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Nonclassical properties of induced states from single-mode squeezed vacuum state related with Hermite excited elementary superposition operation

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Abstract. Based on single-mode squeezed vacuum state (SVS) and Hermite-excited elementary superposition operator $H_m(xa^\dagger + ya)$, we induce two new quantum states, *i.e.*, Hermite-excited squeezed vacuum state (HSVS) and Hermite-excite-orthogonalized squeezed vacuum state (HOSVS). HSVS is obtained by applying the operator on SVS and HOSVS is obtained by applying the orthogonalizer on SVS, where HSVS is just HOSVS for odd m. We study and compare mathematical and nonclassical properties for SVS, HSVS and HOSVS, including photon number distribution, Mandel's Q parameter, quadrature squeezing, and Wigner function. Numerical results show that i) HSVS and HOSVS have only even (odd) photon components for even (odd) m; ii) HSVS and HOSVS can exhibit sub-Poissonian statistics in low-squeezing parameter regime and squeezing effect in large-squeezing parameter regime; iii) moreover, squeezing is always incompatible with sub-Poissonianity; iv) Wigner functions for HSVS and HOSVS have negative values in phase space.

1 Introduction

Quantum states of optical light are important resources for modern quantum technology, such as quantum teleportation and quantum metrology. In order to meet the requirements of engineering, a significant attention has been given to the preparation and manipulation of various nonclassical states. Therefore many theoretical and experimental ways are proposed to generate new quantum states [1]. Meanwhile, studying nonclassical states of light has become a very important topic of quantum optics [2,3]. One theoretical way is to obtain states by superposing two or more known quantum states. For example, Schrodinger cat state is a superposition of two coherent states [4–6]. Another common way is to applying an operator on a known quantum state. For example, the famous photon-added coherent state [7] is obtained by applying the photon-added operator on the coherent state [8,9]. As another example, new quantum states can be generated by subtracting or adding photons from or to traditional quantum states, which have received more attention from both experimentalists and theoreticians [10–16]. Of course, there are many other new ways to generate quantum states.

Recently, finding the orthogonal state $|\psi_\perp\rangle$ for a known pure quantum state $|\psi\rangle$ has become a novel way to generate new states. The generalized orthogonalization procedure to infinite-dimensional, continuous-variable pure state was proposed by Coelho *et al.* [17]. The basic steps include: i) Giving an operator $\hat{C}$ and an pure state $|\psi\rangle$; ii) Calculating the mean value $\langle\hat{C}\rangle_{|\psi\rangle}$ for the state $|\psi\rangle$; iii) Constructing the orthogonalizer $\hat{O}_C \equiv \hat{C} - \langle\hat{C}\rangle_{|\psi\rangle}\hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ is the identity operator; iv) Applying $\hat{O}_C$ on $|\psi\rangle$ and obtain orthogonal state $|\psi_\perp\rangle$. This technique has become a useful tool for quantum state engineering, which can help us to produce custom-made quantum states. That is to say, the orthogonalizer is general enough to work based on any operator $\hat{C}$ and with any pure input state $|\psi\rangle$ [18,19].

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Since more than twenty years ago, Hermit polynomial states have been attracted extensive interest of researchers. Theoretically, these states are constructed by applying a squeezed operator to a state that consists of a Hermite polynomial. There are two viewpoints about these states. One is that they are a subset of the minimum uncertainty states for amplitude-squared squeezing, and another is that they are solutions of an eigenvalue equation which is quadratic in creation and annihilation operators. That is to say, the Hermitian polynomial states are related with the quadrature quantities $\hat{Y}_1 = (a^2 + a^{\dagger 2})/\sqrt{2}$ and $\hat{Y}_2 = (a^2 - a^{\dagger 2})/(i\sqrt{2})$. Some properties for these states, including squeezing effect, sub-Poissonian photon statistics, and quasiprobability distributions have been presented in the works of Bergou [20] and Datta [21]. Some other nonclassicality are also studied by Tan [22].

Inspired by the orthogonalization idea and some works relevant with Hermite polynomial squeezed state, we take $H_m(xa^\dagger + ya)$ as operator $\hat{C}$ and single-mode squeezed vacuum state (SVS) [23] as state $|\psi\rangle$ and theoretically construct two non-Gaussian quantum states in this paper. Hermite-excited squeezed vacuum state (HSVS) is obtained by operating $\hat{C}$ on SVS directly and Hermite-excite-orthogonalized squeezed vacuum state (HOSVS) is obtained by orthogonalization procedure. It should be noted that $xa^\dagger + ya$ is the elementary superposition operator from creation operator $a^\dagger$ and annihilation operator a with corresponding coefficients x and y. Moreover, $H_m(z)$ is the m-th–order Hermite polynomials [24], which can also be expressed as $H_m(z) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} e^{2zt-t^2}|_{t=0}$. Mathematical and physical properties for such states are studied in detail. These nonclassical properties include photon number distribution, Mandel's Q parameter, quadrature squeezing, and Wigner function. By adjusting all relative parameters, we can obtain the different quantum states with abundant properties.

The paper is organized as follows: In sect. 2 we introduce three quantum states considered in this paper. In sect. 3 we present preliminary techniques, which is quite useful in dealing with problems. In sect. 4 we give the uniform expressions of physical quantities which connect all these quantum states. In sect. 4 we analyze the photon number distributions for these states. In sect. 5 the statistical properties are studied including Mandel's Q parameter and quadrature squeezing. Then in sect. 6 we discuss the Wigner functions whose negativity shows the nonclassicality. Conclusion is given in the last section.

2 Three quantum states

In this section, we first introduce three quantum states. One is the single-mode squeezed vacuum state (SVS), which is familiar to everyone in the field of quantum optics. The other two are induced from the SVS by operating the operators related to $\hat{C} = H_m(\hat{V})$ with $\hat{V} = xa^\dagger + ya$ in different ways.

Mathematically, one can generate the SVS by operating S_r on the vacuum state $|0\rangle$,

$$|\psi\rangle = S_r|0\rangle, \tag{1}$$

where $S_r = e^{\frac{r}{2}(a^{\dagger 2} - a^2)}$ is single-mode squeezing operator with real squeezing parameter r.

