\omega(x-\omega^2 t)}\omega^2}{(1+e^{\omega(x-\omega^2 t)})^2}. \tag{48}$$

Application 3. Assume the following space-conformable fractional nonlinear KdVBE:

$$\frac{\partial \psi(x,t)}{\partial t} + \psi(x,t)\frac{\partial^\beta \psi(x,t)}{\partial x^\beta} + \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{\partial^3 \psi(x,t)}{\partial x^3} = 0, \quad 0 < \beta \le 1,\ x,\ t > 0, \tag{49}$$

subject to

$$\psi(x,0) = x^3. \tag{50}$$

Applying the method of RPS, taking $r = \varepsilon = \eta = \mu = \alpha = 1$ and $h_0(x) = x^3$, and using the k-th residual function:

$$\mathrm{Res}_k(x,t) = \frac{\partial \psi_k(x,t)}{\partial t} + \psi_k(x,t)\frac{\partial^\beta \psi_k(x,t)}{\partial x^\beta} + \frac{\partial^2 \psi_k(x,t)}{\partial x^2} + \frac{\partial^3 \psi_k(x,t)}{\partial x^3}, \quad k = 1,2,3. \tag{51}$$

Now, computations by using the same procedure as in sect. 3 and eq. (19) give the following forms for $h_n(x)$, $n = 1,2,3$:

$$\begin{aligned}
h_0(x) &= x^3,\\
h_1(x) &= -3\left(2+2x+x^{6-\beta}\right),\\
h_2(x) &= 3(9-\beta)x^{9-2\beta} + 3\left(38-11\beta+\beta^2\right)x^{4-\beta} + 3\left(126-74\beta+15\beta^2-\beta^3\right)x^{3-\beta},\\
h_3(x) &= -\left(432-108\beta+6\beta^2\right)x^{12-3\beta} - \left(2994-1515\beta+264\beta^2-15\beta^3\right)x^{7-2\beta}\\
&\quad -\left(16092-13572\beta+4230\beta^2-567\beta^3+27\beta^4\right)x^{6-2\beta}\\
&\quad -\left(1440-1194\beta+381\beta^2-54\beta^3+3\beta^4\right)x^{2-\beta}\\
&\quad -\left(5076-6978\beta+3714\beta^2-954\beta^3+120\beta^4-6\beta^5\right)x^{1-\beta}\\
&\quad -\left(2268-5490\beta+4980\beta^2-2223\beta^3+525\beta^4-63\beta^5+3\beta^6\right)x^{-\beta},\\
&\ \ \vdots
\end{aligned} \tag{52}$$

