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Operational-matrix–based algorithm for differential equations of fractional order with Dirichlet boundary conditions

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Abstract. The fractional differential equations (FDEs) are ground-breaking tools to demonstrate the complex-nature scientific systems in the form of non-linear behavior endorsed by the scientific community to develop some new and accurate mathematical methods. The main objective of this paper is the development of an extended mathematical algorithm based on the Gegenbauer wavelet method for the fractional-order problem. The Gegenbauer wavelet operational matrix with their derivative is proposed in our study. Some new operational matrices for the derivative of fractional order with Dirichlet boundary condition is purposed by introducing the piecewise function. Furthermore, a successful use to analyze the solution for the set of algebraic equations governed through the extended Gegenbauer wavelets technique is performed. Analytical solutions of the mentioned problem are effectively obtained, and a comparative study is presented. The outcomes are obtained via the modified Gegenbauer wavelet method by endorsing the accuracy and effectiveness of the mentioned technique. The convergence and error bound analysis are enclosed in our investigation. It is further verified that the algorithm is quite accurate, and an efficient mathematical tool is used to tackle the nonlinear fractional-order complex-nature problems.

1 Introduction

Fractional Calculus (FC) and fractional differential equations (FDEs) are another ground-breaking tool which have been recently utilized to demonstrate complex natural biological, physical or industrial systems with long-term memory and non-linear behavior. This ancient mathematical topic is carried out from a familiar dialogue between two well-known mathematicians (Leibniz and Hospital). However, the area of FDEs is the more generalized form of arbitrary order derivatives [1]. Many famous mathematicians presented a novel definition to evaluate the fractional-ordered derivatives and a detailed review is available in [2]. The comparison among fractional-order and integer-order differential equations witnesses that FDEs shows several advantages over the ordinary differential equations (ODEs). Atangana and Baleanu [3] presented a new fractional derivative with non-local and non-singular kernel. The theory and applications are purely related to heat transfer modeling. Goufo [4] presented the applications of two-parameter derivative with non-singular and non-local kernel in chaotic processes. The simulation of the FDEs arising in physics, system biology, finance, hydrology, chemistry, control theory, and biochemistry is comparatively more efficient and easier as compared to integer-order DEs. The potential application of FDEs are fractance, viscoelasticity, dynamical systems, capacitor theory, diffusion, robotics, optimal portfolio, neurology, economics, signal processing, filtering, electro-analytical chemistry, viscoelastic materials, bio-engineering, electrical circuits, electronics, fluid, solid, and statistical mechanics [5–11].

Because of the wide range of applications, the role of fractional calculus in engineering, industry, and other sciences motivates the scientific community towards the complex area of non-integer and its modeling. The FEDs with Dirichlet boundary conditions (DBCs) are a moderate topic of research and significantly appear in many real-world problems.

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The Dirichlet boundary conditions (DBCs) are perceived as relatively natural in Eulerian structures but are found difficult to impose by Lagrangian procedures. These conditions impose static concentration values at locations, defined as a mass ($\mathrm{d}m$) within a specific volume ($\mathrm{d}V$). In physical-chemistry the appearance of DBCs is due to the local chemical equilibrium by mass transport fit for solubility concentration from donor phase into acceptor phase. The worthy applications of FDEs with DBCs are fluid dynamics, biological system, heat generation, neurology, mass transport, economics, heat transport, chemical reactions, and many other [12]. Ul Hassan [13] developed an algorithm based on soliton theory to analyze the solution of biological population equation of fractional order. Shah *et al.* [14] examined the heat transport in a 2nd-grade fluid over an oscillating perpendicular plate. A fractional Caputo-Fabrizio derivative was reported and an exact (close form) solution of nonlinear FDE model has been presented. In reviewing these applications, problems of the following form are studied or analyzed via different methods:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = F\left(x, y, u, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\gamma u}{\partial y^\gamma}\right). \tag{1}$$

The expression (1) is representing a fractional partial differential equation (FPDE), in which α, β and γ are the order of fractional derivatives. The domains of these fractional orders are: α, $\gamma > 1$ and $\beta < 1$. The problem (1) is subject to the Dirichlet boundary conditions, and the mathematical form is stated below:

$$\begin{cases} u(x,0) = \alpha_1(x), \quad u(0,y) = \alpha_1(y), \\ u(x,1) = \beta_1(x), \quad u(1,y) = \beta_2(y), \end{cases} \tag{2}$$

where, the functions α_i and β_i are twice continuously differentiable functions on the interval $L^2[0,1]$. The problem (1)–(2) mentioned above was discussed previously by Heydari *et al.* [15]. The study is reported for a new extension based on Legendre wavelets and applications to examine the solution of FDEs with DBCs. Later on, the same problem was analyzed by Rahimkhani and Ordokhani [16] by means of Bernoulli collocation-based wavelet scheme. A worthy and application-based study of FDEs is available in the literature [17–19] and references therein.

It is obvious from the literature survey that the solutions of physical problems and the attention of the scientific community towards the area of FDEs is fairly realistic due to its dynamic applications. The development of novel techniques or extensions in existing methods was focused by many researchers due to the complexity of the kernel operators of FDEs. Previously, several mathematicians adopted analytical [20–22], numerical [23–25], soliton [26–28] based techniques to analyze the solutions of FDEs. The focus of scientists towards theoretical investigation over experimental study is realistic because it saves cost, time and apparatus. The usage of orthogonal-basis polynomials, wavelet and some extended wavelet algorithms have opened a new research method in various scientific domains [11,15–19,29]. The wavelets are frequently advents in several field of engineering, biological, physical, and other sciences [15–17,30–36]. The mainly used orthogonal basis functions are Chebyshev, Haar, Laguerre, Legendre, Laurent, Gegenbauer polynomials, and few others [11,15–19,29–36]. The Gegenbauer-polynomial–based wavelet techniques can offer better solution because they reduce the computational cost at a tangible level and provide a better rate of accuracy. Previously, the usage of the Gegenbauer wavelet method (GWM) was not frequent and not considered by the researchers. Currently, the Gegenbauer wavelet method and many modifications have been made to analyze the solution of various kinds of mathematical problems [29,31–36] and references therein. For some recent advanced work and its applications, the readers are referred to [37–40] which are published to demonstrate the basic idea about FDEs.

In reviewing the previous surveys and potential applications of FC and FDEs, herein we are reporting an extension of the Gegenbauer wavelet method to examine the solution of fractional problems with Dirichlet boundary conditions. The purpose of the current study is to develop an algorithm to theoretically analyze a class of nonlinear physical problems. Some new operational matrices for the derivative of fractional order with Dirichlet boundary condition with the help of piecewise functions has been proposed. The extended Gegenbauer wavelets technique convert the given problem into set of algebraic equations. Analytical solutions of problem (1), (2) are effectively obtained and the outcomes are compared with existing results. The outcomes found via Gegenbauer wavelets are validating the effectiveness and accuracy of the suggested method. The analysis of error bound and convergence is enclosed in our investigation to prove the reliability and effectiveness of the mathematical formulation of the algorithm. It is observed that it is an accurate and efficient tool to tackle the non-linear fractional order problems of complex nature and it can be further extended for the major finding of non-linear problems of fractional order. Moreover, one can extend it to analyze some physical or complex-nature problems arising in engineering and physics.

2 Preliminaries of fractional calculus

This section devoted to the brief study regarding fractional order derivative. The definition of Riemann-Liouville and the Caputo's derivative are used most widely.

