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New exact wave solutions of the variable-coefficient $(1+1)$-dimensional Benjamin-Bona-Mahony and $(2+1)$-dimensional asymmetric Nizhnik-Novikov-Veselov equations via the generalized exponential rational function method

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Abstract. In this paper, the variable-coefficient $(1+1)$-dimensional Benjamin-Bona-Mahony (BBM) and $(2+1)$-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equations are investigated via the generalized exponential rational function method (GERFM). This paper proceeds step-by-step with increasing detail about derivation processes, first illustrating the algorithms of the proposed method and then exploiting an even deeper connection between the derived solutions with the GERFM. As a result, versions of variable-coefficient exact solutions are formally generated. The presented solutions exhibit abundant physical phenomena. Particularly, upon choosing appropriate parameters, we demonstrate a variety of traveling waves in figures. Finally, the results indicate that free parameters can drastically influence the existence of solitary waves, their nature, profile, and stability. They are applicable to enrich the dynamical behavior of the $(1+1)$ and $(2+1)$-dimensional nonlinear wave in fluids, plasma and others.

1 Introduction

It is well-known that nonlinear evolution equations (NLEEs) are widely used to describe complex physical phenomena in various science domains, such as fluid physics, plasma physics and nonlinear optics [1–12]. It is worth mentioning that solitary waves play the key role in NLEEs. Therefore, it is necessary to focus on solitary waves in a detailed manner from a mathematical point of view. So far, a variety of powerful methods used for seeking solitary wave solutions have been developed, including the Tanh method, the Hirota method, the linear superposition principle and so on [1–17].

In some cases, the solitary structures exist indefinitely in time, as long as parameters stay constant. But, they will disappear if the values of parameters move outside the possible range of existence of the soliton waves. Moreover, in physical situations the variable-coefficient solutions are important as they can provide much more realistic models than their constant-coefficient counterparts. Consequently, a good understanding of exact solitary solutions with variable coefficients is very useful for researchers to address the nonlinear wave model in real-world applications. At this moment we are trying to search reliable solitary solutions via the newly developed method, called the generalized exponential rational function method (GERFM) [13].

This paper aims to extract new solitary solutions of the variable-coefficient $(1+1)$-dimensional Benjamin-Bona-Mahony (BBM) [14] and $(2+1)$-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equations [15]. The GERFM is employed to help achieve our results. The remainder of this paper is organized as follows. In sect. 2, the algorithms of GERFM are described. In sect. 3, new solitary solutions are generated. In sect. 4, conclusions are given.

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2 The generalized exponential rational function method (GERFM)

In this section, we will briefly explain the fundamental steps involved in the GERFM. Let us take into account the nonlinear partial differential equation (NPDE) in the form

$$L(u_t, u_x, u_{xx}, \ldots) = 0. \tag{1}$$

Using the transformation $\xi = kx + my - ct$, we reduce eq. (1) to the following ordinary differential equation:

$$L\left(u(\xi), \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}, \ldots\right) = 0. \tag{2}$$

Step 1) The key to this method is to suppose that eq. (2) has the formal solution

$$u(\xi) = A_0 + \sum_{q=1}^{M} A_q \Phi^q(\xi) + \sum_{q=1}^{M} B_q \Phi^{-q}(\xi), \tag{3}$$

where

$$\Phi(\xi) = \frac{\rho_1 e^{\kappa_1 \xi} + \rho_2 e^{\kappa_2 \xi}}{\rho_3 e^{\kappa_3 \xi} + \rho_4 e^{\kappa_4 \xi}}\,. \tag{4}$$

The unknown coefficients A_0, A_k and $B_q (1 \leq q \leq M)$ and ρ_j and $\kappa_j (1 \leq j \leq 4)$ are arbitrary real (or complex) constants to be determined, such that the solution (3) satisfies eq. (2). Note that, the positive integer M can be determined by the balancing principle between the higher derivative and the nonlinear terms in eq. (2).

Step 2) After substituting eq. (3) into eq. (2) and collecting all terms, the left-hand side of eq. (2) is converted into a polynomial $P(Y_1, Y_2, Y_3, Y_4)$ in terms of $Y_j = e^{\kappa_j \xi}$ for $j = 1, \ldots, 4$. If we set each coefficient of P to zero, then we derive a set of algebraic equations for ρ_j and κ_j $(1 \leq j \leq 4)$, and for λ, ν, A_0, A_k, and B_q $(1 \leq q \leq M)$. This can be tackled with the aid of symbolic computation, such as Maple or Mathematica.

Step 3) After solving the algebraic equations in step 2, and substituting non-trivial solutions in eq. (3), the wave solutions of eq. (2) could be obtained this way.

3 Applications

3.1 The new solutions for the $(1+1)$-dimensional BBM equation

Consider the BBM equation [14] as

$$u_t + \alpha u_x - \beta u_{xxt} + \gamma (u^2)_x = 0, \tag{5}$$

where α, β, γ are arbitrary constants.

The BBM equation, a regularized version of the KdV equation was derived as a model for the unidirectional propagation of long-crested, surface water waves and well investigated in the explicit literature. Furthermore, it is applied in the analysis of waves arising in several physical fields, such as waves in cold plasma, inharmonic crystals and other [18,19].