Applying $\hat{C}$ on SVS, we obtain Hermite-excited SVS

$$|\psi_C\rangle = \frac{1}{\sqrt{N_C}}\hat{C}|\psi\rangle, \tag{2}$$

where $N_C = \langle\psi|\hat{C}^\dagger\hat{C}|\psi\rangle$ is the normalization coefficient.

By operating the orthogonalizer on SVS according to the orthogonalization procedure, we obtain Hermite-excite-orthogonalized SVS,

$$|\psi_\perp\rangle = \frac{1}{\sqrt{N_\perp}}\left(\hat{C} - \langle\hat{C}\rangle_{|\psi\rangle}\hat{\mathbf{1}}\right)|\psi\rangle, \tag{3}$$

where $N_\perp = N_C - |\langle\hat{C}\rangle_{|\psi\rangle}|^2$ is the normalization coefficient with $\langle\hat{C}\rangle_{|\psi\rangle} = \langle\psi|\hat{C}|\psi\rangle$.

The simple case is $m = 1$. The HSVS and the HOSVS are just $S_r|1\rangle$ (*i.e.*, squeezed one-photon state), which is independent of x and y. Moreover, when m is odd, we find that the identity $\langle\hat{C}\rangle_{|\psi\rangle} = 0$ always holds, which means that HOSVS is the same as HSVS when the integer m is odd. Only when the integer m is even, HOSVS is different from HSVS due to $\langle\hat{C}\rangle_{|\psi\rangle} \neq 0$. There are many adjusted parameters include r, m, x and y for these quantum states. By adjusting these parameters, we can analyze and compare their statistical properties. The key is to analyze their rich nonclassical characteristics.

3 Preliminary techniques

Some useful techniques are introduced in this section, which can help us to calculate normalization factor and all statistical properties.

3.1 Technique for the SVS

The squeezing operator S_r is unitary since $S_r^\dagger = S_{-r}$, which is the inverse of S_r. The factored form of the operator S_r can be written as [25]

$$S_r = (1-\lambda^2)^{1/4} e^{\frac{\lambda}{2}a^{\dagger 2}} \left(\sqrt{1-\lambda^2}\right)^{a^\dagger a} e^{-\frac{\lambda}{2}a^2}, \tag{4}$$

with $\lambda = \tanh r$. Then by operating it on the vacuum state $|0\rangle$, we obtain another expression of SVS with $|\psi\rangle = (1-\lambda^2)^{1/4} e^{\frac{\lambda}{2}a^{\dagger 2}}|0\rangle$. Using the completeness of the coherent state $\int \frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \hat{\mathbf{1}}$ [26], we can write the SVS in the following form:

$$|\psi\rangle = (1-\lambda^2)^{1/4} \int \frac{\mathrm{d}^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{2}+\frac{\lambda}{2}\alpha^{*2}}|\alpha\rangle, \tag{5}$$

which is an integral in the coherent state representation. Similarly, we also have its conjugate state

$$\langle\psi| = (1-\lambda^2)^{1/4} \int \frac{\mathrm{d}^2\beta}{\pi}\langle\beta| e^{-\frac{|\beta|^2}{2}+\frac{\lambda}{2}\beta^{2}}. \tag{6}$$

The fact that the SVS is expressed as an integral in the coherent state representation can help us deal with all subsequent calculations.

3.2 Technique for the operator

Using the so-called Baker-Hausdorff formula $e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B$ with $[[A,B],A] = [[A,B],B] = 0$ [27], we expand $H_m(\hat{V})$ and $H_m^\dagger(\hat{V})$ into the following normal ordering form:

$$H_{m_1}(\hat{V}) = \frac{\mathrm{d}^{m_1}}{\mathrm{d}t^{m_1}} e^{(2xy-1)t^2} e^{2txa^\dagger} e^{2tya}|_{t=0} \tag{7}$$

and

$$H_{m_2}^\dagger(\hat{V}) = \frac{\mathrm{d}^{m_2}}{\mathrm{d}s^{m_2}} e^{(2x^*y^*-1)s^2} e^{2sy^*a^\dagger} e^{2sx^*a}|_{s=0}, \tag{8}$$

respectively. These two normal ordering forms are very useful in the process of calculation.

3.3 Two useful basic expressions

Combinating eqs. (5) and (7), we have

$$H_{m_1}(\hat{V})|\psi\rangle = (1-\lambda^2)^{1/4} \frac{\mathrm{d}^{m_1}}{\mathrm{d}t^{m_1}} e^{(2xy-1)t^2} \int \frac{\mathrm{d}^2\alpha}{\pi} e^{-|\alpha|^2+2ty\alpha+\frac{\lambda}{2}\alpha^{*2}} e^{(2tx+\alpha)a^\dagger}|0\rangle|_{t=0}. \tag{9}$$

Similarly, combinating eqs. (6) and (8), we have

$$\langle\psi|H_{m_2}^\dagger(\hat{V}) = (1-\lambda^2)^{1/4} \frac{\mathrm{d}^{m_2}}{\mathrm{d}s^{m_2}} e^{(2x^*y^*-1)s^2} \int \frac{\mathrm{d}^2\beta}{\pi}\langle 0| e^{(2sx^*+\beta^*)a} e^{-|\beta|^2+2sy^*\beta^*+\frac{\lambda}{2}\beta^2}|_{s=0}. \tag{10}$$

3.4 Normalization coefficients

Using eqs. (9) and (10) with proper m_1 and m_2, we finally obtain

$$\langle\hat{C}\rangle_{|\psi\rangle} = \frac{\mathrm{d}^m}{\mathrm{d}t^m} e^{-t^2+\frac{2\lambda}{1-\lambda^2}t^2(x^2+y^2)+2\frac{1+\lambda^2}{1-\lambda^2}t^2xy}|_{t=0} \tag{11}$$

and

$$\begin{aligned} N_C = \frac{\mathrm{d}^{2m}}{\mathrm{d}s^m\mathrm{d}t^m} & e^{-t^2-s^2+2\frac{1+\lambda^2}{1-\lambda^2}(t^2xy+s^2x^*y^*)} \\ & e^{+\frac{2\lambda}{1-\lambda^2}(t^2(x^2+y^2)+s^2(x^{*2}+y^{*2}))} \\ & e^{+\frac{4}{1-\lambda^2}st(xx^*+\lambda^2yy^*)+\frac{4\lambda}{1-\lambda^2}st(xy^*+x^*y)}|_{s=t=0}. \end{aligned} \tag{12}$$

Then $N_\perp$ can be obtained.