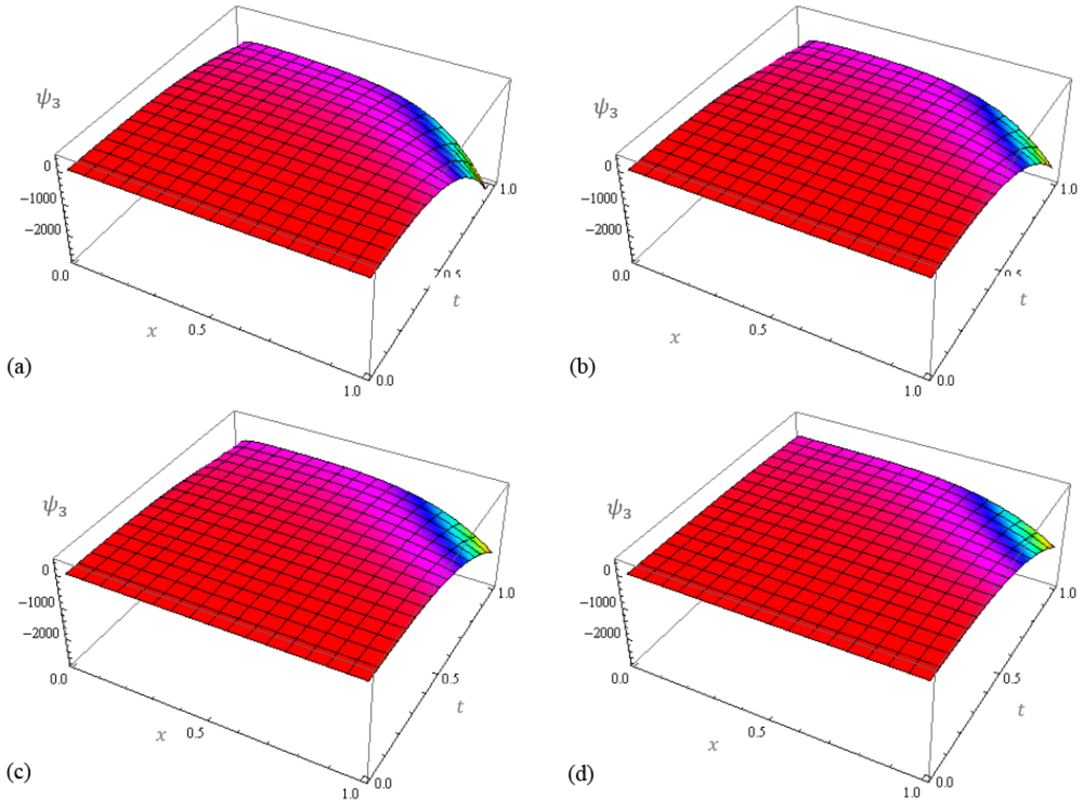

Fig. 3. The graph of the approximate solution $\psi_3(x,t)$ of eq. (49) is given in (53) when: (a) $\beta = 0.3$, (b) $\beta = 0.6$, (c) $\beta = 0.9$, (d) $\beta = 1$.

Consequently, the 3rd approximate of the RPS solution of eqs. (49) and (50) is given as follows:

$$\begin{aligned}\psi_3(x,t) = {}& x^3 - 3\left(2+2x+x^{6-\beta}\right)t + \left(3(9-\beta)x^{9-2\beta} + 3(38-11\beta+\beta^2)x^{4-\beta} + 3\left(126-74\beta+15\beta^2-\beta^3\right)x^{3-\beta}\right)\frac{t^2}{2!}\\ &- \left(\left(432-108\beta+6\beta^2\right)x^{12-3\beta} + \left(2994-1515\beta+264\beta^2-15\beta^3\right)x^{7-2\beta}\right.\\ &+ \left(16092-13572\beta+4230\beta^2-567\beta^3+27\beta^4\right)x^{6-2\beta} + \left(1440-1194\beta+381\beta^2-54\beta^3+3\beta^4\right)x^{2-\beta}\\ &+ \left(5076-6978\beta+3714\beta^2-954\beta^3+120\beta^4-6\beta^5\right)x^{1-\beta}\\ &\left.+ \left(2268-5490\beta+4980\beta^2-2223\beta^3+525\beta^4-63\beta^5+3\beta^6\right)x^{-\beta}\right)\frac{t^3}{3!}\,. \end{aligned} \tag{53}$$

From eq. (49), it is clear that, $\psi_3(x,t) \to -\infty$ as $x \to \infty$ or $x \to 0^+$, for $t > 0$. The simulation of the solution of eq. (49) on $(0,1] \times [0,1]$ in 3-dimensional space are given in the figures below. Figure 3 shows the approximate solution $\psi_3(x,t)$ on the domain $(0,1] \times [0,1]$ for different values of $\beta = 0.3$, $\beta = 0.6$, $\beta = 0.9$, and $\beta = 1$.

We noted from figs. 3 that the surface graph solutions representations decrease steadily as the value of t and x increases steadily on the definite domain, while all surfaces almost agree well in their behavior. Moreover, the solutions are constantly depending on the space-conformable fractional derivative.

The next solution is the 3rd approximate solution of eqs. (49) and (50), where the fractional derivative in the Caputo sense, as mentioned in [18,35,37] is:

$$\begin{aligned}\psi_3(x,t) = {}& x^3 - \left(6+6x+c_1x^{6-\beta}\right)\frac{t}{1!} + \left(c_2x^{9-2\beta}+c_3x^{4-\beta}+c_4x^{3-\beta}\right)\frac{t^2}{2!}\\ &- \left(c_5x^{12-3\beta}+c_6x^{7-2\beta}+c_7x^{6-2\beta}+c_8x^{2-\beta}+c_9x^{1-\beta}+c_{10}x^{-\beta}\right)\frac{t^3}{3!}\,, \end{aligned} \tag{54}$$