By the traveling wave ansatz and $u(\xi) = u(x,t)$, $\xi = kx - ct$ eq. (5) is transformed into

$$-cu_\xi + \alpha k u_\xi + ck^2 \beta u_{\xi\xi\xi} + \gamma k (u^2)_\xi = 0. \tag{6}$$

Preceding the same matter, we can assume the solution as

$$u(\xi) = A_0 + A_1 \Phi(\xi) + A_2 \Phi^2(\xi) + \frac{B_1}{\Phi(\xi)} + \frac{B_2}{\Phi^2(\xi)}\,. \tag{7}$$

Using the GERFM, we obtain the following non-trivial solutions of (5), as listed below.

Family 1. *We obtain* $\rho = [1, 1, -1, 1]$ *and* $\kappa = [1, -1, 1, -1]$*, which gives*

$$\Phi(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)}\,. \tag{8}$$

Case 1) $c=-\frac{\alpha k}{16\beta k^2-1}$, $k=k$, $A_0=-\frac{12\alpha\beta k^2}{16\beta\gamma k^2-\gamma}$, $A_1=0$, $A_2=\frac{6\alpha\beta k^2}{16\beta\gamma k^2-\gamma}$, $B_1=0$, $B_2=\frac{6\alpha\beta k^2}{16\beta\gamma k^2-\gamma}$.

Substituting the above values and eqs. (7), (8) into (6), we have

$$u(\xi)=\frac{6\alpha\beta k^2(\coth^2(\xi)-1)^2}{\gamma(16\beta k^2-1)\coth^2(\xi)}\,.$$

Therefore, an exact solution of eq. (5) is obtained as

$$u_1(x,t)=\frac{6\alpha\beta k^2(\coth^2(kx+\frac{\alpha k}{16\beta k^2-1}t)-1)^2}{\gamma(16\beta k^2-1)\coth^2(kx+\frac{\alpha k}{16\beta k^2-1}t)}\,.$$

Case 2) $c=\frac{\alpha k}{16\beta k^2+1}$, $k=k$, $A_0=-\frac{4\alpha\beta k^2}{16\beta\gamma k^2+\gamma}$, $A_1=0$, $A_2=-\frac{6\alpha\beta k^2}{16\beta\gamma k^2+\gamma}$, $B_1=0$, $B_2=-\frac{6\alpha\beta k^2}{16\beta\gamma k^2+\gamma}$.

Substituting the above values and eq. (7) into (6), we have

$$u(\xi)=-\frac{\alpha(6\coth^4(\xi)+4\coth^2(\xi)+6)\beta k^2}{(16\beta\gamma k^2+\gamma)\coth^2(\xi)}\,.$$

Therefore, an exact solution is obtained as

$$u_2(x,t)=-\frac{\alpha(6\coth^4(kx-\frac{\alpha k}{16\beta k^2+1}t)+4\coth^2(kx-\frac{\alpha k}{16\beta k^2+1}t)+6)\beta k^2}{(16\beta\gamma k^2+\gamma)\coth^2(kx-\frac{\alpha k}{16\beta k^2+1}t)}\,.$$

Family 2. *We obtain* $\rho=[2,0,1,-1]$ *and* $\kappa=[1,0,1,-1]$*, which gives*

$$\Phi(\xi)=\frac{\cosh(\xi)+\sinh(\xi)}{\sinh(\xi)} \tag{9}$$

Case 1) $c=-\frac{\alpha k}{4\beta k^2-1}$, $k=k$, $A_0=0$, $A_1=-\frac{12\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $A_2=\frac{6\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $B_1=0$, $B_2=0$.

Substituting the above values, eqs. (7) and (9) into (6), we have

$$u(\xi)=\frac{6\alpha\beta k^2}{\gamma(4\beta k^2-1)\sinh^2(\xi)}\,.$$

Therefore, an exact solution is obtained as

$$u_3(x,t)=\frac{6\alpha\beta k^2}{\gamma(4\beta k^2-1)\sinh^2(kx+\frac{\alpha k}{4\beta k^2-1}t)}\,.$$

Case 2) $c=\frac{\alpha k}{4\beta k^2+1}$, $k=k$, $A_0=-\frac{4\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$, $A_1=\frac{12\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$, $A_2=-\frac{6\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$, $B_1=0$, $B_2=0$.

Substituting the above values, eqs. (7) and (9) into (6), we have

$$u(\xi)=-\frac{\alpha\beta k^2(4\cosh^2(\xi)+2)}{(4\beta\gamma k^2+\gamma)\sinh^2(\xi)}\,.$$

Therefore, an exact solution is obtained as

$$u_4(x,t)=-\frac{\alpha\beta k^2(4\cosh^2(kx-\frac{\alpha k}{4\beta k^2+1}t)+2)}{(4\beta\gamma k^2+\gamma)\sinh^2(kx-\frac{\alpha k}{4\beta k^2+1}t)}\,.$$

Family 3. *We obtain* $\rho=[-1-i,1-i,-1,1]$ *and* $\kappa=[i,-i,i,-i]$*, which gives*

$$\Phi(\xi)=\frac{\sin(\xi)+\cos(\xi)}{\sin(\xi)}\,. \tag{10}$$

Case 1) $c = \frac{\alpha k}{4\beta k^2+1}$, $k = k$, $A_0 = -\frac{12\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$, $A_1 = 0$, $A_2 = 0$, $B_1 = \frac{24\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$, $B_2 = -\frac{24\alpha\beta k^2}{4\beta\gamma k^2+\gamma}$.