Definition 1. *Fractional derivative of order* $\delta(t)$ *in the Riemann-Liouville sense* [1–13] *is given as follows:*

$$ {}_{0}^{\mathrm{RL}}D_t^{\delta(t)} f(t) = \frac{1}{\Gamma(\gamma-\delta(t))} \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} \int_a^t \frac{1}{(t-s)^{\delta(t)-\gamma+1}} f(s)\mathrm{d}s, \quad \textit{for } \gamma-1 \le \delta(t) < \gamma \in Z^+. \tag{3} $$

Definition 2. *The fractional derivative of* $n-1<\delta(t)<n$ *in Caputo's sense* [1–13] *is given as*

$$ {}_{0}^{C}D_t^{\delta(t)} f(t) = \frac{1}{\Gamma(n-\delta(t))} \int_{0+}^t \frac{1}{(t-s)^{\delta(t)}} f^{(n)}(s)\mathrm{d}s, \quad n \in \mathbb{N}. \tag{4} $$

The operator ${}_{0}^{C}D_t$ *satisfies the following properties:*

$$ \begin{aligned} &1.\quad {}_{0}^{C}D_t^{\delta(t)}(\lambda f(t)+\gamma g(t)) = \lambda {}_{0}^{C}D_t^{\delta(t)} f(t) + \gamma {}_{0}^{C}D_t^{\delta(t)} g(t),\\ &2.\quad {}_{0}^{C}D_t^{\delta(t)} t^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\delta(t)+1)} t^{\beta-\delta(t)}, & \textit{otherwise},\\ 0, & \delta\in\mathbb{N}_0,\ \beta<\delta(t). \end{cases}\\ &3.\quad {}_{0}^{C}D_t^{\delta(t)}\lambda = 0, \end{aligned} \tag{5} $$

where γ *and* λ *are the constants.*

Definition 3. *Caputo fractional-order partial derivatives of order* $\delta(t)>0$ *of the function* $\phi(x,t)$ *w.r.t. its variables* x *and* t *are defined as follows* [15,16]*:*

$$ {}_{0}^{C}D_x^{\delta}\phi(x,t) = \frac{1}{\Gamma(n-\delta)} \int_0^x \frac{1}{(x-s)^{\delta-n+1}} \frac{\partial^n \phi(s,t)}{\partial s^n}\mathrm{d}s, \quad n-1<\delta\le n, $$

and

$$ {}_{0}^{C}D_t^{\delta}\phi(x,t) = \frac{1}{\Gamma(n-\delta)} \int_0^t \frac{1}{(t-r)^{\delta-n+1}} \frac{\partial^n \phi(x,r)}{\partial r^n}\mathrm{d}r, \quad n-1<\delta\le n. $$

3 Gegenbauer wavelets and their properties

Nowadays the wavelets methods are frequently used in various scientific fields including engineering, physical and biological sciences [15–17,30–36]. Basically the wavelets create a set of functions built up from dilation and translation from an individual function $\psi(t)$, also known as mother wavelet. The set of continuous wavelets are given below when a and b are the dilation and translation parameters respectively, continuously varying:

$$ \psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right); \quad a,b\in\mathbb{R},\ a\neq 0. $$

The subsequent set of discrete wavelets is obtained after restricting a and b such that $a=a_0^{-k}$, and $b=nb_0a_0^{-k}$, a_0-1, $b_0>0$:

$$ \psi_{k,n}(t) = |a|^{k/2}\psi(a_0^k t - nb_0); \quad k,n\in\mathbb{Z}. $$

Here the function $\psi_{k,n}(t)$ forms the wavelets basis for $L^2(\mathbb{R})$. When $a_0=2$ and $b_0=1$, then $\psi_{k,n}(t)$ forms an orthonormal basis. Gegenbauer wavelets defined [18,29,33,34,36] on $(0,1]$ are given as follows:

$$ \psi_{p,q}(t) = \begin{cases} \dfrac{2^{k/2}}{\sqrt{L_q^\nu}} G_q^\nu(2^k t - \hat{p}), & \dfrac{\hat{p}-1}{2^k} \le t \le \dfrac{\hat{p}+1}{2^k},\\ 0 & \text{otherwise}. \end{cases} \tag{6} $$

where $p=1,2,3,\ldots,2^{k-1}$, $q=0,1,2,\ldots,M-1$, $\hat{p}=2p-1$, $G_q^\nu(t)$ signifies the q-th–order Gegenbauer polynomials. Gegenbauer polynomials [18,29,33,34,36] can be found with the help of an explicit formula given as

$$ G_q^\nu(t) = \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \frac{(-1)^k \Gamma(q-k+\nu)}{k!\,\Gamma(\nu)(q-2k)!}(2t)^{q-2k}. \tag{7} $$

The relations of the Gegenbauer polynomials [18, 29, 33, 34, 36] with other polynomials are given as:

$$G_0^\nu = 1 = L_0 = T_0 = U_0^*,$$
$$L_q = G_q^{1/2}, \qquad T_q = \frac{q}{2}\lim_{\nu\to 0}\frac{G_q^{1/2}}{\nu}, \qquad U_q^* = G_q^1, \qquad G_q^\nu = \frac{\Gamma(2\nu)_q}{(\nu+1/2)_q}P_q^{(\nu-\frac{1}{2},\nu-\frac{1}{2})},$$

where $T_q(t)$, $U_q^*(t)$, $P_q(t)$ and $L_q(t)$ are the first kind of Chebyshev, second kind of Chebyshev, Jacobi and Legendre polynomials, respectively. The Gegenbauer polynomials $G_q^\nu(t)$ are orthogonal [18,29,33,34,36] w.r.t. L^2 on the interval $[-1,1]$:

$$\int_{-1}^{1} G_p^\nu(t)G_q^\nu(t)\vartheta^\nu(t)\mathrm{d}t = \begin{cases} 0, & \text{for } p \neq q, \\ L_q^\nu, & \text{for } p = q, \end{cases} \tag{8}$$

where $\vartheta^\nu(t)$ and L_q^ν are the weight function and normalizing factor, respectively, and are given as

$$\vartheta^\nu(t) = (1-t^2)^{\nu-1/2},$$
$$L_q^\nu = \frac{2^{1-2\nu}\pi\Gamma(m+2\nu)}{(m+\nu)\Gamma(m+1)[\Gamma(\nu)]^2}\,.$$

The weight function defined in above for the Gegenbauer wavelets [18, 29, 33, 34, 36] is given below after being dilated and translated:

$$\vartheta_p^\nu(t) = (1-(2^k t-\hat{p})^2)^{\nu-1/2}. \tag{9}$$

Theorem 1. *A function $f(t)$ defined in $[0,1)$ from the $L^2(\mathbb{R})$-space can be expanded with the help of the truncated Gegenbauer wavelets $\psi_{p,q}(t)$ [15–17, 30–36] as*

$$f(t) = \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1}\tau_{p,q}\psi_{p,q}(t), \tag{10}$$

where $\tau_{p,q} = \int_{-1}^{1} f(t)\psi_{p,q}(t)\vartheta_p^\nu(t)\mathrm{d}t$. The matrix form of eq. (10) is given as

$$f(t) = \sum_{k=1}^{\hat{m}}\tau_k\psi_k(t) = C^T\Psi(t), \tag{11}$$

where $\hat{m} = 2^{k-1}M$, k can be obtained with the help of the relation $k = M(p-1)+q+1$. Moreover, C and $\Psi(t)$ are the matrices of order $\hat{m}\times 1$ given as

$$C = [\tau_1, \tau_2, \tau_3, \ldots, \tau_{\hat{m}}]^T,$$
$$\Psi(t) = [\psi_1, \psi_2, \psi_3, \ldots, \psi_{\hat{m}}]^T. \tag{12}$$

4 Operational matrices of the derivatives

The detailed study about the derivation of the operational matrices for the derivative of positive integer and fractional order is presented in this section.