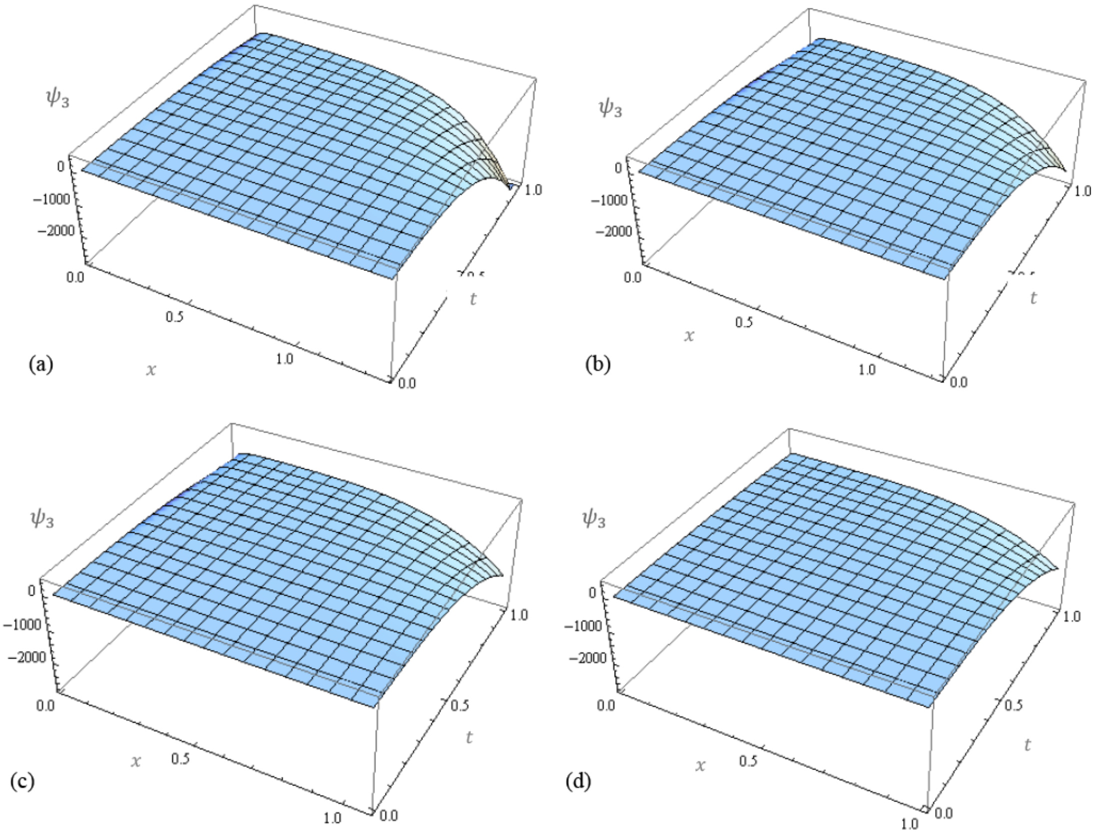

Fig. 4. The graph of the approximate solution $\psi_3(x,t)$ of eq. (49) and (50) is given in (54) when: (a) $\beta = 0.3$, (b) $\beta = 0.6$, (c) $\beta = 0.9$, (d) $\beta = 1$.

where

$$\begin{aligned}
&c_1 = \frac{6}{\Gamma(4-\beta)}\,, \qquad c_2 = c_1\frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)} + c_1^2, \qquad c_3 = 6c_1 + \frac{6}{\Gamma(2-\beta)} + c_1(6-\beta)(5-\beta),\\
&c_4 = 6c_1 + c_1(6-\beta)(5-\beta)(4-\beta), \qquad c_5 = c_2\frac{\Gamma(10-2\beta)}{\Gamma(10-3\beta)} + c_1c_2 + 2c_1\left(c_2 - c_1^2\right),\\
&c_6 = c_3\frac{\Gamma(5-\beta)}{\Gamma(5-2\beta)} + c_1\frac{12}{\Gamma(2-\beta)} + 12(c_2 - c_1^2) + c_1c_3 + (8-2\beta)(9-2\beta)c_2,\\
&c_7 = c_4\frac{\Gamma(4-\beta)}{\Gamma(4-2\beta)} + 12(-c_1^2 + c_2) + c_1c_4 + (7-2\beta)(8-2\beta)(9-2\beta)c_2,\\
&c_8 = \frac{72}{\Gamma(2-\beta)} + (3-\beta)(4-\beta)c_3, \qquad c_9 = \frac{72}{\Gamma(2-\beta)} + (4-\beta)(3-\beta)(2-\beta)c_3 + (2-\beta)(3-\beta)c_4,\\
&c_{10} = (1-\beta)(2-\beta)(3-\beta)c_4.
\end{aligned} \tag{55}$$