Substituting the above values, eqs. (7) and (10) into (6), we have

$$u(\xi) = -\frac{12\alpha\beta k^2}{\gamma(4\beta k^2+1)(2\cos(\xi)\sin(\xi)+1)}\,.$$

Therefore, an exact solution is obtained as

$$u_5(x,t) = -\frac{12\alpha\beta k^2}{\gamma(4\beta k^2+1)(2\cos(kx - \frac{\alpha k}{4\beta k^2+1}t)\sin(kx - \frac{\alpha k}{4\beta k^2+1}t)+1)}\,.$$

Case 2) $c = -\frac{\alpha k}{4\beta k^2-1}$, $k = k$, $A_0 = \frac{8\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $A_1 = 0$, $A_2 = 0$, $B_1 = -\frac{24\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $B_2 = \frac{24\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$.

Substituting the above values, eqs. (7) and (10) into (6), we have

$$u(\xi) = -\frac{8\alpha\beta k^2(\cos(\xi)\sin(\xi)-1)}{\gamma(4\beta k^2-1)(2\cos(\xi)\sin(\xi)+1)}\,.$$

Therefore, an exact solution is obtained as

$$u_6(x,t) = -\frac{8\alpha\beta k^2(\cos(kx + \frac{\alpha k}{4\beta k^2-1}t)\sin(kx + \frac{\alpha k}{4\beta k^2-1}t)-1)}{\gamma(4\beta k^2-1)(2\cos(kx + \frac{\alpha k}{4\beta k^2-1}t)\sin(kx + \frac{\alpha k}{4\beta k^2-1}t)+1)}\,.$$

Family 4. *We obtain* $\rho = [2,0,1,1]$ *and* $\kappa = [-1,0,1,-1]$, *which gives*

$$\Phi(\xi) = \frac{\cosh(\xi)-\sinh(\xi)}{\cosh(\xi)}\,. \tag{11}$$

Case 1) $c = \frac{\sqrt{2}\alpha}{6\sqrt{\beta}}$, $k = \frac{\sqrt{2}}{2\sqrt{\beta}}$, $A_0 = -\frac{2\alpha}{3\gamma}$, $A_1 = \frac{2\alpha}{\gamma}$, $A_2 = -\frac{\alpha}{\gamma}$, $B_1 = 0$, $B_2 = 0$.

Substituting the above values, eqs. (7) and (11) into (6), we have

$$u(\xi) = -\frac{\alpha(2\cosh^2(\xi)-3)}{3\gamma\cosh^2(\xi)}\,.$$

Therefore, an exact solution is obtained as

$$u_7(x,t) = -\frac{\alpha(2\cosh^2(\frac{\sqrt{2}}{2\sqrt{\beta}}x - \frac{\sqrt{2}\alpha}{6\sqrt{\beta}}t)-3)}{3\gamma\cosh^2(\frac{\sqrt{2}}{2\sqrt{\beta}}x - \frac{\sqrt{2}\alpha}{6\sqrt{\beta}}t)}\,.$$

Case 2) $c = -\frac{\alpha k}{4\beta k^2-1}$, $k = k$, $A_0 = 0$, $A_1 = -\frac{12\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $A_2 = \frac{6\alpha\beta k^2}{4\beta\gamma k^2-\gamma}$, $B_1 = 0$, $B_2 = 0$.

Substituting the above values, eqs. (7) and (11) into (6), we have

$$u(\xi) = -\frac{6\alpha\beta k^2}{\gamma(4\beta k^2-1)\cosh^2(\xi)}\,.$$

Therefore, an exact solution is obtained as

$$u_8(x,t) = -\frac{6\alpha\beta k^2}{\gamma(4\beta k^2-1)\cosh^2(kx + \frac{\alpha k}{4\beta k^2-1}t)}\,.$$

Family 5. *We obtain* $\rho = [-3,-2,1,1]$ *and* $\kappa = [0,1,0,1]$, *which gives*

$$\Phi(\xi) = \frac{\cosh(\xi)-\sinh(\xi)}{\cosh(\xi)}\,. \tag{12}$$

Case 1) $c=\frac{\sqrt{\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}$, $k=-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{\gamma B_2-216\alpha}}$, $A_0=\frac{B_2}{6}$, $A_1=0$, $A_2=0$, $B_1=\frac{5B_2}{6}$, $B_2=B_2$.

Substituting the above values, eqs. (7) and (12) into (6), we have

$$u(\xi)=-\frac{\mathrm{e}^{\xi}B_2}{6(3+2\mathrm{e}^{\xi})^2}\,.$$

Therefore, an exact solution is obtained as

$$u_9(x,t)=-\frac{\mathrm{e}^{\left(-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{\gamma B_2-216\alpha}}-\frac{\sqrt{\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}t\right)}B_2}{6\left(3+2\mathrm{e}^{\left(-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{\gamma B_2-216\alpha}}-\frac{\sqrt{\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}t\right)}\right)^2}\,.$$

Family 6. *We obtain* $\rho=[2,3,1,1]$ *and* $\kappa=[1,0,1,0]$, *which gives*

$$\Phi(\xi)=\frac{2+3\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}\,. \tag{13}$$

Case 1) $c=\frac{\sqrt{-\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}$, $k=-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{-\gamma B_2-216\alpha}}$, $A_0=\frac{37B_2}{216}$, $A_1=0$, $A_2=0$, $B_1=-\frac{5B_2}{6}$, $B_2=B_2$.

