Eur. Phys. J. Plus (2019) **134**: 303
DOI 10.1140/epjp/i2019-12679-9

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Solitons and lump wave solutions to the graphene thermophoretic motion system with a variable heat transmission

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Received: 24 February 2019 / Revised: 26 March 2019
Published online: 27 June 2019

Abstract. Graphene is one of the thinnest and hardest elastic nanoscale materials and has opened new horizons in the field of material science for its versatile applications. The thermophoretic motion system is under investigation which describes the propagation of solitons in substrate-supported graphene sheets. A test function of the extended three-soliton method is used to construct the new soliton solutions. The one-order and mixed-order solutions involving solitons and lump waves are constructed. The dynamical behaviour of solitons under reflection, periodic distribution and interaction is depicted. Moreover, the bright and mixed type lump wave soliton propagation and interaction are discussed.

1 Introduction

Graphene conatins a layered structure consisting of a two-dimensional honeycomb lattice. It is considered to be the best nanoscale material possessing very promising mechanical, thermal, magnetic, optical, and electronic properties. Graphene has opened new horizons in the field of material science and has been reported for versatile industrial applications [1–3]. Keeping in view its tremendous applications, it is believed that new features on graphene materials will be developed by theoretical considerations and experimentation.

Thermophoresis is an important phenomenon, in which a body immersed in a fluid experiencing a force, independent of convection, drifts from hot to cold [4]. This phenomenon is being reported as a novel technique for manipulation of nano particles. Materials such as graphene with good thermal conductivity and low surface friction are the best candidates for solid-solid transportations or manipulations. In this study, we employ nonequilibrium molecular dynamics simulations to explore the feasibility of utilizing a thermal gradient on a large graphene substrate to control the motion of a small graphene nanoflake on it. Recently, a lot of interests has been observed for methods to control nanoscale transport and manipulation [5]. Wrinkle-like waves may be formed by inductivities of many ripples, chemical functional groups, defects, or mechanical strains. In contrast to the ripples, wrinkle-like waves are directional, self-similar and hyperboloidal strips which go through the whole single-layered graphene sheets [6,7]. To the best of our knowledge, up to now little attention has been paid to the research of wrinkle-like soliton interactions in graphene sheets. Most of previous works have focused on the numerical analysis of wrinkle-like soliton dynamics by molecule dynamics simulation. Hence, in this article, the exact soliton and lump wave solutions are discussed analytically. For this, a test function of extended three soliton method is used to construct the new solitons. The one order and mixed order lump wave solutions are also constructed. The dynamical behaviour of solitons under reflection, periodic distribution and

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interaction is discussed. The system [8] with a variable heat transmission for a greaphene thermophoretic motion reads

$$(u_t + uu_x + u_{xxx}) + [\alpha(t) + b - C_w]u_x = 0, \tag{1}$$

where u is a thermophoretic moving variable, x and t are the independent variable stand for displacement and time, respectively, whereas α and b are system thermal conductive coefficients, and C_w is a system parameter adjusted to u. In the growing scientific world the study of soliton and lump waves has received great attention of researchers in the field [9–16]. In recent years, many forms of lump waves and new exact solutions have been studied. The study of linear and nonlinear partial differential equations with lump wave and new exact solutions is a very hot topic of research [17–21]. Further, this technique leads to obtaining some new exact wave solutions. The transformation, where u_0 is an arbitrary constant and $F(x,t)$ is a real valued function

$$u = u_0 + 12[\ \ln F(x,t)\ ]_{xx}, \tag{2}$$

connected with the following bilinear operator [22,23],

$$D_x^m D_t^n\Big(f(x,t)\cdot g(x,t)\Big) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)\cdot g(x',t')|_{t=t',x=x'}, \tag{3}$$

is given by

$$\Big(D_x D_t + D_x^4 + [u_o + \alpha(t) + b - C_w]D_x^2\Big)F\cdot F = 0. \tag{4}$$

After simplification and using the billinear derivative operators defined by eq. (3), eq. (4) is written as

$$F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + [u_o + \alpha(t) + b - C_w]F_{xx}F - [u_o + \alpha(t) + b - C_w]F_x^2 = 0. \tag{5}$$

In the following section, the exact soliton solutions are discussed.