4 Uniform expressions of physical quantities

Uniform expression relevant with any considered quantity $\hat{M}$ can be expressed as

$$F(\hat{M};m_1,m_2)=\langle\psi|H^{\dagger}_{m_2}(\hat{V})\hat{M}H_{m_1}(\hat{V})|\psi\rangle. \tag{13}$$

As long as we know the expression in eq. (13), we can obtain the expectation value $\langle\hat{M}\rangle$ for SVS, HSVS and HOSVS by choosing the appropriate integers m_1, m_2. The procedures are illustrated as follows.

I) For SVS, we have

$$\langle\hat{M}\rangle_{|\psi\rangle}=F(\hat{M};0,0). \tag{14}$$

II) For HSVS, we have

$$\langle\hat{M}\rangle_{|\psi_C\rangle}=\frac{F(\hat{M};m,m)}{N_C}. \tag{15}$$

III) For HOSVS, we have

$$\langle\hat{M}\rangle_{|\psi_\perp\rangle}=\frac{F(\hat{M};m,m)}{N_\perp}-\frac{F(\hat{M};0,m)\langle\hat{C}\rangle_{|\psi\rangle}}{N_\perp}-\frac{\langle\hat{C}\rangle^*_{|\psi\rangle}F(\hat{M};m,0)}{N_\perp}+\frac{\langle\hat{C}\rangle^*_{|\psi\rangle}F(\hat{M};0,0)\langle\hat{C}\rangle_{|\psi\rangle}}{N_\perp}. \tag{16}$$

In our following work, we shall take a quantity $\hat{M}$ for each considered property.

5 Photon number distribution

In this section, we discuss photon number distribution (PND) of SVS, HSVS and HOSVS. The PND is the probability of finding n photons in the optical field ρ, which can be given by

$$P(n)=\langle n|\rho|n\rangle=\mathrm{Tr}\left[|n\rangle\langle n|\rho\right]. \tag{17}$$

Taking $\hat{M}_n=|n\rangle\langle n|$, which can be further expressed as

$$\hat{M}_n=\frac{1}{n!}\frac{\mathrm{d}^{2n}}{\mathrm{d}\mu^n\mathrm{d}\nu^n}e^{\mu a^\dagger}|0\rangle\langle 0|e^{\nu a}|_{\mu=\nu=0}, \tag{18}$$

substituting it into eq. (13), and using eqs. (9) and (10), we finally obtain

$$\begin{aligned}F(\hat{M}_n;m_1,m_2)=&\frac{(1-\lambda^2)^{1/2}}{n!}\frac{\mathrm{d}^{m_1+m_2+2n}}{\mathrm{d}t^{m_1}\mathrm{d}s^{m_2}\mathrm{d}\mu^n\mathrm{d}\nu^n}\\&e^{-t^2-s^2+\frac{\lambda}{2}(\mu^2+\nu^2)+2(t^2xy+s^2x^*y^*)+2\lambda(t^2y^2+s^2y^{*2})}\\&e^{+2(\nu tx+\mu sx^*)+2\lambda(\nu ty+\mu sy^*)}|_{\mu=\nu=s=t=0}.\end{aligned} \tag{19}$$

Using eqs. (14) and (19), we obtain the PND of SVS, which consists of a superposition only of the even-number states, as one can see from fig. 1(a). Combinating eq. (19) with eqs. (15) and (16), we obtain the PNDS of HSVS and HOSVS, respectively. In the special case with $m=1$, HSVS and HOSVS are just $S_r|1\rangle$, whose PND consists of a superposition of only the odd-number states (see from fig. 1(b)). By fixing $m=2$ and $r=0.5$, we plot the PNDs of HSVS and HOSVS with different x and y in fig. 2, which consist of only even-number states. Similarly, we plot the PNDs of HSVS and HOSVS by fixing $m=3$ and $r=0.5$ in fig. 3, which consist of only odd-number states.

6 Quantum statistical properties

In order to discuss the properties of SVS, HSVS and HOSVS, we derive the general expression of $\langle a^{\dagger k}a^l\rangle$ for each quantum states, where k, l can be chosen cleverly according to the circumstance. Substituting $\hat{M}=a^{\dagger k}a^l$ with

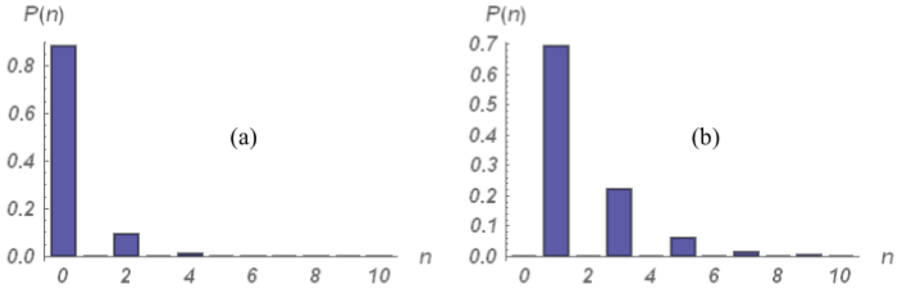

Fig. 1. PNDs for (a) $S_r|0\rangle$ and (b) $S_r|1\rangle$ with $r = 0.5$.