Theorem 2. *Suppose that the Gegenbauer polynomials is defined in $[-1,1]$, then the Gegenbauer polynomials must fulfill the following relation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}[G_q^\nu(t)] = \sum_{\substack{k=0 \\ q+k\ \mathrm{odd}}}^{q-1} 2(k+\nu)G_k^\nu(t). \tag{13}$$

Proof. Assume a function, say $g(t)$, which is approximated by the Gegenbauer polynomials as

$$g(t) = \sum_{k=0}^{\infty}\tilde{g}_k G_k^\nu(t). \tag{14}$$

Expression (14) takes the following form, after differentiating the above expression on both sides w.r.t. t:

$$g'(t) = \sum_{k=0}^{\infty} \tilde{g}_k^{(1)} G_k^{\nu}(t). \tag{15}$$

In eq. (15), $\tilde{g}_k^{(1)}$ is given as

$$\tilde{g}_k^{(1)} = 2(k+\nu) \sum_{\substack{p=q+1 \\ p+k \text{ odd}}}^{\infty} \hat{g}_p.$$

Now, considering that $g(t) = G_q^{\nu}(t)$ in eq. (14), we attained $\hat{g}_i = 0$ for $i \neq q$ and $\hat{g}_q = 1$, subsequently we have,

$$\tilde{g}_k^{(1)} = \begin{cases} 2(k+\nu), & \text{for } q+k \text{ is odd}, \ k \leq q-1, \\ 0, & \text{otherwise.} \end{cases}$$

By means of the value of $\tilde{g}_k^{(1)}$ in eq. (15), we get the following required expression:

$$\frac{\mathrm{d}}{\mathrm{d}t}[G_q^{\nu}(t)] = \sum_{\substack{k=0 \\ q+k \text{ odd}}}^{q-1} 2(k+\nu) G_k^{\nu}(t).$$

Theorem 3. *The derivative of vector $\Psi(t)$ defined in eq. (12) w.r.t. t must satisfy the following relation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(t)] = \mathbf{D}\Psi(t). \tag{16}$$

In eq. (16) the matrix $\mathbf{D}$ is the operational matrix of the derivative of $\hat{m} \times \hat{m}$ order and is given below:

$$\mathbf{D} = \begin{bmatrix} \boldsymbol{\Phi} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{\Phi} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{\Phi} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \boldsymbol{\Phi} \end{bmatrix}. \tag{17}$$

In the above $\boldsymbol{\Phi}$ is a matrix of order $M \times M$ having the (i,j)-th elements given as

$$\Phi_{i,j} = \begin{cases} \dfrac{2^{k+1}(j+\nu-1)\sqrt{(i-1+\nu)\Gamma(i)\Gamma(j-1+2\nu)}}{\sqrt{(j-1+\nu)\Gamma(j)\Gamma(i-1+2\nu)}}, & i = 2,3,\dots,M, \ j = 1,2,\dots,i-1 \ \textit{and} \ (i+j) \ \textit{odd}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. First we consider the i-th element of the Gegenbauer wavelets vector $\Psi(t)$ given as follows:

$$\psi_i(t) = \psi_{p,q}(t) = \frac{2^{k/2}}{\sqrt{L_q^{\nu}}} G_q^{\nu}(2^k t - \hat{p}) \chi_{[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]}, \quad \text{for } i = 1,2,\dots,\hat{m}. \tag{18}$$

In eq. (18), $\chi_{[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]}$ denotes the characteristic function which is given as

$$\chi_{[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]} = \begin{cases} 1, & t \in \left[\dfrac{\hat{p}-1}{2^k}, \dfrac{\hat{p}+1}{2^k}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Now eq. (18) takes the following form, after differentiating w.r.t. t:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\psi_i(t)] = \frac{2^{k/2}}{\sqrt{L_q^{\nu}}} [G_q^{\nu}(2^k t - \hat{p})]' \chi_{[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]}. \tag{19}$$

The characteristic function is zero outside the interval $[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]$, therefore the Gegenbauer wavelets expansion includes such elements of $\Psi(t)$, which are not zero in $[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]$ that are $\psi_r(t)$; $r = M(p-1)+1, M(p-1)+2, \ldots, M(p-1)+M$. The Gegenbauer wavelets expansion takes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\psi_i(t)] = \sum_{r=M(p-1)+1}^{Mp} a_r \psi_r(t).$$

The operational matrix $\mathbf{D}$ present in (17) is obtained with the help of the above expression. Moreover, $[G_0^\nu(t)]' = 0$ yields $[\psi_i(t)]' = 0$, for $i = 1, M+1, 2M+1, \ldots, (2^{k-1}-1)M+1$. Hence the first row of matrix $\boldsymbol{\Phi}$ is zero. By means of expression (13) into (19), we get the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\psi_i(t)] = \frac{2^{k/2}}{\sqrt{L_q^\nu}} 2^{k+1} \sum_{\substack{l=0 \\ l+q \,\mathrm{odd}}}^{q-1} (l+\nu) G_l^\nu(2^k t - \hat{p})\, \chi_{[\frac{\hat{p}-1}{2^k}, \frac{\hat{p}+1}{2^k}]}. \tag{20}$$

The required result is achieved as follows, after expanding the expression (20) into the Gegenbauer wavelets basis $\psi(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\psi_i(t)] = 2^{k+1} \sum_{\substack{j=0 \\ i+j \,\mathrm{odd}}}^{i-1} (j+\nu-1) \frac{\sqrt{(i-1+\nu)\Gamma(i)\Gamma(j-1+2\nu)}}{\sqrt{(j-1+\nu)\Gamma(j)\Gamma(i-1+2\nu)}} \psi_{M(p-1)+j}(t).$$

Consider $\boldsymbol{\Phi}_{i,j}$ such that

$$\boldsymbol{\Phi}_{i,j} = \begin{cases} \dfrac{2^{k+1}(j+\nu-1)\sqrt{(i-1+\nu)\Gamma(i)\Gamma(j-1+2\nu)}}{\sqrt{(j-1+\nu)\Gamma(j)\Gamma(i-1+2\nu)}}, & i = 2,3,\ldots,M,\ j = 1,2,\ldots,i-1 \text{ and } (i+j) \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

For particular the values $k = 2$ and $M = 3$, we obtain the following operational matrix $\mathbf{D}$:

$$\mathbf{D} = 4\sqrt{2(\nu+1)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{\dfrac{\nu+2}{2\nu+1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{\dfrac{\nu+2}{2\nu+1}} & 0 \end{bmatrix}.$$

Corollary.. *The operational matrix* $\mathbf{D}$ *for the* n*-th–order derivative of vector* $\Psi(t)$*, can be defined as follows with the help of eq.* (16)*:*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\Psi(t)] = \mathbf{D}^n \Psi(t).$$

Now to find the derivative operational matrix of fractional order we are going to introduce the following family of piecewise functions defined on $[0,1]$*:*

$$\omega_{p,q} = \begin{cases} t^q & t \in \left[\dfrac{\hat{p}-1}{2^k}, \dfrac{\hat{p}+1}{2^k}\right], \\ 0 & \textit{otherwise}. \end{cases} \tag{21}$$

Here in the above $q = 0,1,2,\ldots,M-1$, $p = 1,2,3,\ldots,2^{k-1}$*. The piecewise functions given in expression* (21) *are not normalized, the* m*-th piecewise functions can be stated as*

$$\Theta = [\omega_1, \omega_2, \omega_3, \ldots, \omega_{\hat{m}}]. \tag{22}$$

In the above $\omega_i = \omega_{p,q}$ *and the index* i *can be obtained with the help of the relation* $i = M(p-1)+q+1$.