2 Exact soliton solutions

For the analytical study of eq. (1), the novel test function for extended three soliton method is used in this case. Thus, we have

$$F(x,t) = \beta_1\cos\xi_1 + \beta_2\cosh\xi_2 + \exp(-\xi_3) + \beta_3\exp(\xi_3), \tag{6}$$

where $\xi_i = k_i x + w_i t$ for $i = 1,2,3$ and for the amplitude of the soliton of the ith position, k_i is the real parameter, the wave speed is represented by w_i and β_i are taken as arbitrary constants. From eqs. (5) and (6), one can get a system of equations after comparing the different coefficients of all the power of $\sin\xi_1\exp(\pm\xi_3)$, $\sin\xi_1\sinh\xi_2$, $\cos\xi_1\exp(\pm\xi_3)$, $\cos\xi_1\cosh\xi_2$, $\sinh\xi_2\exp(\pm\xi_3)$, $\cosh\xi_2\exp(\pm\xi_3)$ and $\exp(0)$. This system of different algebraic equations is obtained for β_i, k_i, w_i, for $(i = 1,2,3)$. After solving this system, we have the following different exact soliton solutions.

Case I. For $\beta_1 = 0$, $\beta_2 = 0$, $\omega_3 = -4k_3^3 - k_3[u_o + \alpha(t) + b - C_w]$ and β_3, k_3 are free parameters. We put eq. (6) into eq. (5) with eq. (2), then the bright soliton solution of eq. (1) can be obtained as

$$u_1 = u_o + 12k_3^2\,\mathrm{sech}^2(k_3x - (4k_3^3 + k_3[u_o + \alpha(t) + b - C_w])t + \ln\sqrt{\beta_3}),$$

where $\xi_3 = k_3x - (4k_3^2 + k_3[u_o + \alpha(t) + b - C_w])t$, β_3, k_3 are arbitrary constants; K_3 is the wave number of the x-direction and the wave speed is represented as $w_3 = -(4k_3^4 + k_3[u_o + \alpha(t) + b - C_w])t$. Figures 1(a) and (b) show the propagation of the soliton with periodic distribution against the $(x,\,t)$-axes, where the soliton waves are controlled by coefficient $\alpha(t)$ using the value $\alpha(t) = \cos(t)$. In figs. 1(c) and (d), instead, the soliton propagation under reflection is illustrated. In this case, soliton waves are controlled by the coefficient $\alpha(t)$ with value $\alpha(t) = \mathrm{sn}(t, 0.5) + 5\tanh(t)$. Moreover, after reflection, solitons change their directions at $t = 0$.

Case II. The coefficients β_2, k_1, k_2, k_3, w_1 are free parameters which are associated with the amplitudes and wave speed of the soliton. When $\beta_1 = 0$, $\beta_3 = \frac{1}{4}\frac{\beta_2^2k_2^2}{k_3^2}$, $w_2 = -3k_2k_3^2 - k_2[u_o + \alpha(t) + b - C_w] - k_2^3$ and $w_3 = -k_3(k_3^2 + [u_o + \alpha(t) + b - C_w] + 3k_2^2)$, by putting eq. (6) into eq. (5) with eq. (2), we obtain the solution of eq. (1) as follows:

$$u_2 = u_o + \frac{12\beta_2\cosh(\xi_2)[B(k_2^2 + k_3^2) + \beta_3(k_2^2 + k_3^2)^2A] + 4\beta_3k_3^2}{[\beta_2\cosh(\xi_2x) + B + \beta_3A]^2} + \frac{\beta_2^2k_2^2 + 2\beta_2\sinh(\xi_2)k_2k_3[B - \beta_3A]}{[\beta_2\cosh(\xi_2) + B + \beta_3A]^2},$$

where $A = \cosh(\xi_3) + \sinh(\xi_3)$ and $B = \cosh(\xi_3) - \sinh(\xi_3)$.