Substituting the above values, eqs. (7) and (13) into (6), we have

$$u(\xi)=\frac{B_2(9\mathrm{e}^{2\xi}-24\mathrm{e}^{\xi}+4)}{216(3\mathrm{e}^{\xi}+2)^2}\,.$$

Therefore, an exact solution is obtained as

$$u_{10}(x,t)=\frac{B_2\left(9\mathrm{e}^{2\left(-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{-\gamma B_2-216\alpha}}x-\frac{\sqrt{-\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}t\right)}-24\mathrm{e}^{\left(-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{-\gamma B_2-216\alpha}}x-\frac{\sqrt{-\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}t\right)}+4\right)}{216\left(3\mathrm{e}^{\left(-\frac{\sqrt{\gamma}\sqrt{B_2}}{\sqrt{\beta}\sqrt{-\gamma B_2-216\alpha}}x-\frac{\sqrt{-\gamma B_2-216\alpha}\sqrt{\gamma}\sqrt{B_2}}{216\sqrt{\beta}}t\right)}+2\right)^2}\,.$$

It is worth noting that based on solitary solutions u_{1-10} the free parameters α, β, γ are revealed as non-zero arbitrary constants.

3.2 The new solutions for the $(2+1)$-dimensional ANNV equation

The ANNV equation is considered as

$$u_t+\omega\left(u\int u_x\mathrm{d}_y\right)_x+u_{xxx}=0. \tag{14}$$

It is notable that eq. (14) is also called the coupled KdV equation [15,20,21] due to the fact that it can be transformed as

$$\begin{aligned} u_t+\omega u_x v+\omega u v_x+u_{xxx}&=0,\\ \int u_x\mathrm{d}_y&=v, \end{aligned} \tag{15}$$

where ω is a non-zero arbitrary constant. When $u=v$, $x=y$ and $\omega=3$ eq. (14) is reduced to the original KdV equation

$$u_t+6uu_x+u_{xxx}=0. \tag{16}$$

The ANNV equation is derived as the model for an incompressible fluid, where u and v are the components of the dimensionless velocity. In [20] the lump soliton, mixed lump stripe and periodic lump solutions were obtained.

Using the traveling ansatz $\xi=kx+my-ct$, $U(\xi)=u(x,y,t)$ and $V(\xi)=v(x,y,t)$ eq. (15) are thus transformed into

$$-cU_{\xi}+\omega kU_{\xi}V+\omega kUV_{\xi}+k^3U_{\xi\xi\xi}=0, \tag{17}$$

$$kU_{\xi}=mV_{\xi}. \tag{18}$$

Proceeding as before we assume the solution as

$$U(\xi) = A_0 + A_1\Phi(\xi) + A_2\Phi^2(\xi) + \frac{B_1}{\Phi(\xi)} + \frac{B_2}{\Phi^2(\xi)}\,. \tag{19}$$

Via the GERFM, we obtain the following non-trivial solutions of (15), as listed below.

Family 1. *We obtain* $\rho = [1-i, -1-i, -1, 1]$ *and* $\kappa = [i, -i, i, -i]$, *which gives*

$$\Phi(\xi) = \frac{\cos(\xi) - \sin(\xi)}{\sin(\xi)}\,. \tag{20}$$

Case 1) $c = -4k^3$, $k = k$, $m = -\frac{\omega A_2}{12k}$, $A_0 = 2A_2$, $A_1 = 2A_2$, $A_2 = A_2$, $B_1 = 0$, $B_2 = 0$.
Substituting the above values, eqs. (19) and (20) into (17), (18), we have

$$U(\xi) = \frac{A_2}{\sin^2(\xi)}\,,$$
$$V(\xi) = -\frac{12k^2}{\omega\sin^2(\xi)}\,.$$

Therefore, the exact solutions of the ANNV equations are obtained as

$$u_1(x,y,t) = \frac{A_2}{\sin^2(kx - \frac{\omega A_2}{12k}y + 4k^3t)}\,,$$
$$v_1(x,y,t) = -\frac{12k^2}{\omega\sin^2(kx - \frac{\omega A_2}{12k}y + 4k^3t)}\,.$$

Case 2) $c = 4k^3$, $k = k$, $m = -\frac{\omega B_1}{48k}$, $A_0 = \frac{B_1}{3}$, $A_1 = 0$, $A_2 = 0$, $B_1 = B_1$, $B_2 = B_1$.
Substituting the above values, eqs. (19) and (20) into (17), (18), we have

$$U(\xi) = -\frac{B_1(\cos(\xi)\sin(\xi) + 1)}{6\cos(\xi)\sin(\xi) - 3}\,,$$
$$V(\xi) = \frac{48k^2(\cos(\xi)\sin(\xi) + 1)}{\omega(6\cos(\xi)\sin(\xi) - 3)}\,.$$

Therefore, the exact solutions are obtained as

$$u_2(x,y,t) = -\frac{B_1(\cos(kx - \frac{\omega B_1}{48k}y - 4k^3t)\sin(kx - \frac{\omega B_1}{48k}y - 4k^3t) + 1)}{6\cos(kx - \frac{\omega B_1}{48k}y - 4k^3t)\sin(kx - \frac{\omega B_1}{48k}y - 4k^3t) - 3}\,,$$
$$v_2(x,y,t) = \frac{48k^2(\cos(kx - \frac{\omega B_1}{48k}y - 4k^3t)\sin(kx - \frac{\omega B_1}{48k}y - 4k^3t) + 1)}{\omega(6\cos(kx - \frac{\omega B_1}{48k}y - 4k^3t)\sin(kx - \frac{\omega B_1}{48k}y - 4k^3t) - 3)}\,.$$

Family 2. *We obtain* $\rho = [i, -i, 1, 1]$ *and* $\kappa = [i, -i, i, -i]$, *which gives*