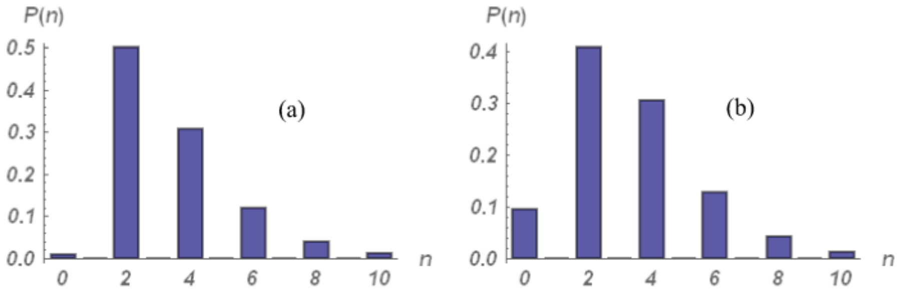

Fig. 2. PNDs for (a) HSVS and (b) HOSVS with $m = 2$, $r = 0.5$ and $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$.

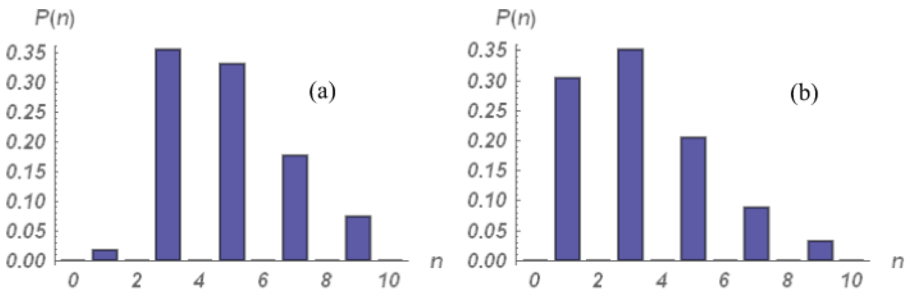

Fig. 3. PNDs for HSVS (or HOSVS) with $m = 3$, $r = 0.5$ but (a) $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$ and (b) $x = -i/\sqrt{2}$, $y = i/\sqrt{2}$.

$a^{\dagger k} = \frac{\mathrm{d}^k}{\mathrm{d}\mu^k} e^{\mu a^\dagger}|_{\mu=0}$ and $a^l = \frac{\mathrm{d}^l}{\mathrm{d}\nu^l} e^{\nu a}|_{\nu=0}$ into eq. (13), and using eqs. (9) and (10), we obtain the general expression of the form

$$\begin{aligned} F(a^{\dagger k}a^l; m_1, m_2) = {} & \frac{\mathrm{d}^{m_1+m_2+k+l}}{\mathrm{d}t^{m_1}\mathrm{d}s^{m_2}\mathrm{d}\mu^k d\nu^l} e^{-t^2-s^2+\frac{1}{2}\frac{\lambda}{1-\lambda^2}(\mu^2+\nu^2)+\frac{\lambda^2}{1-\lambda^2}\mu\nu} \\ & e^{+\frac{2\lambda}{1-\lambda^2}(t^2(x^2+y^2)+s^2(x^{*2}+y^{*2}))+2\frac{1+\lambda^2}{1-\lambda^2}(t^2xy+s^2x^*y^*)} \\ & e^{+2\frac{\nu+\lambda\mu}{1-\lambda^2}tx+2\frac{\mu+\lambda\nu}{1-\lambda^2}sx^*+\frac{2\lambda(\nu+\lambda\mu)}{1-\lambda^2}ty+\frac{2\lambda(\lambda\nu+\mu)}{1-\lambda^2}sy^*} \\ & e^{+\frac{4\lambda}{1-\lambda^2}st(xy^*+x^*y)+\frac{4}{1-\lambda^2}st(xx^*+\lambda^2yy^*)}|_{\mu=\nu=s=t=0}. \end{aligned} \tag{20}$$

The above expression is of importance for further investigating the properties of SVS, HSVS and HOSVS. According to eq. (14), eq. (15) and eq. (16), we can obtain the analytical expressions of $\langle a^{\dagger k}a^l\rangle$ for SVS, HSVS and HOSVS, respectively. Of course, we must notice the choice of m_1 and m_2.

6.1 Mandel's Q parameter

As pointed out by Bergou, Hermite polynomial states may or may not have sub-Poissonian photon statistics [20]. This prompts us to discuss the sub-Poissonianity of the HSVS and the HOSVS. A simple way to gauge the nature of the photon statisitcs of any quantum state is to calculate the Mandel's Q parameter [28],

$$Q_M = \frac{\langle a^{\dagger 2}a^2\rangle}{\langle a^\dagger a\rangle} - \langle a^\dagger a\rangle, \tag{21}$$

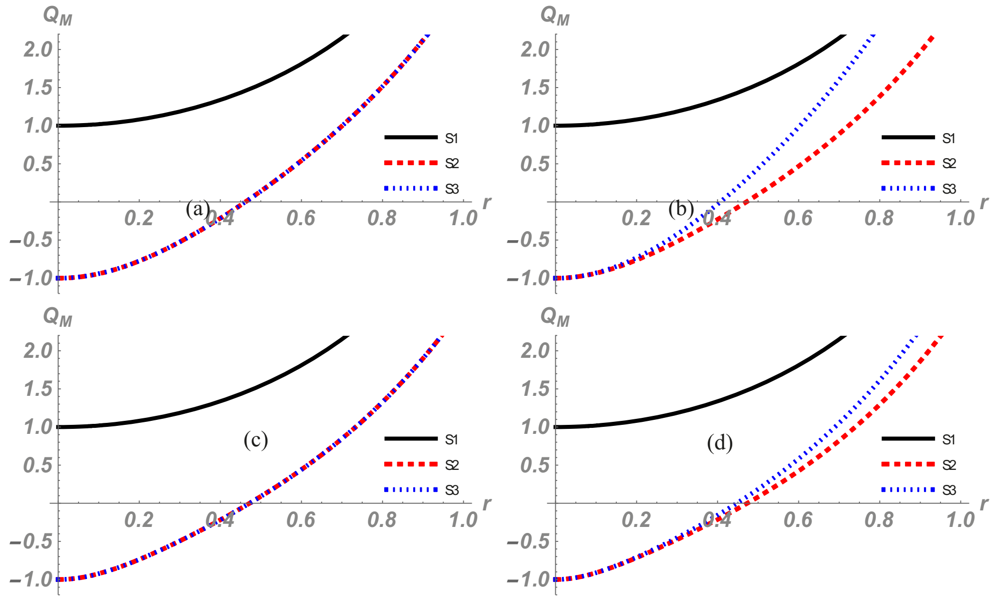

Fig. 4. Q_M *versus* r with fixed $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$ for SVS (black solid), HSVS (red dashed), HOSVS (blue dotted), respectively: (a) $m = 1$; (b) $m = 2$; (c) $m = 3$; (d) $m = 4$.