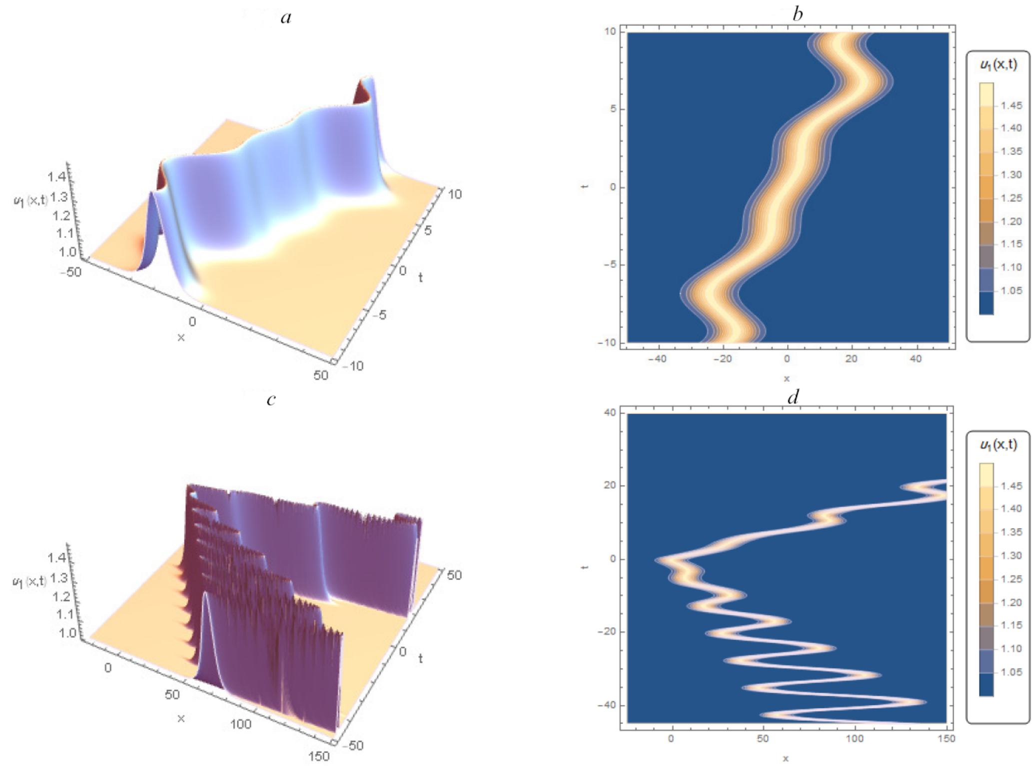

Fig. 1. 3D plots and contourplots for soliton propagation of $u_1(x,t)$ along the t-axis. Panels (a) and (b): $\beta_3 = 1.2$, $b = 2$, $k_3 = 0.2$, $u_0 = 1$, $C_w = 0.5$, and $\alpha(t) = \cos(t)$. Panels (c) and (d): the same as in (a) and (b), except for $\alpha(t) = \mathrm{sn}(t, 0.5) + 5\tanh(t)$.

Case III. The coefficients β_1, k_1, k_2, k_3, w_2 are free parameters and $\beta_2 = 0$. We also have the following values of β_3, w_1 and w_3:

$$\beta_3 = \frac{-1}{4}\,\frac{\beta_1^2 k_1^2(3k_3^4 + 3k_1^4 + 2[u_o + \alpha(t) + b - C_w][k_2k_3 - k_3^2] + 6k_1^2k_3^2)}{k_3(-2k_1^2k_2[u_o + \alpha(t) + b - C_w] + 2[u_o + \alpha(t) + b - C_w]k_1^2k_3 + 3k_3^5 + 6k_1^2k_3^2 + 3k_1^4k_3)},$$

$$w_1 = \frac{-k_1(2k_1^2k_3^2 - k_1^4 + 3k_3^4 - [u_o + \alpha(t) + b - C_w][k_3^2 - k_1^2 - 2k_2k_3])}{k_3^2 + k_1^2},$$

$$w_3 = \frac{-(-2k_1^2K_3^3 - 3k_1^4k_3 + k_3^5 + [u_o + \alpha(t) + b - C_w][2k_1^2k_2 + k_3^3 - k_1^2k_3])}{k_3^2 + k_1^2}.$$