Theorem 4. *Consider* $\Theta(t)$ *to be a vector present in eq.* (22) *and*

$$\Theta(t) = \boldsymbol{\Delta}\Psi(t), \tag{23}$$

where $\boldsymbol{\Delta}$ *is a matrix having order* $\hat{m} \times \hat{m}$ *which is stated as*

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_1 & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{\Delta}_2 & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{\Delta}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \boldsymbol{\Delta}_{2^{k-1}} \end{bmatrix}, \tag{24}$$

and

$$\boldsymbol{\Delta}_p = \begin{bmatrix} \xi(0,0) & \xi(0,1) & \xi(0,2) & \dots & \xi(0,M-1) \\ \xi(1,0) & \xi(1,1) & \xi(1,2) & \dots & \xi(1,M-1) \\ \xi(2,0) & \xi(2,1) & \xi(2,2) & \dots & \xi(2,M-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi(M-1,0) & \xi(M-1,1) & \xi(M-1,2) & \dots & \xi(M-1,M-1) \end{bmatrix}.$$

Then the following relation must satisfied:

$$\xi(q,j) = \frac{2^{\frac{k}{2}}}{2^{(q+1)k}} \sqrt{\frac{(j+\nu)[\Gamma(\nu)]^2\Gamma(j+1)}{2^{1-2\nu}\pi\,\Gamma(j+2\nu)}} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^k 2^{j-2k}\Gamma(j-k+\nu)}{\Gamma(\nu)k!(j-2k)!} \\ \times \sum_{l=0}^{q} \binom{q}{l} \frac{\hat{p}^{q-l}}{2} \frac{(1-(-1)^{j-2k+l})\Gamma(\frac{j}{2}-k+\frac{l}{2}+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+\frac{j}{2}-k+\frac{l}{2}+1)}. \tag{25}$$

Proof. Since

$$\Theta(t) = \boldsymbol{\Delta}\Psi(t),$$

by means of theorem 1, we obtain the following relation:

$$\xi(q,j) = \int_{-1}^{1} \omega_{p,q}(t)\psi_{q,j}(t)\vartheta_p^{\nu}(t)\mathrm{d}t = 2^{\frac{k}{2}} \sqrt{\frac{(j+\nu)[\Gamma(\nu)]^2\Gamma(j+1)}{2^{1-2\nu}\pi\Gamma(j+2\nu)}} \\ \times \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^k 2^{j-2k}\Gamma(j-k+\nu)}{\Gamma(\nu)k!(j-2k)!} \int_{\frac{\hat{p}-1}{2^k}}^{\frac{\hat{p}+1}{2^k}} (2(2^k t-\hat{p}))^{j-2k} t^q (1-(2^k t-\hat{p})^2)^{\nu-\frac{1}{2}}\mathrm{d}t. \tag{26}$$

Now, let us consider that $\eta = 2^k t - \hat{p}$ which implies that $\mathrm{d}t = 2^{-k}\mathrm{d}\eta$. Therefore, consider the integrating part of eq. (26) as

$$\int_{\frac{\hat{p}-1}{2^k}}^{\frac{\hat{p}+1}{2^k}} (2(2^k t-\hat{p}))^{j-2k} t^q (1-(2^k t-\hat{p})^2)^{\nu-\frac{1}{2}}\mathrm{d}t = \frac{2^{j-3k}}{2^{kq}} \int_{-1}^{1} \eta^{j-2k}(\eta+\hat{p})^q(1-\eta^2)^{\nu-\frac{1}{2}}\mathrm{d}\eta \\ = \frac{1}{2^{(q+1)k}} \sum_{l=0}^{q} \binom{q}{l} \frac{\hat{p}^{q-l}}{2} \frac{(1-(-1)^{j-2k+l})\Gamma(\frac{j}{2}-k+\frac{l}{2}+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+\frac{j}{2}-k+\frac{l}{2}+1)}, \quad p = 1,2,\dots,2^{k-1}. \tag{27}$$

Using the expression (27) into (26), we get the required results (25). The matrix $\boldsymbol{\Delta}$ for $k = 2$, $M = 3$ takes the following form:

$$\boldsymbol{\Delta} = \frac{\sqrt{\Gamma(\nu+\frac{1}{2})}\sqrt[4]{\pi}}{2\sqrt{\Gamma(\nu+1)}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{2}}{8\sqrt{1+\nu}} & 0 & 0 & 0 & 0 \\ \frac{2\nu+3}{32(\nu+1)} & \frac{\sqrt{2}}{16\sqrt{1+\nu}} & \frac{\sqrt{2\nu+1}}{32(\nu+1)\sqrt{2+\nu}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{\sqrt{2}}{8\sqrt{1+\nu}} & 0 \\ 0 & 0 & 0 & \frac{18\nu+19}{32(\nu+1)} & \frac{3\sqrt{2}}{16\sqrt{1+\nu}} & \frac{\sqrt{2\nu+1}}{32(\nu+1)\sqrt{2+\nu}} \end{bmatrix},$$

Lemma 1. The fractional differentiation of order δ of expression (21) is defined as follows, where $(\gamma-1) < \delta < \gamma$ is a positive function:

$${}_0^C D_t^{\delta} \omega_{p,q}(t) = \begin{cases} \dfrac{q!}{\Gamma(q-\delta+1)} t^{q-\delta}, & q = \gamma, \gamma+1, \dots, M-1,\ t \in \left[\dfrac{\hat{p}-1}{2^k}, \dfrac{\hat{p}+1}{2^k}\right], \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The proof is very simple by using eq. (5).

Lemma 2. The fractional differentiation of order δ of expression (22) in the Caputo sense is defined as follows, where $(\gamma-1) < \delta < \gamma$ is a positive function:

$${}_0^C D_t^{\delta}[\Theta(t)] = \mathbf{P}^{\delta} \Omega(t).$$

Here $\mathbf{P}^{\delta}$ is a matrix of order $\hat{m} \times \hat{m}$ that is given as

$$\mathbf{P}^{\delta} = t^{-\delta} \begin{bmatrix} \mathbf{Q}^{\delta} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{Q}^{\delta} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{Q}^{\delta} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{Q}^{\delta} \end{bmatrix}, \tag{28}$$

and the matrix $\mathbf{Q}^{\delta}$ is square having order $M \times M$ defined as

$$\mathbf{Q}^{\delta} = \begin{bmatrix} 0 \ 0 \ \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots \ \vdots \ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \ 0 \ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 \dots 0 & \frac{\gamma!}{\Gamma(\gamma-\delta+1)} & 0 & 0 & \dots & 0 \\ 0 \dots\dots & 0 & \frac{\gamma!}{\Gamma(\gamma-\delta+2)} & 0 & 0 & 0 \\ \vdots \ \vdots \ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 \ 0 \ \dots & 0 & 0 & 0 & \frac{\gamma!}{\Gamma(\gamma-\delta-1)} & 0 \\ 0 \ 0 \ \dots & 0 & 0 & 0 & 0 & \frac{\gamma!}{\Gamma(\gamma-\delta)} \end{bmatrix}.$$

Proof. The proof is very simple by means of lemma 1.