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}\,. \tag{21}$$

Case 1) $c = \frac{\omega^3 {A_0}^3}{432m^3}$, $k = -\frac{\omega A_0}{12m}$, $m = m$, $A_0 = A_0$, $A_1 = 0$, $A_2 = A_0$, $B_1 = 0$, $B_2 = 0$.
Substituting the above values, eqs. (19) and (21) into (17), (18), we have

$$U(\xi) = \frac{A_0}{\cos^2(\xi)}\,,$$
$$V(\xi) = -\frac{\omega {A_0}^2}{12m^2\cos^2(\xi)}\,.$$

Therefore, the exact solutions are obtained as

$$u_3(x,y,t) = \frac{A_0}{\cos^2(-\frac{\omega A_0}{12m}x + my - \frac{\omega^3 {A_0}^3}{432m^3}t)},$$
$$v_3(x,y,t) = -\frac{\omega {A_0}^2}{12m^2\cos^2(-\frac{\omega A_0}{12m}x + my - \frac{\omega^3 {A_0}^3}{432m^3}t)}.$$

Case 2) $c = 4k^3$, $k = k$, $m = -\frac{\omega A_2}{12k}$, $A_0 = \frac{A_2}{3}$, $A_1 = 0$, $A_2 = A_2$, $B_1 = 0$, $B_2 = 0$.
Substituting the above value, eqs. (19) and (21) into (17), (18), we have

$$U(\xi) = -\frac{(2\cos^2(\xi) - 3)A_2}{3\cos^2(\xi)},$$
$$V(\xi) = \frac{4k^2(2\cos^2(\xi) - 3)}{\omega\cos^2(\xi)}.$$

Therefore, the exact solutions are obtained as

$$u_4(x,y,t) = -\frac{(2\cos^2(kx - \frac{\omega A_2}{12k}y - 4k^3t) - 3)A_2}{3\cos^2(kx - \frac{\omega A_2}{12k}y - 4k^3t)},$$
$$v_4(x,y,t) = \frac{4k^2(2\cos^2(kx - \frac{\omega A_2}{12k}y - 4k^3t) - 3)}{\omega\cos^2(kx - \frac{\omega A_2}{12k}y - 4k^3t)}.$$

Family 3. *We obtain* $\rho = [2,1,1,1]$ *and* $\kappa = [1,0,1,0]$, *which gives*

$$\Phi(\xi) = \frac{2\mathrm{e}^{\xi} + 1}{\mathrm{e}^{\xi} + 1}. \tag{22}$$

Case 1) $c = -\frac{\omega^3 {B_2}^3}{110592m^3}$, $k = -\frac{\omega B_2}{48m}$, $m = m$, $A_0 = \frac{B_2}{2}$, $A_1 = 0$, $A_2 = 0$, $B_1 = -\frac{3B_2}{2}$, $B_2 = B_2$.
Substituting the above values, eqs. (19) and (22) into (17), (18), we have

$$U(\xi) = -\frac{\mathrm{e}^{\xi}B_2}{2(2\mathrm{e}^{\xi} + 1)^2},$$
$$V(\xi) = \frac{\omega {B_2}^2\mathrm{e}^{\xi}}{96m^2(2\mathrm{e}^{\xi} + 1)^2}.$$

Therefore, the exact solutions are obtained as

$$u_5(x,y,t) = -\frac{\mathrm{e}^{\left(-\frac{\omega B_2}{48m}x + my + \frac{\omega^3 {B_2}^3}{110592m^3}t\right)}B_2}{2\left(2\mathrm{e}^{\left(-\frac{\omega B_2}{48m}x + my + \frac{\omega^3 {B_2}^3}{110592m^3}t\right)} + 1\right)^2},$$
$$v_5(x,y,t) = \frac{\omega {B_2}^2\mathrm{e}^{\left(-\frac{\omega B_2}{48m}x + my + \frac{\omega^3 {B_2}^3}{110592m^3}t\right)}}{96m^2\left(2\mathrm{e}^{\left(-\frac{\omega B_2}{48m}x + my + \frac{\omega^3 {B_2}^3}{110592m^3}t\right)} + 1\right)^2}.$$

Family 4. *We obtain* $\rho = [-3,-2,1,1]$ *and* $\kappa = [0,1,0,1]$, *which gives*

$$\Phi(\xi) = \frac{-3 - 2\mathrm{e}^{\xi}}{1 + \mathrm{e}^{\xi}}. \tag{23}$$

Case 1) $c = \frac{\omega^3 {B_2}^3}{80621568m^3}$, $k = -\frac{\omega B_2}{432m}$, $m = m$, $A_0 = \frac{37B_2}{216}$, $A_1 = 0$, $A_2 = 0$, $B_1 = \frac{5B_2}{6}$, $B_2 = B_2$.

Substituting the above values, eqs. (19) and (23) into (17), (18), we have

$$U(\xi) = -\frac{B_2(-4e^{2\xi} + 24e^{\xi} - 9)}{216(3 + 2e^{\xi})^2},$$
$$V(\xi) = \frac{\omega {B_2}^2(-4e^{2\xi} + 24e^{\xi} - 9)}{93312m^2(3 + 2e^{\xi})^2}.$$

Therefore, the exact solutions are obtained as

$$u_6(x,y,t) = -\frac{B_2\left(-4e^{2\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)} + 24e^{\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)} - 9\right)}{216\left(3 + 2e^{\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)}\right)^2},$$
$$v_6(x,y,t) = \frac{\omega {B_2}^2\left(-4e^{2\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)} + 24e^{\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)} - 9\right)}{93312m^2\left(3 + 2e^{\left(-\frac{\omega B_2}{432m}x+my-\frac{\omega^3 {B_2}^3}{80621568m^3}t\right)}\right)^2}.$$

Family 5. *We obtain* $\rho = [-1-i, 1-i, 1, -1]$ *and* $\kappa = [i, -i, i, -i]$, *which gives*

$$\Phi(\xi) = \frac{\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{24}$$

Case 1) $c = \frac{\omega^3 {A_0}^3}{3456m^3}$, $k = -\frac{\omega A_0}{24m}$, $m = m$, $A_0 = A_0$, $A_1 = 0$, $A_2 = 0$, $B_1 = -2A_0$, $B_2 = 2A_0$.