which measures the deviation of the PND of the optical field state under consideration from the Poissonian distribution. For a state with Q_M in the range of $-1 \le Q_M < 0$, the statistics is sub-Poissonian, and if $Q_M > 0$, the statistics is super-Poissonian. Obviously, $Q_M = 0$ for a coherent state with Poissonian photon statistics.

Some special cases: 1) For the SVS, we know $Q_M|_{|\psi\rangle} = \cosh 2r$; 2) If $m = 1$, then $|\psi_C\rangle = |\psi_\perp\rangle = S_r|1\rangle$ and $Q_M|_{S_r|1\rangle} = -\frac{2}{3} + \frac{8}{3-9\cosh 2r} + \cosh 2r$, which is independent of x, y. As is shown in fig. 4, with fixed $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$, one can see clearly that Q_M of SVS is always larger than 1, which shows that SVS has no possibility of sub-Poissonian distribution. While for HSVS and HOSVS, Q_M is negative when the squeezing parameter r is between 0 and a certain threshold value r_c, which shows the sub-Poissonian characteristic. For instance, $r_c \simeq 0.470836$ for HSVS and $r_c \simeq 0.406702$ for HOSVS in fig. 4(b).

6.2 Quadrature squeezing effects

There are many ways to definte squeezing. As pointed out by Bergou [20], the Hermite polynomial states may or may not be squeezed in the normal sense. This normal squeezing is just the quadrature squeezing we will consider in this subsection. Clearly, the SVS being a squeezed vacuum is squeezed for all $r \neq 0$. It is of interest to determine where the HSVS and the HOSVS are squeezed in the normal sense.

Coordinate operator $\hat{X} = (a + a^\dagger)/\sqrt{2}$ and momentum operator $\hat{P} = (a - a^\dagger)/(i\sqrt{2})$ are the quadrature operators of quantum fields, satisfying $[\hat{X}, \hat{P}] = i$. Their variances $\Delta^2\hat{X} = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$ and $\Delta^2\hat{P} = \langle\hat{P}^2\rangle - \langle\hat{P}\rangle^2$ can further be expressed as

$$\Delta^2\hat{X} = \langle a^\dagger a\rangle - |\langle a^\dagger\rangle|^2 + Re\left(\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2\right) + \frac{1}{2} \tag{22}$$

and

$$\Delta^2\hat{P} = \langle a^\dagger a\rangle - |\langle a^\dagger\rangle|^2 - Re\left(\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2\right) + \frac{1}{2}, \tag{23}$$

which satisfy the uncertainty relation of quantum mechanics $\Delta\hat{X}\Delta\hat{P} \ge 1/2$. Obviously, the variances $\Delta^2\hat{X}$ and $\Delta^2\hat{P}$ for the coherent state (or vacuum state) is equal to $1/2$. Squeezing has, by definition, a characteristic that $\Delta^2\hat{X}$ or $\Delta^2\hat{P}$ is smaller than $1/2$ [29].

Next, let us analyze some special cases. For the SVS (*i.e.* $S_r|0\rangle$), we know $\Delta^2\hat{X}|_{|\psi\rangle} = \frac{1}{2}e^{2r}$, $\Delta^2\hat{P}|_{|\psi\rangle} = \frac{1}{2}e^{-2r}$, and $\Delta\hat{X}\Delta\hat{P} = 1/2$, which is just a standard squeezing case. Moreover, when $m = 1$, HSVS and HOSVS are corresponding to $S_r|1\rangle$, which lead to $\Delta^2\hat{X}|_{|\psi_C\rangle} = \Delta^2\hat{X}|_{|\psi_\perp\rangle} = \frac{3}{2}e^{2r}$, $\Delta^2\hat{P}|_{|\psi_C\rangle} = \Delta^2\hat{P}|_{|\psi_\perp\rangle} = \frac{3}{2}e^{-2r}$, and $\Delta\hat{X}\Delta\hat{P} = 3/2$. The results show that $\Delta^2\hat{X}$ and $\Delta^2\hat{P}$ are independent of x, y in this case and that HSVS and HOSVS are squeezed in the "p-direction" when $\frac{3}{2}e^{-2r} < \frac{1}{2}$, *i.e.* $r > \frac{\ln 3}{2} \approx 0.549306$.