Theorem 5. *The fractional differentiation of order δ of expression* (12) *in the Caputo sense is defined as follows, where* $(\gamma-1)<\delta<\gamma$ *is a positive function:*

$$ {}_0^C D_t^{\delta}[\Psi(t)] = \mathbf{K}^{\delta}\Psi(t) = (\boldsymbol{\Delta}^{-1}\mathbf{P}^{\delta}\boldsymbol{\Delta})\Psi(t), $$

where $\boldsymbol{\Delta}$ and $\mathbf{P}^{\delta}$ are the matrices present in expressions (24) *and* (28), *respectively, and $\mathbf{K}^{\delta(t)}$ is known as the fractional operational matrix of order δ for the Gegenbauer wavelets.*

Proof. By means of the relation (23), we obtain the following form:

$$ \Psi(t) = \boldsymbol{\Delta}^{-1}\Theta(t). $$

Differentiating both sides of the above expression w.r.t. t of order δ and using lemma 2, we obtain the following form:

$$ {}_0^C D_t^{\delta}[\Psi(t)] = \boldsymbol{\Delta}^{-1}{}_0^C D_t^{\delta}\Theta(t) = (\boldsymbol{\Delta}^{-1}\mathbf{P}^{\delta}\boldsymbol{\Delta})\Psi(t), $$

which completes the proof. For $\nu=1$, $k=2$, $M=3$ the matrix $\mathbf{K}^{\delta}$ is given as below:

$$ \mathbf{K}^{\delta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{2}{\Gamma(2-\delta)} & \dfrac{1}{\Gamma(2-\delta)} & 0 & 0 & 0 & 0 \\ \dfrac{8}{\Gamma(2-\delta)}+\dfrac{10}{\Gamma(3-\delta)} & -\dfrac{2}{\Gamma(2-\delta)}+\dfrac{8}{\Gamma(3-\delta)} & \dfrac{2}{\Gamma(3-\delta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{6}{\Gamma(2-\delta)} & \dfrac{2-\delta}{\Gamma(3-\delta)} & 0 \\ 0 & 0 & 0 & -\dfrac{72}{\Gamma(2-\delta)}+\dfrac{74}{\Gamma(3-\delta)} & -\dfrac{12}{\Gamma(2-\delta)}+\dfrac{24}{\Gamma(3-\delta)} & \dfrac{2}{\Gamma(3-\delta)} \end{bmatrix}. $$

5 Error bound and convergence analysis

In this section, the error bound and convergence analysis of the modified Gegenbauer wavelets method are discussed comprehensively.

Theorem 6. *Consider a real-valued function $g(t)\in C^M[0,1)$, such that*

$$ g(t) = \sum_{p=1}^{2^{k-1}} g_p(t). $$

Also suppose that $\Upsilon_p = \mathrm{span}\{\psi_{p,0},\psi_{p,1},\psi_{p,2},\ldots,\psi_{p,M-1}\}$, $p=1,2,3,\ldots 2^{k-1}$. If $C_p^T\Psi_p(t)$ is the best approximation of $g_p(t)$ and from Υ_p, here the vectors C_p and $\Psi_p(t)$ are define as

$$ C_p = [\tau_{p,0},\tau_{p,1},\tau_{p,2},\ldots,\tau_{p,M-1}], \qquad \Psi_p(t) = [\psi_{p,0},\psi_{p,1},\psi_{p,2},\ldots,\psi_{p,M-1}]. $$

Then $g_{N,M}(t)=f(t)=C^T\Psi(t)$ is the approximation of $g(t)$, which has the error bound as given below:

$$ \|g(t)-f_{N,M}(t)\| \le \left(\frac{\beta}{M!}\right)\frac{\sqrt{2}}{2^{M(k-1)}\sqrt{2M+1}}, $$

where $\beta=\max_{t\in[0,1)}|g^M(t)|$.

Proof. Taylor's series of $g_p(t)$ is given as follows for $\frac{\hat{p}-1}{2^k}\le t\le\frac{\hat{p}+1}{2^k}$;

$$ \bar{g}_p(t) = g_p\left(\frac{\hat{p}-1}{2^k}\right) + g_p'\left(\frac{\hat{p}-1}{2^k}\right)\left(t-\frac{\hat{p}-1}{2^k}\right) + \ldots + g_p^{(M-1)}\left(\frac{\hat{p}-1}{2^k}\right)\frac{(t-\frac{\hat{p}-1}{2^k})^{M-1}}{(M-1)!}. $$

We know that

$$ |g_p(t)-\bar{g}_p(t)| \le |g^M(t)|\frac{(t-\frac{\hat{p}-1}{2^k})^M}{M!}, \quad t\in\left[\frac{\hat{p}-1}{2^k},\frac{\hat{p}+1}{2^k}\right), \quad p=1,2,3,\ldots,2^{k-1}. \tag{29} $$

Since the best approximation of $g_p(t)$ is $C_p\Psi_p(t)$ and $\bar{g}_p(t) \in \Upsilon$, by means of the relation (29) we get

$$\|g_p(t) - C_p\Psi_p(t)\| \leq \|g_p(t) - \bar{g}_p(t)\|_2^2 \leq \int_{\frac{\hat{p}-1}{2^k}}^{\frac{\hat{p}+1}{2^k}} |g_p(t) - \bar{g}_p(t)|^2 \mathrm{d}t,$$

$$\leq \int_{\frac{\hat{p}-1}{2^k}}^{\frac{\hat{p}+1}{2^k}} \left[\frac{g^M(t)\left(t - \frac{\hat{p}-1}{2^k}\right)^M}{M!}\right]^2 \mathrm{d}t \leq \left(\frac{\beta}{M!}\right)^2 \int_{\frac{\hat{p}-1}{2^k}}^{\frac{\hat{p}+1}{2^k}} \left(t - \frac{\hat{p}-1}{2^k}\right)^{2M} \mathrm{d}t = \left(\frac{\beta}{M!}\right)^2 \frac{2^{2M+1}}{2^{2kM}(2M+1)}.$$

Now

$$\|g(t) - C^T\Psi(t)\|_2^2 \leq \sum_{p=1}^{2^{k-1}} \|g_p(t) - C_p^T\Psi_p(t)\|_2^2 \leq \left(\frac{\beta}{M!}\right)^2 \frac{2^{2M+1}}{2^{2kM}(2M+1)}. \tag{30}$$

The required result is obtain after taking the square root of eq. (30). Moreover, we conclude that as $k, M \to \infty$, $C^T\Psi(t) \to g(t)$.

Theorem 7. *The approximate solution* $\tilde{f}(t) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \tau_{p,q}\psi_{p,q}(t)$ *tends to the exact solution* $f(t)$ *as* $M, k \to \infty$.

Proof. The inner product of $f(t)$ and $\psi_{p,q}(t)$ w.r.t. the $\vartheta_p^\nu(t)$ is given as

$$\tau_{p,q} = \langle f(t), \psi_{p,q}(t)\rangle = \int_0^1 f(t)\psi_{p,q}(t)\vartheta_p^\nu(t)\mathrm{d}t.$$

Assume that $\hat{r} = 2^{k-1}$, $r_1 = 2^{d-1}$, $\hat{s} = M$ and $s_1 = N$. M and N denote the order of the Gegenbauer polynomials, respectively the resolution level represented by k and d. Consider that $S_{\hat{r},\hat{s}}^\nu$ is the partial sum of $\tau_{p,q}\psi_{p,q}(t)$. Now, we need to show that the $S_{\hat{r},\hat{s}}^\nu$ is a Cauchy sequence in $L^2[0,1)$. Then, we have to prove that $S_{\hat{r},\hat{s}}^\nu$ converges to $f(t)$ as $\hat{r}, \hat{s} \to \infty$.