Substituting the above values, eqs. (19) and (24) into (17), (18), we have

$$U(\xi) = \frac{A_0}{2\cos(\xi)\sin(\xi) + 1},$$
$$V(\xi) = -\frac{\omega {A_0}^2}{24m^2(2\cos(\xi)\sin(\xi) + 1)}.$$

Therefore, the exact solutions are obtained as

$$u_7(x,y,t) = \frac{A_0}{2\cos(-\frac{\omega A_0}{24m}x + my - \frac{\omega^3 {A_0}^3}{3456m^3}t)\sin(-\frac{\omega A_0}{24m}x + my - \frac{\omega^3 {A_0}^3}{3456m^3}t) + 1},$$
$$v_7(x,y,t) = -\frac{\omega {A_0}^2}{24m^2(2\cos(-\frac{\omega A_0}{24m}x + my - \frac{\omega^3 {A_0}^3}{3456m^3}t)\sin(-\frac{\omega A_0}{24m}x + my - \frac{\omega^3 {A_0}^3}{3456m^3}t) + 1)}.$$

Family 6. *We obtain* $\rho = [1, 1, -1, 1]$ *and* $\kappa = [1, -1, 1, -1]$, *which gives*

$$\Phi(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)}. \tag{25}$$

Case 1) $c = \frac{\omega^3 {B_2}^3}{108m^3}$, $k = -\frac{\omega B_2}{12m}$, $m = m$, $A_0 = \frac{2B_2}{3}$, $A_1 = 0$, $A_2 = B_2$, $B_1 = 0$, $B_2 = B_2$.

Substituting the above values, eqs. (19) and (25) into (17), (18), we have

$$U(\xi) = \frac{B_2(3\coth^4(\xi) + 2\coth^2(\xi) + 3)}{3\coth^2(\xi)},$$
$$V(\xi) = -\frac{\omega {B_2}^2(3\coth^4(\xi) + 2\coth^2(\xi) + 3)}{36m^2\coth^2(\xi)}.$$

Therefore, the exact solutions are obtained as

$$u_8(x,y,t) = \frac{B_2(3\coth^4(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t) + 2\coth^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t) + 3)}{3\coth^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t)},$$
$$v_8(x,y,t) = -\frac{\omega {B_2}^2(3\coth^4(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t) + 2\coth^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t) + 3)}{36m^2\coth^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{108m^3}t)}.$$

Family 7. *We obtain* $\rho = [-1, 1, 1, 1]$ *and* $\kappa = [1, -1, 1, -1]$, *which gives*

$$\Phi(\xi) = -\frac{\sinh(\xi)}{\cosh(\xi)}\,. \tag{26}$$

Case 1) $c = \frac{\omega^3 {B_2}^3}{432 m^3}$, $k = -\frac{\omega B_2}{12m}$, $m = m$, $A_0 = -\frac{B_2}{3}$, $A_1 = 0$, $A_2 = 0$, $B_1 = 0$, $B_2 = B_2$.
Substituting the above values, eqs. (19) and (26) into (17), (18), we have

$$U(\xi) = -\frac{B_2(\tanh^2(\xi) - 3)}{3\tanh^2(\xi)}\,,$$
$$V(\xi) = \frac{\omega {B_2}^2(\tanh^2(\xi) - 3)}{36 m^2 \tanh^2(\xi)}\,.$$

Therefore, the exact solutions are obtained as

$$u_9(x, y, t) = -\frac{B_2(\tanh^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{432m^3}t) - 3)}{3\tanh^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{432m^3}t)}\,,$$
$$v_9(x, y, t) = \frac{\omega {B_2}^2(\tanh^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{432m^3}t) - 3)}{36m^2\tanh^2(-\frac{\omega B_2}{12m}x + my - \frac{\omega^3 {B_2}^3}{432m^3}t)}\,.$$

Family 8. *We obtain* $\rho = [1, 1, 1, -1]$ *and* $\kappa = [1, -1, 1, -1]$, *which gives*

$$\Phi(\xi) = \frac{\cosh(\xi)}{\sinh(\xi)}\,. \tag{27}$$

Case 1) $c = 16k^3$, $k = k$, $m = -\frac{\omega A_2}{12k}$, $A_0 = -2A_2$, $A_1 = 0$, $A_2 = A_2$, $B_1 = 0$, $B_2 = A_2$.
Substituting the above values, eqs. (19) and (27) into (17), (18), we have

$$U(\xi) = \frac{A_2}{\cosh^2(\xi)\sinh^2(\xi)}\,,$$
$$V(\xi) = -\frac{12k^2}{\omega\cosh^2(\xi)\sinh^2(\xi)}\,.$$

Therefore, the exact solutions read as

$$u_{10}(x, y, t) = \frac{A_2}{\cosh^2(kx - \frac{\omega A_2}{12k}y - 16k^3t)\sinh^2(kx - \frac{\omega A_2}{12k}y - 16k^3t)}\,,$$

$$v_{10}(x, y, t) = -\frac{12k^2}{\omega\cosh^2(kx - \frac{\omega A_2}{12k}y - 16k^3t)\sinh^2(kx - \frac{\omega A_2}{12k}y - 16k^3t)}\,.$$