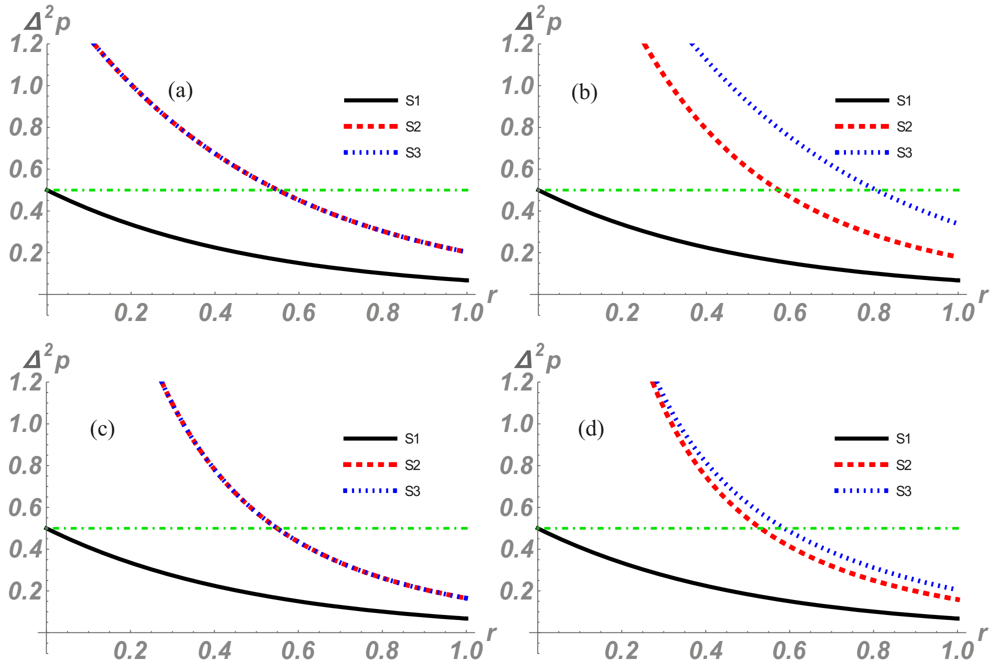

Fig. 5. $\Delta^2\hat{P}$ *versus* r with fixed $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$ for SVS (black solid), HSVS (red dashed), HOSVS (blue dotted), respectively: (a) $m = 1$; (b) $m = 2$; (c) $m = 3$; (d) $m = 4$.

In order to see clearly the value $\Delta^2\hat{P}$ with other parameter m, we plot $\Delta^2\hat{P}$ *versus* r in fig. 5, with fixed $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$. From these figures, one can see that $\Delta^2\hat{P}$ for SVS is always less than $1/2$ and $\Delta^2\hat{P}$ for HSVS (and HOSVS) is less than $1/2$ only when the squeezing parameter r exceeds a certain threshold value r_c. For instance, $r_c \simeq 0.572106$ for HSVS and $r_c \simeq 0.804719$ for HOSVS in fig. 5(b). That is to say, only when $r > r_c$, there exists the squeezing effects for HSVS and HOSVS. Furthermore, one can see clearly that $\Delta^2\hat{P}$ for HSVS is same as that for HOSVS for odd number m and $\Delta^2\hat{P}$ for HSVS is different from that for HOSVS for even number m.

Figure 4 shows that the Mandel parameter for HSVS and HOSVS can be negative for the low squeezing parameter, whereas fig. 5 shows that squeezing happens for the high squeezing parameter. Is it possible to find such domains of parameters (r, m, x, y) where squeezing and sub-Poissonianity could co-exist, or squeezing is always incompatible with sub-Poissonianity? By using the scientific computing software MATHEMATICA, we try our best to find the domains of parameters (r, m, x, y) satisfying both $-1 \leq Q_M < 0$ and $\Delta^2\hat{P} < 1/2$, but fail. Thus, the fact is that squeezing is always incompatible with sub-Poissonianity for HSVS and HOSVS.

7 Wigner function

The Wigner function is a powerful tool with which to investigate the nonclassicality of optical fields. Its partial negativity implies the highly nonclassical properties of quantum states [30–33]. In this section, we derive the analytical expressions of the Wigner functions for SVS, HSVS and HOSVS and then make numerical simulation to analyze their nonclassicality.

Wigner function $W(\zeta)$ for a quantum state ρ can be measured through a combination of coherent displacements and parity measurement, that is

$$W(\zeta) = \frac{2}{\pi}\,\mathrm{Tr}\left[\hat{\Pi} D^\dagger(\zeta)\rho D(\zeta)\right], \tag{24}$$

where $D(\zeta) = e^{\zeta a^\dagger - \zeta^* a}$ is the usual displacement operator with $\zeta = \frac{q+ip}{\sqrt{2}}$, and $\hat{\Pi} = (-1)^{a^\dagger a}$ is the parity operator. In order to uniform the calculation of the Wigner function for SVS, HSVS and HOSVS, we assume $\hat{M}_W = \frac{2}{\pi} D(\zeta)\hat{\Pi} D^\dagger(\zeta)$. According to $D(\zeta) a D^\dagger(\zeta) = a - \zeta$ and $(-1)^{a^\dagger a} = e^{i\pi a^\dagger a} =: e^{-2a^\dagger a}:$, we immediately obtain the normal ordering form as follows:

$$\hat{M}_W = \frac{2}{\pi} :\, \exp\left(-2\left(a^\dagger - \zeta^*\right)(a - \zeta)\right) : , \tag{25}$$

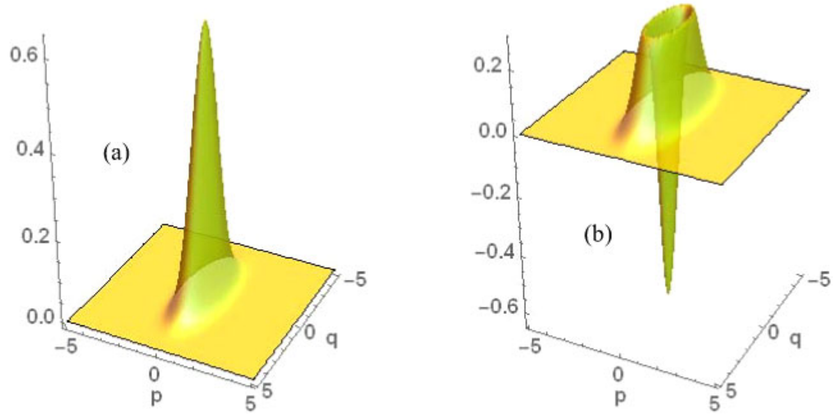

Fig. 6. Wigner functions for (a) $S_r|0\rangle$ and (b) $S_r|1\rangle$ with $r = 0.5$.