First, we consider any sum $S_{r,s}^\nu$ of $\tau_{p,q}\psi_{p,q}(t)$ with $\hat{r} > r$ and $\hat{s} > s$, to show that $S_{\hat{r},\hat{s}}^\nu$ is a Cauchy sequence:

$$\|S_{\hat{r},\hat{s}}^\nu - S_{r,s}^\nu\|^2 = \left\|\sum_{p=r+1}^{\hat{r}} \sum_{q=s}^{\hat{s}-1} \tau_{p,q}\psi_{p,q}(t)\right\|^2 = \left\langle \sum_{p=r+1}^{\hat{r}} \sum_{q=s}^{\hat{s}-1} \tau_{p,q}\psi_{p,q}(t), \sum_{n=r+1}^{\hat{r}} \sum_{m=s}^{\hat{s}-1} \tau_{n,m}\psi_{n,m}(t)\right\rangle,$$

$$= \sum_{p=r+1}^{\hat{r}} \sum_{q=s}^{\hat{s}-1} \sum_{n=r+1}^{\hat{r}} \sum_{m=s}^{\hat{s}-1} \tau_{p,q}\bar{\tau}_{n,m}\langle\psi_{p,q}(t), \psi_{n,m}(t)\rangle = \sum_{p=r+1}^{\hat{r}} \sum_{q=s}^{\hat{s}-1} |\tau_{p,q}|^2.$$

By means of the well-known Bessel's inequality, we found that the above relation is convergent and therefore,

$$\|S_{\hat{r},\hat{s}}^\nu - S_{r,s}^\nu\|^2 \to 0, \quad \text{as } \hat{s}, \hat{r}, s, r \to \infty,$$

which means that $S_{\hat{r},\hat{s}}^\nu$ is a Cauchy sequence and $S_{\hat{r},\hat{s}}^\nu$ converges to a function say $g(t) \in L^2[0,1)$. Now we need to prove that $g(t) = f(t)$, for this

$$\langle g(t) - f(t), \psi_{p,q}(t)\rangle = \langle g(t), \psi_{p,q}(t)\rangle - \langle f(t), \psi_{p,q}(t)\rangle,$$
$$= \lim_{\hat{r},\hat{s}\to\infty} \langle S_{\hat{r},\hat{s}}^\nu, \psi_{p,q}(t)\rangle - \tau_{p,q} = \tau_{p,q} - \tau_{p,q} = 0.$$

So, $\sum_{p=1}^{\hat{r}} \sum_{q=0}^{\hat{s}-1} \tau_{p,q}\psi_{p,q}(t)$ converges to $f(t)$ as $\hat{r}, \hat{s} \to \infty$.

6 Proposed methodology

An operation matrix-based algorithm is designed in this section to investigate the approximate solutions of the partial differential equations associated with Dirichlet boundary conditions. The proposed method has the following steps.

Step 1) Consider the partial differential equation as [16,17]

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \mu_1\frac{\partial^\beta u}{\partial x^\beta} + \mu_2\frac{\partial^\gamma u}{\partial y^\gamma} + \mu_3 u(x,y) = F(x,y), \quad 1 < \alpha,\ 0 < \beta,\ \gamma \leq 1, \tag{31}$$

associated with the following Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= \alpha_1(x), \qquad u(0,y) = \alpha_2(y), \\ u(x,1) &= \beta_1(x), \qquad u(1,y) = \beta_2(y). \end{aligned} \tag{32}$$

Step 2) For the solution of problem (31), (32) via the proposed method, we consider the following trial solution:

$$\tilde{u}(x,y) = \Psi(x)^T \mathbf{C} \Psi(y). \tag{33}$$

Here, $\mathbf{C} = [a_{p,q}]$, is a matrix having order $\hat{m} \times \hat{m}$, which essential to be determined and Ψ is a vector present in eq. (12). We approximate each term present in eq. (31) with the help of the operational matrices defined in sect. 4:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \Psi(x)^T [\mathbf{K}^\alpha]^T \mathbf{C} \Psi(y), \quad \frac{\partial^\beta u}{\partial x^\beta} = \Psi(x)^T [\mathbf{K}^\beta]^T \mathbf{C} \Psi(y), \quad \frac{\partial^\gamma u}{\partial y^\gamma} = \Psi(x)^T \mathbf{C} \mathbf{K}^\gamma \Psi(y). \tag{34}$$

Step 3) After incorporating the expressions obtained in the last step into eq. (31), (32), we get the following matrix equations:

$$\Psi(x)^T \left([\mathbf{K}^\alpha]^T \mathbf{C} + \mu_1 [\mathbf{K}^\beta]^T \mathbf{C} + \mu_2 \mathbf{C} \mathbf{K}^\gamma + \mu_3 \mathbf{C} \right) \Psi(y) - F(x,y) \triangleq \Lambda(x,y) \simeq 0, \tag{35}$$

and the Dirichlet boundary conditions take the following form:

$$\begin{aligned} \Phi_1(x) &= \Psi(x)^T \mathbf{C} \Psi(0) - \alpha_1(x) \simeq 0, \qquad \Phi_2(y) = \Psi(0)^T \mathbf{C} \Psi(y) - \alpha_2(y) \simeq 0, \\ \Phi_3(x) &= \Psi(x)^T \mathbf{C} \Psi(1) - \beta_1(x) \simeq 0, \qquad \Phi_4(y) = \Psi(1)^T \mathbf{C} \Psi(y) - \beta_2(y) \simeq 0. \end{aligned} \tag{36}$$

Step 4) To find the matrix $\mathbf{C}$, we need $\hat{m}^2$ algebraic equations. Therefore, we collocate the matrix equations (35)–(36) as

$$\Lambda(x_i, y_j) = 0, \quad \text{for } i = j = 2, 3, 4, \dots \hat{m} - 1, \tag{37}$$

where $x_i = \frac{i}{\hat{m}-1}$, $y_j = \frac{j}{\hat{m}-1}$, used as collocation points and expression (37) gives $(\hat{m} - 2)^2$ algebraic equations. The remaining $4(\hat{m} - 1)$ algebraic equations can be attained by setting expressions (36) as

$$\begin{aligned} \Phi_1(x_i) &= \Phi_3(x_i) = 0, \quad \text{for } i = 2, 3, 4, \dots, \hat{m} - 1, \\ \Phi_2(y_i) &= \Phi_4(y_i) = 0, \quad \text{for } i = 1, 2, 3, \dots, \hat{m}, \end{aligned} \tag{38}$$

where $x_i = \frac{i}{\hat{m}}$, and $y_j = \frac{j}{\hat{m}}$.

Step 5) Finally, the required unknown matrix $\mathbf{C}$ is achieved after solving the $\hat{m}^2$ system of algebraic equations with the aid of Maple 2015. The approximate solution of problem (1), (2) is obtained after inserting the matrix $\mathbf{C}$ into the trial solution.

7 Test problems

In this section some numerical problems are taken to expose the efficiency of the Gegenbauer wavelets method. For the numeric computation we used Maple 2015. The absolute-error, root mean square-error L_2 and maximum absolute-error L_∞ of the proposed technique are stated as

$$\begin{aligned} |\mathbf{E}(x_i, y_j)| &= |u(x_i, y_j) - \tilde{u}(x_i, y_j)|, \\ L_2 &= \sqrt{\frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}} |u(x_i, y_i) - \tilde{u}(x_i, y_i)|^2}, \\ L_\infty &= \max_{1 \le i \le \hat{m}} |u(x_i, y_i) - \tilde{u}(x_i, y_i)|. \end{aligned} \tag{39}$$

Problem 1. *Consider the following linear partial differential equation* (31) *for* $\mu_1 = \mu_2 = \mu_3 = 1$ *as* [15,16]

$$\frac{\partial^{3/2} u}{\partial x^{3/2}} + \frac{\partial^{3/4} u}{\partial x^{3/4}} + \frac{\partial^{4/3} u}{\partial y^{4/3}} + u(x,y) = F(x,y), \tag{40}$$

the Dirichlet boundary conditions associated with the above problem is given as

$$\begin{aligned} u(x,0) &= x^2, \qquad u(0,y) = y, \\ u(x,1) &= 1 + x^2, \qquad u(1,y) = 1 + y. \end{aligned} \tag{41}$$

where $F(x,y) = x^2 + y + 4\sqrt{x/\pi} + 16x^{5/4}\Gamma(3/4)\sqrt{2}/5\pi$. *The analytic solution of this problem is* $u(x,y) = x^2 + y$.