So far, new exact solutions of BBM and ANNV equations are concisely obtained. They are all checked for accuracy via Maple. It is easily found that all free parameters α, β, γ, ω are non-zero arbitrary constants. Here, a brief summary is given.

i) The obtained solutions can be presented as various versions of traveling waves by specifying the parameters α, β, γ, ω, including the solitary waves. They are helpful to simulate and elaborate a lot of experimental situations, as shown in figs. 1–9. The resultant solutions and figures may provide significant supplements to the studies in cold plasma and incompressible fluids.
ii) All free parameters α, β, γ, ω directly affect the amplitude and speed of the traveling waves to BBM and ANNV equations.

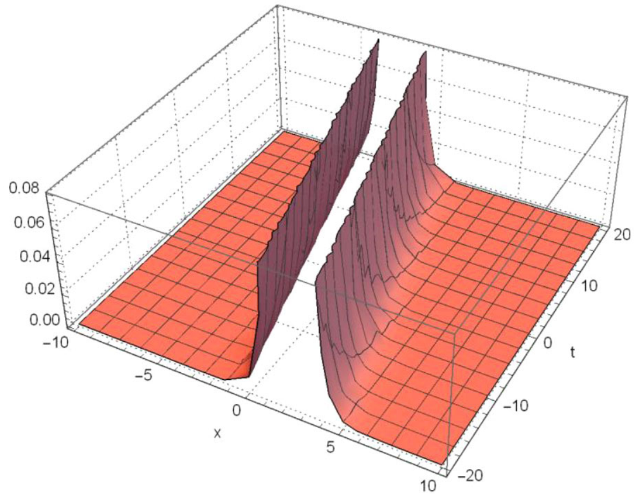


Fig. 1. The traveling waves to the BBM equation in the case of $\alpha = 1.2$, $\beta = 15$, $k = 0.6$, $\gamma = 4$ via the solution u_1.

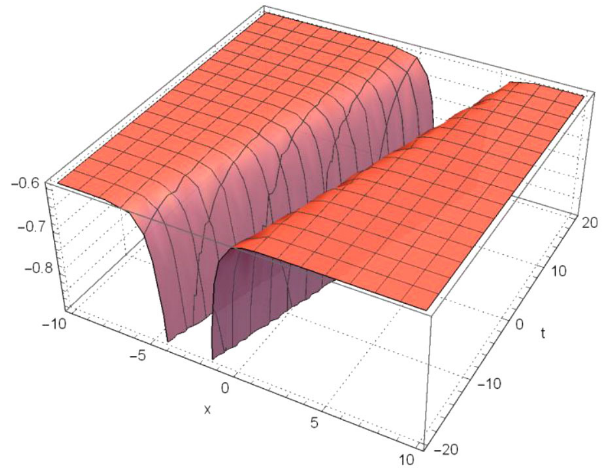


Fig. 2. The traveling waves to the BBM equation in the case of $\alpha = -1$, $\beta = 2$, $k = 0.5$, $\gamma = 1.5$ via the solution u_2.

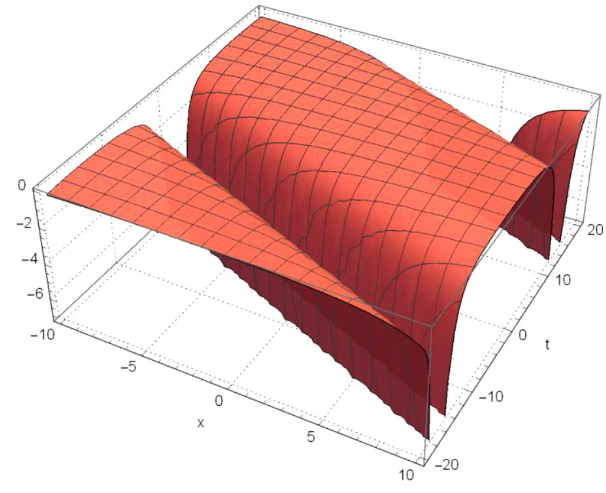


Fig. 3. The traveling waves to the BBM equation in the case of $\alpha = -1.6$, $\beta = -3$, $k = 0.09$, $\gamma = -1.2$ via the solution u_5.

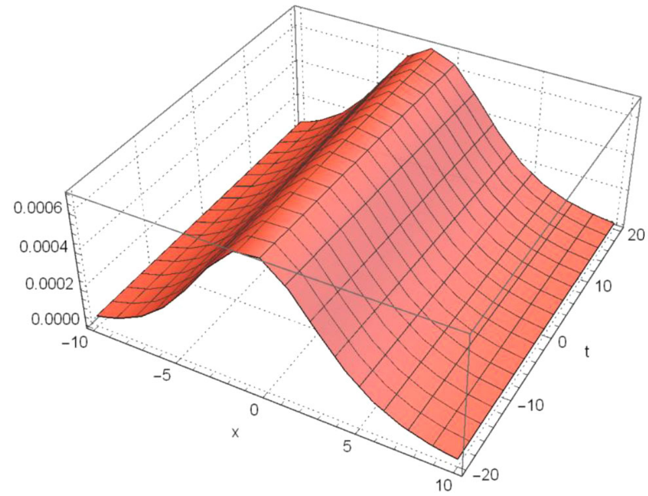


Fig. 4. The traveling waves to the BBM equation in the case of $\alpha = -0.1$, $\beta = 3.2$, $B_2 = 0.1$, $\gamma = 41.2$ via the solution u_9.