where the symbol : : denotes normal ordering. Substituting it into eq. (13) and using eqs. (9) and (10), we finally obtain the explicit expression

$$\begin{aligned} F(\hat{M}_W; m_1, m_2) = {} & \frac{2}{\pi} e^{-2\frac{1+\lambda^2}{1-\lambda^2}|\zeta|^2 + \frac{2\lambda}{1-\lambda^2}(\zeta^2+\zeta^{*2})} \\ & \frac{\mathrm{d}^{m_1+m_2}}{\mathrm{d}t^{m_1}\mathrm{d}s^{m_2}} e^{-t^2-s^2-\frac{4}{1-\lambda^2}st(xx^*+\lambda^2 yy^*)-\frac{4\lambda}{1-\lambda^2}st(xy^*+x^*y)} \\ & e^{\frac{4}{1-\lambda^2}(t\zeta^* x+s\zeta x^*)-\frac{4\lambda^2}{1-\lambda^2}(t\zeta y+s\zeta^* y^*)-\frac{4\lambda}{1-\lambda^2}(t(\zeta x-\zeta^* y)+s(\zeta^* x^*-\zeta y^*))} \\ & e^{\frac{2\lambda}{1-\lambda^2}(t^2(x^2+y^2)+s^2(x^{*2}+y^{*2}))+2\frac{1+\lambda^2}{1-\lambda^2}(t^2 xy+s^2 x^* y^*)}\Big|_{s=t=0}. \end{aligned} \tag{26}$$

According to eq. (14) and choosing $m_1 = m_2 = 0$, eq. (26) can be reduced to the Wigner function of the SVS with the following form:

$$W_{|\psi\rangle}(\zeta) = \frac{2}{\pi} e^{-2\frac{1+\lambda^2}{1-\lambda^2}|\zeta|^2 + \frac{2\lambda}{1-\lambda^2}(\zeta^2+\zeta^{*2})}. \tag{27}$$

Obviously, this is a real function and is Gaussian in phase space (see fig. 6(a)).

In special case $m = 1$, the Wigner functions for HSVS and HOSVS have the following form:

$$W_{S_r|1\rangle}(\zeta) = \frac{2}{\pi} G(\lambda;\zeta) e^{-2\frac{1+\lambda^2}{1-\lambda^2}|\zeta|^2 + \frac{2\lambda}{1-\lambda^2}(\zeta^2+\zeta^{*2})}, \tag{28}$$

with

$$G(\lambda;\zeta) = \frac{4(1+\lambda^2)|\zeta|^2}{1-\lambda^2} - \frac{4\lambda(\zeta^2+\zeta^{*2})}{1-\lambda^2} - 1. \tag{29}$$

It corresponds to the Wigner function of $S_r|1\rangle$. This is a non-Gaussian function because of the existence of the term $G(\lambda;\zeta)$ and if the region satisfies $G(\lambda;\zeta) < 0$, the Wigner function will show negativity (see fig. 6(b)).

For the even-m case, the HSVS is different from the HOSVS. While for the odd-m case, the HSVS is same as the HOSVS. In order to see the differences more clearly, the phase-space Wigner distributions of HSVS and HOSVS are depicted in figs. 7 and 8 for several different parameter values of x, y, m and fixed $r = 0.5$. In fig. 7, we show the Wigner functions of the HSVS and the HOSVS with same parameters ($m = 2$, $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$). In fig. 8, we show the Wigner functions for HSVS or HOSVS with $m = 3$ but different x, y. Obviously, squeezing in one of quadratures, as evidence of the nonclassicality of the state, is clear in these plots. In additions, there are some negative regions of the Wigner functions in the phase space, which is another indicator of the nonclassicality of these states.

8 Conclusion and discussion

In summary, we study and compare properties for SVS, HSVS and HOSVS, which are related to the SVS $S_r|0\rangle$ and the operator $H_m(xa^\dagger + ya)$. Mathematical and physical properties of SVS, HSVS and HOSVS are discussed in detail. Some useful techniques are employed in the processes of deriving the analytical expressions of normalization coefficients and all physical quantities. For example, the SVS is expressed as an integral in the coherent state representation and the operator is expressed in normal ordering form. These techniques are helpful in the calculation of whole work.

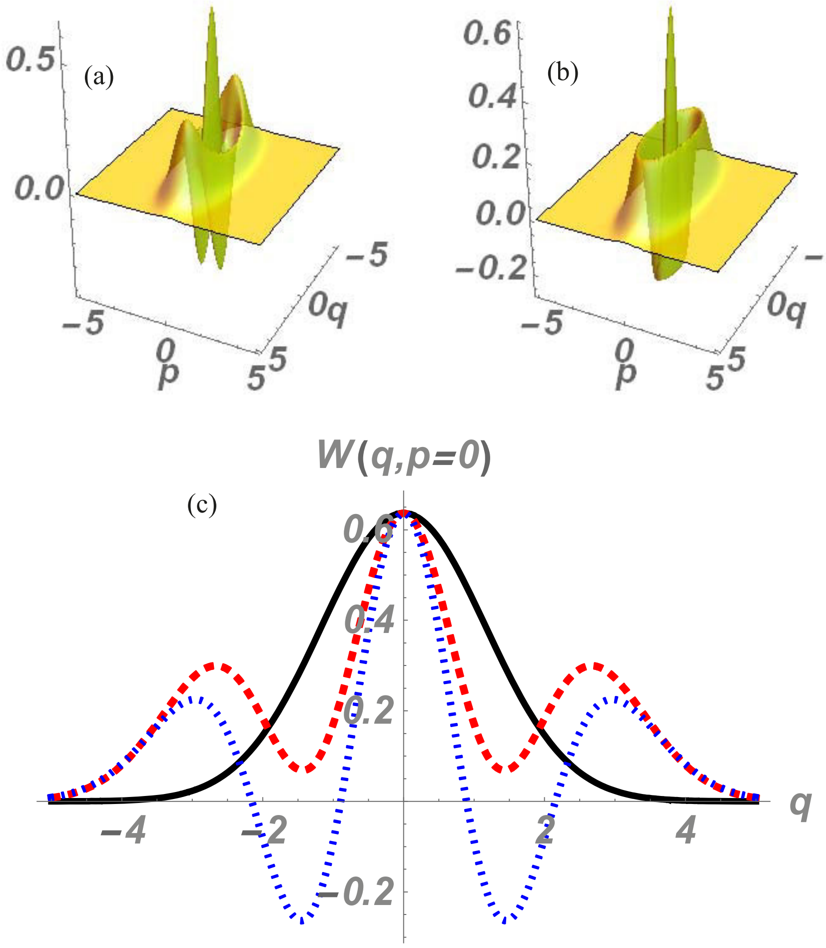

Fig. 7. Wigner functions for (a) HSVS and (b) HOSVS with $m = 2$, $r = 0.5$ and $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$. (c) Cross-sections for Wigner functions. Here, the black solid, red dashed, and blue dotted lines are corresponding to SVS, case (a), and case (b), respectively.