Then by putting eq. (6) into eq. (5) with eq. (2), we obtain the lump wave solution of soliton as follows:

$$u_3 = u_o + 12\frac{(-\beta_1\cos(\xi_1)k_1^2 + k_3^2B + \beta_3k_3^2A) - (-\beta_1\sin(\xi_1)k_1 - k_3B + \beta_3k_3A)^2}{\beta_1\cos(\xi_1) + B + \beta_3A},$$

where $A = \cosh(\xi_3) + \sinh(\xi_3)$ and $B = \cosh(\xi_3) - \sinh(\xi_3)$. Figures 2(a) and (b) show the propagation of lump wave solitons with periodic distribution against the (x, t)-axes; in this case soliton waves are controlled by the coefficient $\alpha(t)$ using the value $\cos(t)$.

Case IV. Another solution is also obtained for β_2, k_1, k_2, k_3, w_1, w_2, w_3 as free parameters and $\beta_1 = 0$ along with

$$\beta_3 = -\frac{\beta_2^2(k_2w_2 + 4k_2^4 + [u_o + \alpha(t) + b - C_w]k_2^2)}{4(k_3w_3 + 4k_3^4 + [u_o + \alpha(t) + b - C_w]k_3^2)}.$$

Thus, the substitution of eq. (6) into eq. (5) yields another solution of eq. (2) as follows:

$$u_4 = u_o + 12\left(\frac{\beta_2^2k_2^2 + 4\beta_3k_3^2 + 4\beta_2\beta_3\cosh(\xi_2)(k_2^2 + k_3^2)\cosh^2(\xi_3 + \ln\sqrt{\beta_3})}{[\beta_2\cosh(\xi_2) + 4\beta_3\cosh^2(\xi_3 + \ln\sqrt{\beta_3})]^2}\right.$$
$$\left. + \frac{48\beta_2\beta_3k_2k_3\sinh(\xi_2)(2\cosh^2(\xi_3 + \ln\sqrt{\beta_3}) - \exp(\xi_3))}{[\beta_2\cosh(\xi_2) + 4\beta_3\cosh^2(\xi_3 + \ln\sqrt{\beta_3})]^2}\right).$$

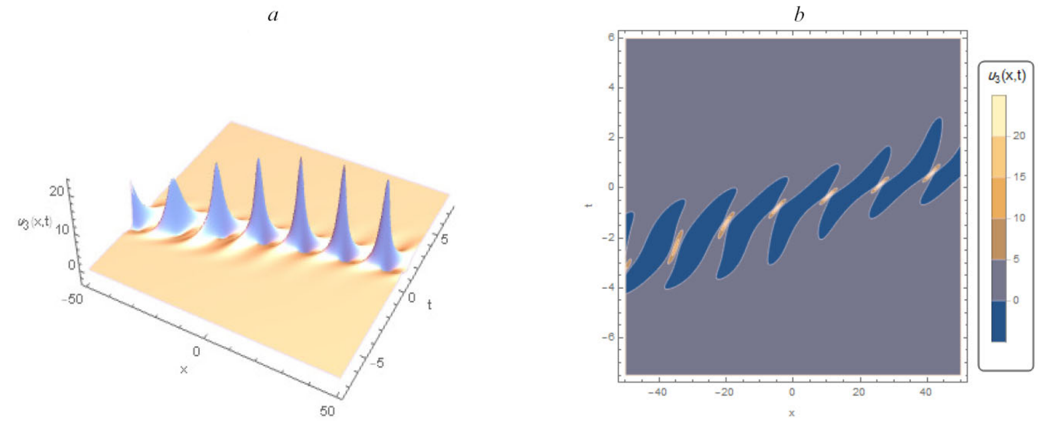

Fig. 2. 3D plot and contourplot for the lump soliton solution of $u_3(x,t)$ for parameters $k_1 = 0.5$, $k_2 = 0.4$, $k_3 = 0.1$, $\beta_1 = -0.2$, $b = 0.2$, $u_0 = 0.1$, $C_w = -4$, and $\alpha(t) = \cos(t)$.