The solution as in eq. (54) is obviously more complex than the solution in eq. (53), despite the similarity in generic terms, but the difference in the two solutions appears in the coefficients. In spite of all this, figs. 3 and 4 show that the graph representation of the solution in eq. (54) is similar, and can even be said to match the representation of the approximate solution in eq. (53). This is another example where it is clear that there is no significant difference between the behavior of the surface graph of the KdVBE solution in case the fractional derivative is considered in the sense of Caputo or in the sense of conformable derivative. Therefore, we can say that the CFD is a suitable tool for modeling the KdVBE.

Application 4. Assume the following space-conformable fractional nonlinear KdVBE:

$$\frac{\partial\psi(x,t)}{\partial t} + \psi^2(x,t)\frac{\partial^\beta\psi(x,t)}{\partial x^\beta} + \frac{\partial^3\psi(x,t)}{\partial x^3} = 0, \quad x,\ t>0,\ 0<\beta\le 1, \tag{56}$$

subject to

$$\psi(x,0) = x^2. \tag{57}$$

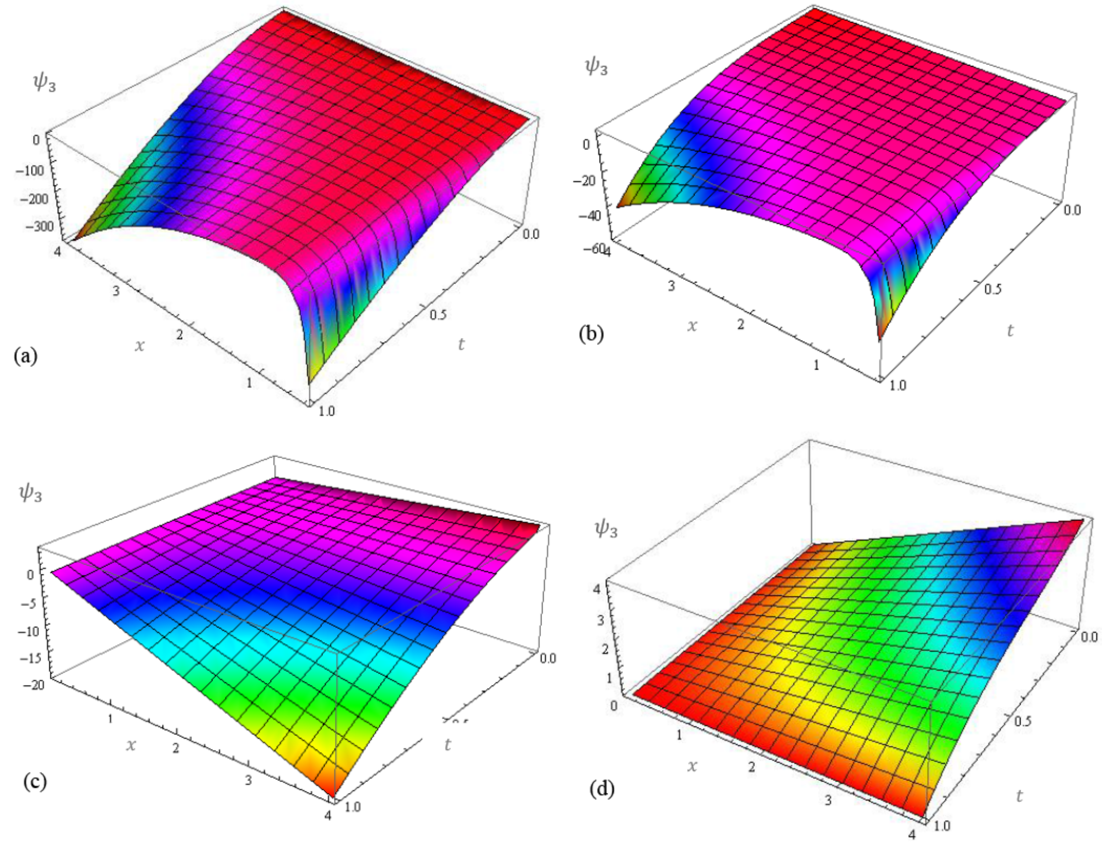

Fig. 5. The graph of the approximate solution $\psi_3(x,t)$ of eq. (61) when: (a) $(\alpha,\beta)=(0.5,0.5)$, (b) $(\alpha,\beta)=(1,0.5)$ (c) $(\alpha,\beta)=(0.5,1)$, (d) $(\alpha,\beta)=(1,1)$.

Applying the method of RPS, taking $r=2$, $\varepsilon=\mu=\alpha=1$, $\eta=0$ and initial function $f_0(x)=x^2$, with the k-th residual function:

$$\mathrm{Res}_k(x,t)=\frac{\partial\psi_k(x,t)}{\partial t}+\psi_k^2(x,t)\frac{\partial^\beta\psi_k(x,t)}{\partial x^\beta}+\frac{\partial^3\psi_k(x,t)}{\partial x^3},\quad k=1,2,3. \tag{58}$$

Now by using the same process as in sect. 3 and eq. (19), computations give the subsequent $h_n(x)$, $n=1,2,3$ forms:

$$\begin{aligned}
&h_0(x)=x^2,\\
&h_1(x)=-2x^{6-\beta},\\
&h_2(x)=-2\left(x^{10-2\beta}(-10+\beta)+x^{3-\beta}\left(-120+74\beta-15\beta^2+\beta^3\right)\right),\\
&h_3(x)=-2(2x^{14-3\beta}(98-21\beta+\beta^2)+3x^{7-2\beta}(2680-2066\beta+581\beta^2-70\beta^3+3\beta^4)\\
&\qquad+x^{-\beta}\left(720-1764\beta+1624\beta^2-735\beta^3+175\beta^4-21\beta^5+\beta^6\right)),\\
&\qquad\vdots
\end{aligned} \tag{59}$$

Consequently, the 3rd approximate of the RPS solution of eqs. (56) and (57) has the following expansion:

$$\begin{aligned}
\psi_3(x,t)&=x^2-2x^{6-\beta}t-2\left(x^{10-2\beta}(-10+\beta)+x^{3-\beta}(-120+74\beta-15\beta^2+\beta^3)\right)\frac{t^2}{2!}-2\left(2x^{14-3\beta}(98-21\beta+\beta^2)\right.\\
&\left.+3x^{7-2\beta}(2680-2066\beta+581\beta^2-70\beta^3+3\beta^4)+x^{-\beta}\left(720-1764\beta+1624\beta^2-735\beta^3+175\beta^4-21\beta^5+\beta^6\right)\right)\frac{t^3}{3!}.
\end{aligned} \tag{60}$$

Application 5. Assume the following time-space-conformable fractional nonlinear KdVBE:

$$\frac{\partial^\alpha\psi(x,t)}{\partial t^\alpha}+\psi(x,t)\frac{\partial^\beta\psi(x,t)}{\partial x^\beta}+\frac{\partial^2\psi(x,t)}{\partial x^2}+\frac{\partial^3\psi(x,t)}{\partial x^3}=0,\quad 0<\alpha,\ \beta\le 1,\ x,\ t>0, \tag{61}$$