Step 1) According to the proposed methodology, we assume the following trial solution for $k = 1$, $M = 3$ to find the solution of the problem (40), (41):

$$\tilde{u}(x, y) = \Psi(x)^T \mathbf{C}\Psi(y), \tag{42}$$

where $\mathbf{C} = [a_{p,q}]$, is a square matrix of order 3×3 which is essential to be determined.

Step 2) After incorporating the expressions obtained in last step into eq. (40), (41), we get the following matrix equations:

$$\Psi(x)^T \left([\mathbf{K}^{3/2}]^T \mathbf{C} + [\mathbf{K}^{3/4}]^T \mathbf{C} + \mathbf{C}\mathbf{K}^{4/3} + \mathbf{C} \right) \Psi(y) - F(x, y) \triangleq \Lambda(x, y) \simeq 0, \tag{43}$$

and the Dirichlet boundary conditions takes the following form:

$$\begin{aligned} &\Phi_1(x) = \Psi(x)^T \mathbf{C}\Psi(0) - x^2 \simeq 0, \qquad \Phi_2(y) = \Psi(0)^T \mathbf{C}\Psi(y) - y \simeq 0, \\ &\Phi_3(x) = \Psi(x)^T \mathbf{C}\Psi(1) - 1 - x^2 \simeq 0, \qquad \Phi_4(y) = \Psi(1)^T \mathbf{C}\Psi(y) - 1 - y \simeq 0, \end{aligned} \tag{44}$$

where $\mathbf{K}^{3/2}$, $\mathbf{K}^{3/4}$, $\mathbf{K}^{4/3}$, $\mathbf{C}$, $\Psi(x)$ and $\Psi(y)$ are given as

$$\mathbf{K}^{\frac{3}{2}} = \frac{4x^{-\frac{3}{2}}}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 4 & 1 \end{bmatrix}, \qquad \mathbf{K}^{\frac{3}{4}} = \frac{2\sqrt{2}x^{-\frac{3}{4}}\Gamma(3/4)}{5\pi} \begin{bmatrix} 0 & 0 & 0 \\ 10 & 5 & 0 \\ 0 & 12 & 8 \end{bmatrix}, \qquad \mathbf{K}^{\frac{4}{3}} = \frac{3t^{-\frac{4}{3}}}{\Gamma(2/3)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 4 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}, \qquad \Psi(x) = \frac{2}{\sqrt{\pi}} \begin{bmatrix} 1 \\ 2(2x-1) \\ 16x^2 - 16x + 3 \end{bmatrix}, \qquad \Psi(y) = \frac{2}{\sqrt{\pi}} \begin{bmatrix} 1 \\ 2(2y-1) \\ 16y^2 - 16y + 3 \end{bmatrix}.$$

Step 3) To find $\mathbf{C}$, set the matrices equations (43), (44) as follows to construct a system of algebraic equations:

$$\begin{cases} \Lambda\left(\dfrac{i}{\hat{m}-1}, \dfrac{j}{\hat{m}-1}\right) = 0, & \text{for } i = j = 2, \\ \Phi_1\left(\dfrac{i}{\hat{m}}\right) = \Phi_3\left(\dfrac{i}{\hat{m}}\right) = 0, & \text{for } i = 2, \\ \Phi_2\left(\dfrac{j}{\hat{m}}\right) = \Phi_4\left(\dfrac{j}{\hat{m}}\right) = 0, & \text{for } j = 1, 2, 3. \end{cases} \tag{45}$$

Step 4) Finally, after solving the system of equations (45) we obtained $\mathbf{C}$ and the exact solution is achieved after inserting $\mathbf{C}$ into trial solution as

$$\tilde{u}(x, y) = \frac{4\pi}{\pi} \left[1 \;\; 2(2x-1) \;\; 16x^2 - 16x + 3 \right] \begin{bmatrix} \frac{13}{64} & \frac{1}{16} & 0 \\ \frac{1}{16} & 0 & 0 \\ \frac{1}{64} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2(2y-1) \\ 16y^2 - 16y + 3 \end{bmatrix} = x^2 + y.$$

Table 1 is constructed for comparison purpose: L_2 and L_∞ errors of the modified Gegenbauer wavelets method are zero while L_2 and L_∞ errors of the Legendre wavelets method [15] are non-zero. This table witnesses that the proposed algorithm is very simple but highly effective.

Problem 2. *Consider the following Laplace equation* (31) *for* $\mu_2 = 1$, $\mu_1 = \mu_3 = 0$, *as* [15]

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\gamma u}{\partial y^\gamma} = 0, \tag{46}$$

the Dirichlet boundary conditions associated with the above problem is given as

$$\begin{aligned} &u(x, 0) = 0, \qquad u(0, y) = 0, \\ &u(x, 1) = 0, \qquad u(1, y) = \sinh(2\pi)\sin(2\pi y). \end{aligned} \tag{47}$$

Table 1. Comparison of L_2 and L_∞ errors of the obtained method with the Legendre wavelets method [15] for different values of y.

y	L_2		L_∞	
	LWM [15]	Our method	LWM [15]	Our method
0.1	8.64×10^{-7}	0	4.90×10^{-7}	0
0.3	8.06×10^{-6}	0	1.47×10^{-6}	0
0.5	1.38×10^{-6}	0	2.28×10^{-6}	0
0.7	7.50×10^{-7}	0	1.17×10^{-6}	0
0.9	8.60×10^{-8}	0	6.45×10^{-6}	0
1.0	0	0	0	0

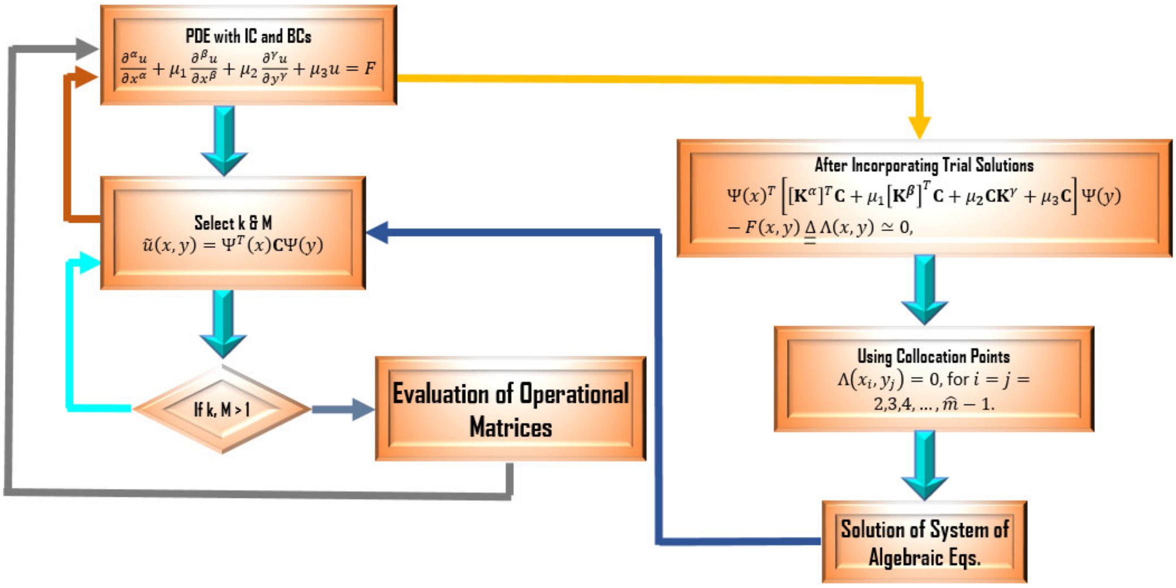

Fig. 1. Flow chart of the proposed method.