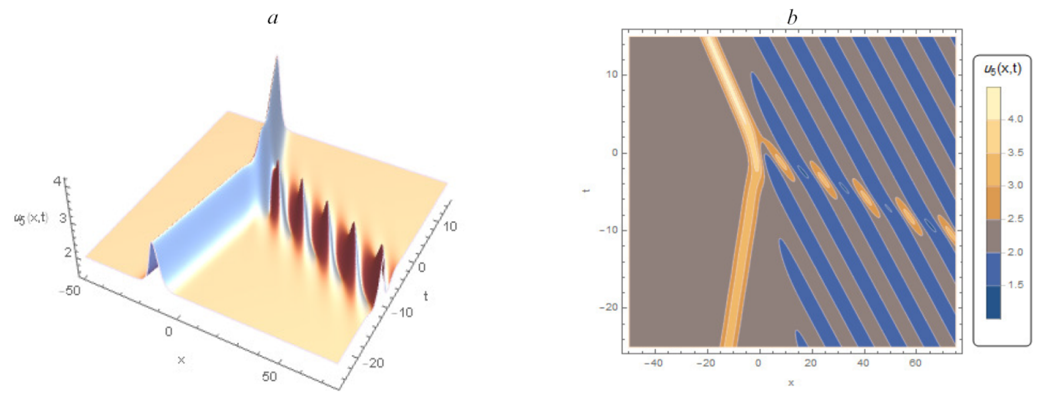

Fig. 3. 3D plot and contourplot for the lump soliton solution of $u_5(x,t)$ for parameters $w_1 = 0.8$, $w_2 = 0.7$, $w_3 = 0.4$, $k_1 = 0.5$, $k_2 = 0.1$, $k_3 = 0.75$, $\beta_1 = 0.7$, $b = 4$, $u_0 = 2$, $C_w = -4$, and $\alpha(t) = \cos(t)$.

Case V. Another solution is obtained after substituting eq. (6) into eq. (4), which yields the solution of eq. (2). Thus, we have

$$u_5 = u_o + 12\frac{-\beta_1^2 k_1^2 + \beta_2^2 k_2^2 - \beta_1 \cos(\xi_1) M}{(\beta_1 \cos\xi_1 + \beta_2 \cosh\xi_2 + \exp(-\xi_3))^2}$$
$$+ 12\frac{\beta_2 \cosh\xi_2 \exp(-\xi_3)(k_2^2 + k_3^2) + 2\beta_1\beta_2 k_1 k_2 \sin\xi_1 \sinh\xi_2 - 2k_3 \exp(-\xi_3) N}{(\beta_1 \cos\xi_1 + \beta_2 \cosh\xi_2 + \exp(-\xi_3))^2},$$

where we suppose $M = \beta_2 \cosh\xi_2(k_1^2 - k_2^2) + \exp(-\xi_3)(k_1^2 - k_3^2)$, $N = \beta_1 k_1 \sin\xi_1 - \beta_2 k_2 \sinh\xi_2$ and β_1, k_1, k_2, k_3, w_1, w_2, w_3 are free parameters. The values of coefficients $\beta_3 = 0$, and

$$\beta_2 = \beta_1 \sqrt{\frac{k_1 w_1 - 4k_1^4 + [u_o + \alpha(t) + b - C_w]k_1^2}{k_2 w_2 + 4k_2^4 + [u_o + \alpha(t) + b - C_w]k_2^2}}.$$

Figures 3(a) and (b) show the dynamical behaviour of the interaction of bright solitons and lump solitons. Also, they show that interactions are not completely elastic (or inelastic) and almost show a fusion.