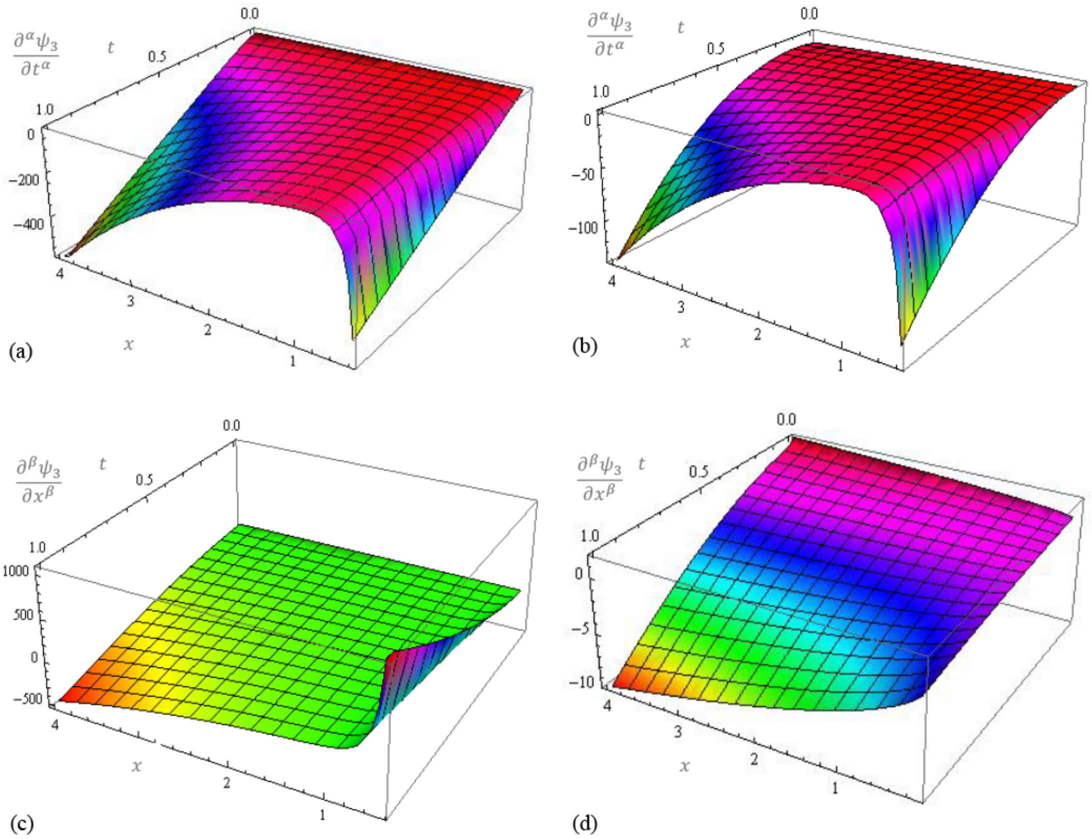

Fig. 6. The graph of the time-space-CFD of the approximate solution $\psi_3(x,t)$ of eq. (61): (a) $\frac{\partial^\alpha \psi_3(x,t)}{\partial t^\alpha}$ when $(\alpha,\beta)=(0.5,0.5)$, (b) $\frac{\partial^\alpha \psi_3(x,t)}{\partial t^\alpha}$ when $(\alpha,\beta)=(1,0.5)$ (c) $\frac{\partial^\beta \psi_3(x,t)}{\partial x^\beta}$ when $(\alpha,\beta)=(0.5,0.5)$, (d) $\frac{\partial^\beta \psi_3(x,t)}{\partial x^\beta}$ when $(\alpha,\beta)=(0.5,1)$.

subject to

$$\psi(x,0)=x. \tag{62}$$

Applying the method of RPS, taking $\varepsilon=1$, $\eta=1$, $\mu=1$, $r=1$, $0<\alpha\le 1$, $0<\beta\le 1$ and the initial function $f_0(x)=x$, with the k-th residual function:

$$\mathrm{Res}_k(x,t)=\frac{\partial^\alpha \psi_k(x,t)}{\partial t^\alpha}+\psi_k(x,t)\frac{\partial^\beta \psi_k(x,t)}{\partial x^\beta}+\frac{\partial^2 \psi_k(x,t)}{\partial x^2}+\frac{\partial^3 \psi_k(x,t)}{\partial x^3}\,,\quad k=1,2,3. \tag{63}$$