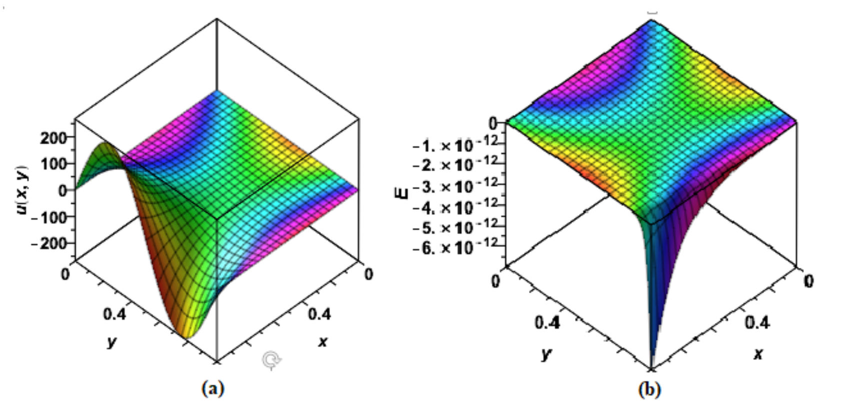

Fig. 2. (a) Approximate solutions for problem 2 for $\alpha = \gamma = 2$. (b) Absolute error analysis for $\alpha = \gamma = 2$.

The exact solution of problem (46), (47) is $u(x, y) = \sinh(2\pi x)\sin(2\pi y)$ for $\alpha = \gamma = 2$. Consider the trial solution $\tilde{u}(x, t) = \Psi(x)^T \mathbf{C}\Psi(t)$ for solving the problem (46), (47). Figure 1 is constructed for $k = 2$, $M = 5$. It is observed that our proposed method is highly effective and the graphical behavior of the absolute error shows that the suggested algorithm converges rapidly. Figure 2 depicts the approximate solution for various values of γ when $\alpha = 2$, $x = 1$. As γ approaches to 2 our obtained solution matches the graph of the exact solution. Figures 3(a), (b) show the approximate solution and absolute error for different values of γ. In short the proposed modified Gegenbauer wavelets method well-matches the solution of such type of physical problems.

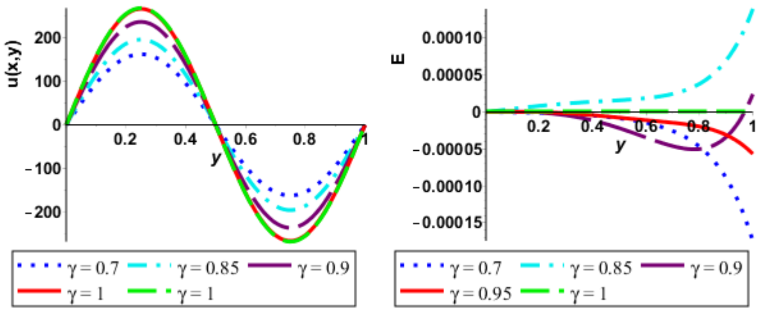

Fig. 3. (a) Approximate solution for different values of γ. (b) Absolute error analysis for different values where the green line shows the exact solution of γ.

Table 2. Comparison of L_2 and L_∞ errors of the obtained method with the Legendre wavelets method [15] for different values of y.

y	L_2		L_∞	
	LWM [15]	Our method	LWM [15]	Our method
0.1	5.50×10^{-4}	1.54×10^{-6}	8.64×10^{-7}	7.65×10^{-6}
0.3	9.61×10^{-5}	3.58×10^{-7}	1.47×10^{-6}	9.65×10^{-7}
0.5	9.43×10^{-6}	2.14×10^{-7}	2.28×10^{-6}	6.35×10^{-8}
0.7	5.41×10^{-6}	8.56×10^{-8}	1.17×10^{-6}	4.10×10^{-8}
0.9	6.53×10^{-7}	7.65×10^{-9}	6.45×10^{-6}	1.11×10^{-9}
1.0	0	0	0	0

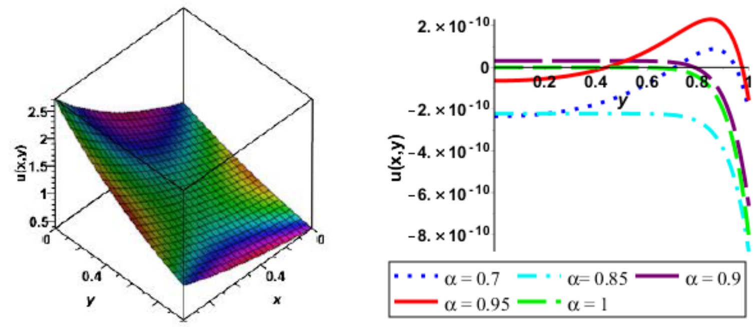

Fig. 4. (a) Approximate solution for $\alpha = \beta = 2$, $\gamma = 1$. (b) Absolute error for different values of α.

Problem 3. *Consider the following non-linear fractional partial differential equation* (31) *for* $2\mu_1 = -\mu_2 = 2\mu_3 = 2$ [15]*:*

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\beta u}{\partial x^\beta} - 2\frac{\partial^\gamma u}{\partial y^\gamma} + \ln(u(x,y)) = F(x,y). \tag{48}$$

The Dirichlet boundary conditions subject to (48) *are given as*

$$\begin{aligned} u(x,0) &= \mathrm{e}^{x}, \qquad u(0,y) = \mathrm{e}^{-y}, \\ u(x,1) &= \mathrm{e}^{x-1}, \qquad u(1,y) = \mathrm{e}^{1-y}. \end{aligned} \tag{49}$$

In the above $F(x,y) = x - y$. The analytic solution of this problem when $\alpha = \gamma = 2$, $\beta = 1$ is $u(x,y) = \mathrm{e}^{x-y}$. Suppose the trial solution $\tilde{u}(x,t) = \Psi(x)^T \mathbf{C} \Psi(t)$ for $k = 2$, $M = 3$. Table 2 represents the comparison of L_2 and L_∞ errors with the Legendre wavelets method [15]. This table evidences that the suggested method is very efficient and compatible. Figure 4(a) represents the surface plot of the approximate solution via the proposed method. Figure 4(b) is plotted to exhibit the absolute error for different values of α. Our obtained results are in excellent agreement with published work and exact solution.

8 Conclusion

The article deals with a new mathematical algorithm based on the Gegenbauer wavelet method for the fractional-ordered problems. The major motivation of the current study is to develop a Gegenbauer wavelet operational matrix of the derivative. Some new operational matrices for the derivative of fractional order with Dirichlet boundary condition has been developed with the help of piecewise functions. The extended Gegenbauer wavelets technique converts the given problem into a set of algebraic equations. Analytical solutions of the problem mentioned below are effectively obtained and the outcomes are compared with existing results. The outcomes found via Gegenbauer wavelets are endorsing the accuracy and effectiveness of the suggested technique. The convergence and error bound analysis is enclosed in our investigation as an evidence for the consistency and it also supports the mathematical formulation of the algorithm. It is observed that an accurate and efficient mathematical tool is used to tackle the non-linear fractional-order problems of complex nature and this method can further be extended to finding the non-linear dynamical problems.

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