3 Mixed lump solutions

In this section, the one-order and mixed-order lump wave solutions are constructed. To construct the one-order lump wave solution, we apply the following transformation to eq. (5):

$$F = h_o + \xi_1^2 + \xi_2^2 + \xi_3^2, \quad \text{where } \xi_i = k_i x + w_i t, \quad \text{for } i = 1, 2, 3. \tag{7}$$

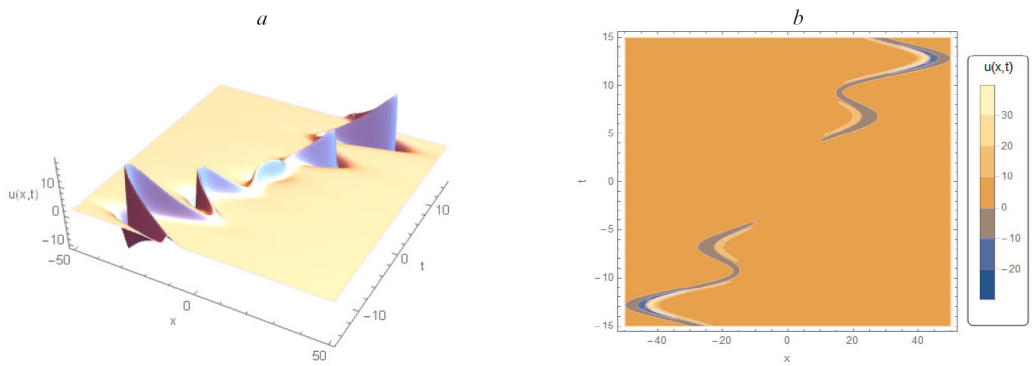

Fig. 4. 3D plot and contourplot for the solution $u(x,t)$ of eq. (8) for parameters $k_2 = 0.3$, $k_3 = 0.4$, $h_1 = 0.9$, $h_2 = 0.8$, $b = 0.5$, $u_0 = 2$, $C_w = 0.1$ and $\alpha(t) = \cos(t)$.

Putting all values in and collecting the terms with the same power of ξ_1^2, ξ_2^2, ξ_3^2, $\xi_1\xi_2$, $\xi_1\xi_3$, $\xi_2\xi_3$ and $\xi_1^0\xi_2^0$ and letting their coefficients be zero, we get a set of algebraic equations. After solving the set of algebraic equations using the computation for k_1, k_2, w_1, w_2, h_o, one can get the following values of constants for lump wave solution:

$$\begin{aligned}
&[u_o + \alpha(t) + b - C_w] = \frac{-w_3}{k_3}\,,\\
&h_o = \frac{6k_3(k_1^2 + k_2^2 + k_3^2)}{w_3(k_3^2 - 1 + k_1^2 + k_2^2)}\,,\\
&k_1 = k_1, \quad k_2 = k_2,\\
&w_1 = \frac{k_1 w_3}{k_3}\,, \quad w_2 = \frac{k_2 w_3}{k_3}\,,
\end{aligned}$$

where $k_3, h_1, h_2, h_3, w_3, h_3 \neq 0$, $w_3 \neq 0$ are constants. Thus the one-order lump wave solution is obtained:

$$u(x,t) = u_o + \frac{G_1}{H_1^2}\,, \tag{8}$$

where

$$\begin{aligned}
G_1 = 24\Big\{&(k_1^2 + k_2^2 + k_3^2)h_o k_3^2 + [k_2^2 + k_3^2 - k_1^2][k_1k_3x + k_1w_3t]^2\\
&+ [k_1^2 + k_3^2 - k_2^2][k_2k_3x + k_2w_3t]^2 + k_3^2[k_1^2 + k_2^2 - k_3^2][k_3x + w_3t]^2\\
&- 4k_1k_2[k_1k_3x + k_1w_3t][k_2k_3x + k_2w_3t] - 4k_1k_3^2[k_1k_3x + k_1w_3t][k_3x + w_3t]\\
&- 4k_1k_3^2[k_2k_3x + k_2w_3t][k_3x + w_3t]\Big\}
\end{aligned}$$

and

$$H_1 = [h_o k_3^2 + (k_1k_3x + k_1w_3t)^2 + (k_3k_2x + k_2w_3t)^2 + k_3^2(k_3x + w_3t)^2].$$

The dynamical behaviour of this solution has been depicted in fig. 4. Figures 4(a) and (b) illustrate the propagation of the lump soliton. This figure shows that the lump soliton wave is periodic along the t-axis but with gaps.