According to the process in the previous section and eq. (19), some simple computations give the subsequent $h_n(x)$, $n=1,2,3$ forms:

$$\begin{aligned}
h_0(x)&=x,\\
h_1(x)&=-x^{2-\beta},\\
h_2(x)&=(-x^{3-2\beta}(-3+\beta)+x^{-\beta}(2-3\beta+\beta^2)-x^{-1-\beta}\beta(2-3\beta+\beta^2)),\\
h_3(x)&=\left(-2x^{4-3\beta}\left(8-6\beta+\beta^2\right)-x^{-2-\beta}\beta\left(2-\beta-2\beta^2+\beta^3\right)\right.\\
&\quad+x^{1-2\beta}\left(-20+41\beta-26\beta^2+5\beta^3\right)+2x^{-3-\beta}\beta\left(4-5\beta^2+\beta^4\right)\\
&\quad-3x^{-2\beta}\left(6-24\beta+32\beta^2-17\beta^3+3\beta^4\right)\\
&\quad\left.-x^{-4-\beta}\beta\left(12+4\beta-15\beta^2-5\beta^3+3\beta^4+\beta^5\right)\right),\\
&\ \ \vdots
\end{aligned} \tag{64}$$

Now, from eq. (19) select the first three terms with initial guess approximation $\psi_0(x,t) = h_0(x)$. Thus, the approximate solution of the 3rd approximate of the RPS solution of eqs. (61) and (62) agree well with the next expansion:

$$\begin{aligned}\psi_3(x,t) = {} & x - x^{2-\beta}\frac{t^\alpha}{\alpha} + (-x^{3-2\beta}(-3+\beta) + x^{-\beta}(2-3\beta+\beta^2) - x^{-1-\beta}\beta(2-3\beta+\beta^2))\frac{t^{2\alpha}}{2!\alpha^2}\\ & + \big(-2x^{4-3\beta}(8-6\beta+\beta^2) - x^{-2-\beta}\beta(2-\beta-2\beta^2+\beta^3) + x^{1-2\beta}(-20+41\beta-26\beta^2\\ & + 5\beta^3) + 2x^{-3-\beta}\beta(4-5\beta^2+\beta^4) - 3x^{-2\beta}(6-24\beta+32\beta^2-17\beta^3+3\beta^4)\\ & - x^{-4-\beta}\beta(12+4\beta-15\beta^2-5\beta^3+3\beta^4+\beta^5)\big)\frac{t^{3\alpha}}{3!\alpha^3}\,.\\ & \ddots \end{aligned} \tag{65}$$

Our goal here is to illustrate the behavior of the acquired estimate solution of eq. (61) geometrical and its time-space-CFD. To achieve that, the approximate solution $\psi_3(x,t)$ surface graph is plotted for $(\alpha,\beta) = (0.5, 0.5)$, $(\alpha,\beta) = (1, 0.5)$, $(\alpha,\beta) = (0.5, 1)$, and $(\alpha,\beta) = (1, 1)$ as given in fig. 5, whereas fig. 6 exhibits the surface graph of the time-CFD $\frac{\partial^\alpha \psi_3(x,t)}{\partial t^\alpha}$ and the space-CFD $\frac{\partial^\beta \psi_3(x,t)}{\partial x^\beta}$ when $(\alpha,\beta) = (0.5, 0.5)$, $(\alpha,\beta) = (1, 0.5)$, respectively, on the domain $[0.45, 4] \times [0, 1]$.

5 Conclusion

The fractional KdVBE is formulated by using the conformable fractional concept. An approximate solution of conformable fractional KdVBE is presented by using the RPS method and some important and interesting applications with graphical results are also given and discussed. The comparisons between the approximate solutions for KdVBEs in the case of conformable and Caputo derivatives show that thebehaviors of the surface graph in both cases are similar, but in the CFD case, the surface graph is smooth on all its domain compared with the surface graph in Caputo's case. Moreover, the solution in the case of CFD was much simpler than in the case of the Caputo derivative. Therefore, the CFD is a good alternative tool to model KdVBEs instead of the Caputo and other fractional derivatives. Finally, computations show that the RPS method for solving the conformable fractional KdV-Burgers equation does not have mathematical requirements which implies that it is very effective for predicting solutions and more flexible in choosing the initial guesses approximations. How to extend our proposed method and the utilization of a conformable fractional derivative for modeling some real-life problems such as physical phenomena and engineering problems still needs further researches.