Furthermore, we give a unified relationship for each property to connect SVS, HSVS and HOSVS because of their links. This is another success point in this paper. HSVS and HOSVS have their respective special properties which is different from the traditional Gaussian SVS. Since HSVS and HOSVS are related to many parameters, one can choose them appropriately to present their peculiar nonclassicality.

The properties we discussed in this paper include PND, Mandel's Q parameter, quadrature squeezing, and Wigner function. Some results are summarized as follows: 1) HSVS and HOSVS are just squeezed one-photon state $S_r|1\rangle$ in the case of $m = 1$, which is independent of x and y; 2) HSVS and HOSVS are same for the odd-m case; 3) HSVS and HOSVS have only even-(odd-) photon components when m is even (odd); 4) HSVS and HOSVS can exhibit sub-Poissonian character in the low-squeezing parameter range, but SVS has no chance to exhibit sub-Poissonian; 5) HSVS and HOSVS can exhibit squeezing character in the large-squeezing parameter range, but SVS always has squeezing for any nonzero r; 6) HSVS and HOSVS can exhibit their non-Gaussian and nonclassical properties, which can be judged from their Wigner functions in phase space.

Indeed, every new quantum state induced from original state has its own properties. We can compare and analyze the properties of these three quantum states considered in this paper. The main features are as follows: i) the SVS always shows squeezing for any nonzero squeezing parameter r, but the HSVS and the HOSVS show squeezing only for large squeezing parameter r. Moreover, the degrees of squeezing for HSVS and HOSVS are even smaller than that for SVS. ii) The HSVS and the HOSVS have sub-Poissonian photon statistics in low squeezing parameter range, but the SVS has no sub-Poissonian photon statistics for whole squeezing parameter range. iii) The Wigner function for the SVS has no negative values, but the Wigner functions for the HSVS and the HOSVS may have negative values.

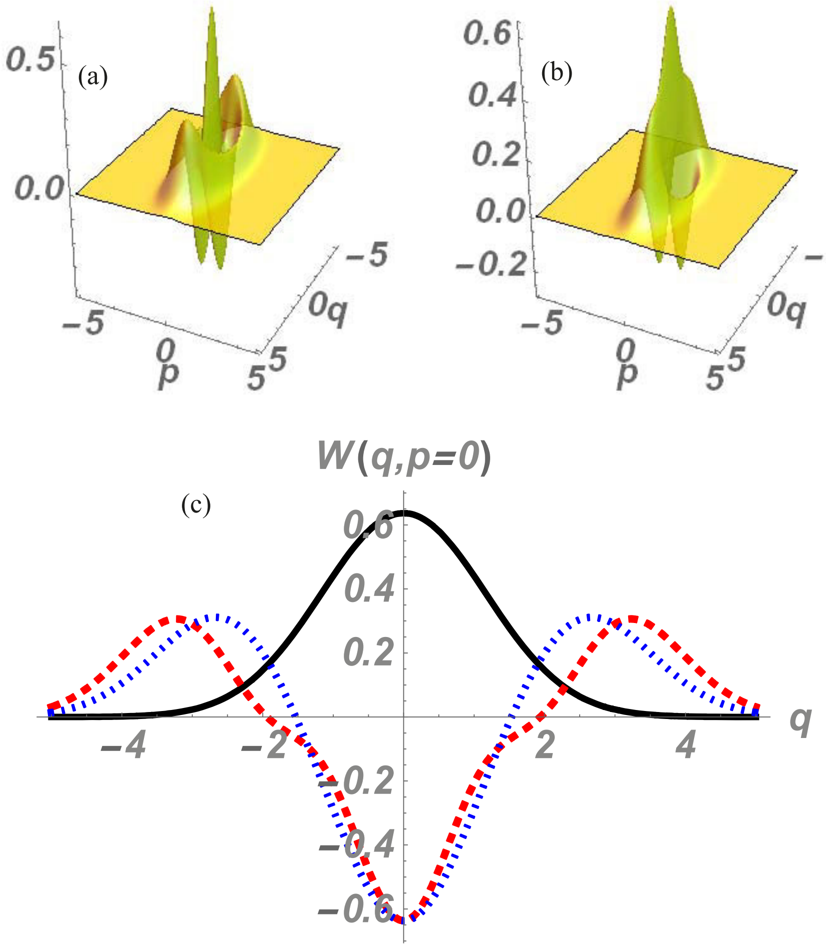

Fig. 8. Wigner functions for HSVS (or HOSVS) with $m = 3$, $r = 0.5$ but (a) $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$ and (b) $x = -i/\sqrt{2}$, $y = i/\sqrt{2}$. (c) Cross-sections for Wigner functions. Here, the black solid, red dashed, and blue dotted lines are corresponding to SVS, case (a), and case (b), respectively.

The studies on quantum states can not only deepen our understanding of the nature of quantum fields, but also help us to realize these fields for possible future applications. Our goal is to produce a complicated orthogonal quantum state (HOSVS) for the familiar squeezing vacuum state from the view of physics. Here we use the generalized orthogonalization procedure and choose a complicated operator relating to Hermitian Polynomials. Of course, we also discuss SVS and HSVS for comparison. Fortunately, Vanner *et al.* illustrated how to achieve experimental orthogonalization using the Jaynes-Cumming or beamsplitter interaction [19]. Following the work of Vanner *et al.*, one can expect that some more concrete experimental scheme will be proposed to produce these quantum states in terms of the technique of this paper in the future.

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