In order to find the mixed-order lump wave solution, we choose F in following manner:

$$F = \lambda_o + \xi_1^2 + \xi_2^2 + e^{\xi_3}, \tag{9}$$

where $\xi_i = k_i x + w_i t$ for $i = 1, 2, 3$. Moreover, constants k_i and w_i are to be determined later. We put this transformation in eq. (5) and after computation work, we get the set of algebraic equations. After simplification we get different values of the constants as follows:

$$\begin{aligned}
&\lambda_o = \frac{4k_1^2}{k_3^2}\,,\\
&k_1 = k_2,\\
&w_1 = w_2 = -k_1(3k_3^2 + [u_o + \alpha(t) + b - C_w]),\\
&w_3 = -k_3(k_3^2 + [u_o + \alpha(t) + b - C_w]).
\end{aligned}$$

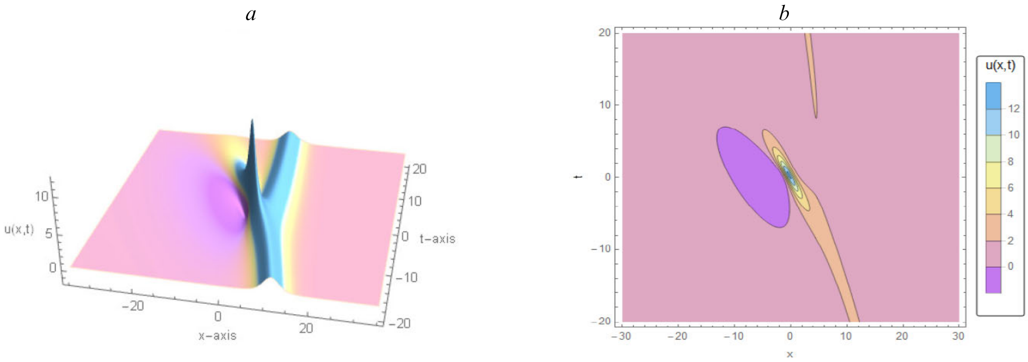

Fig. 5. 3D plot and contourplot for the mixed soliton solution of $u(x,t)$ of eq. (10) for parameters $u_0 = 0.2$, $k_1 = 1.2$, $k_2 = 0.1$, $k_2 = 0.8$, $h_3 = 0.1$, $w_1 = 0.8$, $w_1 = 0.7$, $w_1 = 0.2$ and $\alpha(t) = \sin(t)$.

Thus the mixed-order lump wave solution is obtained:

$$u(x,t) = u_o + \frac{G_2}{H_2^2}\,, \tag{10}$$

where

$$G_2 = 24(k_1^2 + k_2^2)(\lambda_o + e^{\xi_3}) + 12k_3^2 e^{\xi_3}(\lambda_o + \xi_1^2 + \xi_2^2) + 24(k_2^2 - k_1^2)(\xi_1^2 - \xi_2^2) - 96k_1k_2\xi_1\xi_2 - 48k_3 e^{\xi_3}(k_1\xi_1 + k_2\xi_2),$$
$$H_2 = \lambda_o + \xi_1^2 + \xi_2^2 + e^{\xi_3}.$$

The dynamical behaviour of this solution is shown in fig. 5. From figs. 5(a) and (b), it can be observed that the interaction between lump soliton and bright soliton occurs at $t = 0$. Also, figs. 4 and 5 show the mixed-type lump waves propagations. This research could contribute to a deeper understanding of the nonlinear structure and propagation behaviour for the system, and has potential applications in graphene materials.

4 Conclusion

The article studied the dynamical behaviour of a thermophoretic motion system which describes the propagation of solitons in substrate-supported graphene sheets. The system is considered with a variable heat transmission and thermal conductive coefficients. A test function of the extended three-soliton method was used. The results show the dynamical behaviour of solitons under reflection, periodic distribution and interaction. Furthermore, the bright and mixed-type lump wave soliton propagation and interaction are discussed. The study of soliton and lump wave solutions has received great attention of the researchers in this field, due to the potential applications in graphene materials.

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