References

1. L. Wang, X. Che, Entropy **17**, 6519 (2015).
2. S. Rida, A. El-Sayed, A. Arafa, Commun. Nonlinear Sci. **15**, 3847 (2010).
3. S. Kumar, A. Yildirim, Y. Khan, H. Jafari, K. Sayevand, L. Wei, J. Fract. Calc. Appl. **2**, 1 (2012).
4. D.J. Kordeweg, G. de Vries, Philos. Mag. **39**, 422 (1895).
5. C.H. Suand, C.S. Gardner, J. Math. Phys. **10**, 536 (1969).
6. K. Maleknejad, E. Babolian, A.J. Shaerlar, M. Jahnagir, Aust. J. Basic Appl. Sci. **5**, 602 (2011).
7. Q. Wang, Chaos, Solitons Fractals **35**, 843 (2008).
8. A.K. Golmankhaneh, D. Baleanu, Rom. Rep. Phys. **63**, 609 (2011).
9. A.H. Bhrawy, M.A. Zakyand, D. Baleanu, Rom. Rep. Phys. **67**, 340 (2015).
10. M. Wang, Phys. Lett. A **213**, 279 (1996).
11. M.M. Hassan, Chaos, Solitons Fractals **19**, 1201 (2004).
12. A.A. Soliman, Chaos, Solitons Fractals **41**, 1034 (2009).
13. A.A. Soliman, Chaos, Solitons Fractals **29**, 294 (2006).
14. D. Kaya, Appl. Math. Comput. **152**, 279 (2004).
15. S. Momani, Chaos, Solitons Fractals **28**, 930 (2006).
16. C. Rong-Jun, C. Yu-Min, Chin. Phys. B **20**, 70206 (2011).
17. H.N.A. Ismail, T.M. Rageh, G.S.E. Salem, Appl. Math. Comput. **234**, 58 (2014).
18. A. El-Ajou, Q. Abu Arqub, S. Momani, J. Comput. Phys. **293**, 81 (2015).
19. A. El-Ajou, O. Abu Arqub, Z. Al-Zhour, S. Momani, Entropy **15**, 5305 (2013).
20. O. Abu Arqub, A. El-Ajou, Z. Al-Zhour, S. Momani, Entropy **16**, 471 (2014).

21. O. Abu Arqub, A. El-Ajou, S. Momani, J. Comput. Phys. **293**, 385 (2015).
22. A. El-Ajou, O. Abu Arqub, S. Momani, Appl. Math. Comput. **257**, 119 (2015).
23. Abu Arqub, J. Adv. Res. Appl. Math. **5**, 31 (2013).
24. A. El-Ajou, O. Abu Arqub, S. Momani, Appl. Math. Comput. **256**, 851 (2015).
25. Y. Zhang, A. Kumar, S. Kumar, D. Baleanu, X.J. Yang, J. Nonlinear Sci. Appl. **9**, 5821 (2016).
26. A. Kumar, S. Kumar, S.P. Yan, Fundam. Inform. **151**, 213 (2017).
27. A. Kumar, S. Kumar, Nonlinear Eng. **5**, 235 (2016).
28. S. Kumar, A. Kumar, D. Baleanu, Nonlinear Dyn. **85**, 699 (2016).
29. V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems* (Cambridge Academic, Cambridge, 2009).
30. B. West, M. Bologna, P. Grigolini, *Physics of Fractal Operators* (Springer, New York, 2003).
31. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, Yverdon, 1993).
32. M. Inc, J. Math. Anal. Appl. **345**, 476 (2008).
33. S. Momani, Z. Odibat, A. Alawneh, Numer. Methods Partial Differ. Equ. **24**, 262 (2008).
34. M. Safari, D.D. Ganji, M. Moslemi, Comput. Math. Appl. **58**, 2091 (2009).
35. S. Momani, Math. Comput. Simul. **70**, 110 (2005).
36. Q. Wang, Appl. Math. Comput. **190**, 1795 (2007).
37. L. Song, H. Zhang, Phys. Lett. A **367**, 88 (2007).
38. R. Khalil, M.A. Horani, A. Yousef, M. Sababheh, J. Comput. Appl. Math. **264**, 65 (2014).
39. T. Abdeljawad, J. Comput. Appl. Math. **279**, 57 (2015).
40. M. Abu Hammad, R. Khalil, Inter. J. Pure Appl. Math. **94**, 215 (2014).
41. Z. Ayati, J. Biazar, M. Ilei, Int. J. Appl. Math. Res. **6**, 49 (2017).
42. L. Wang, J. Fu, Chin. Phy. B **25**, 4501 (2016).