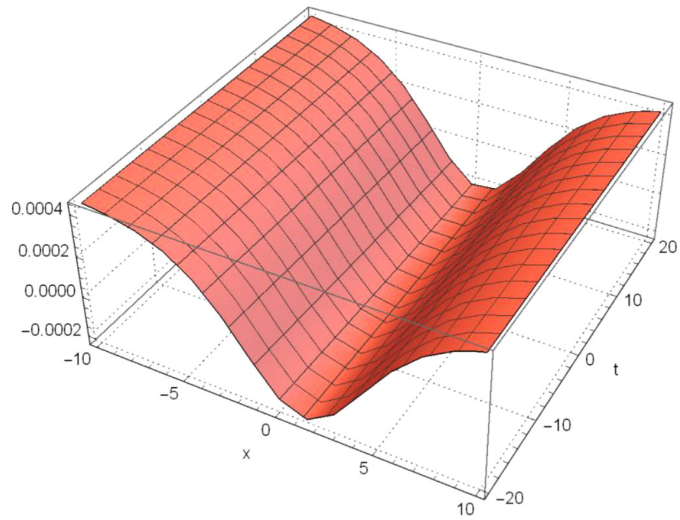


Fig. 5. The traveling waves to the BBM equation in the case of $\alpha = -0.1$, $\beta = 3.2$, $B_2 = 0.1$, $\gamma = 41.2$ via the solution u_{10}.

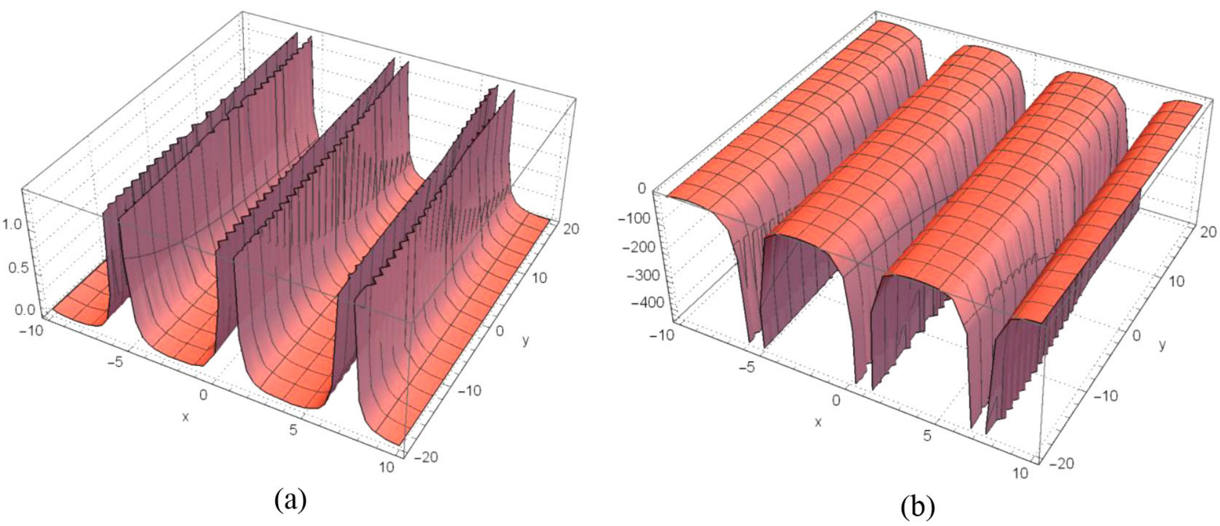
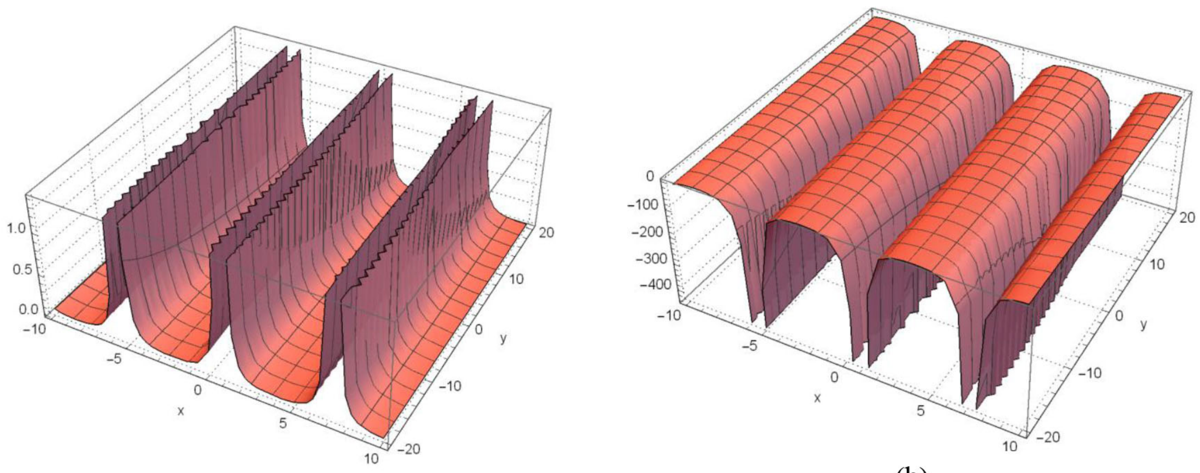
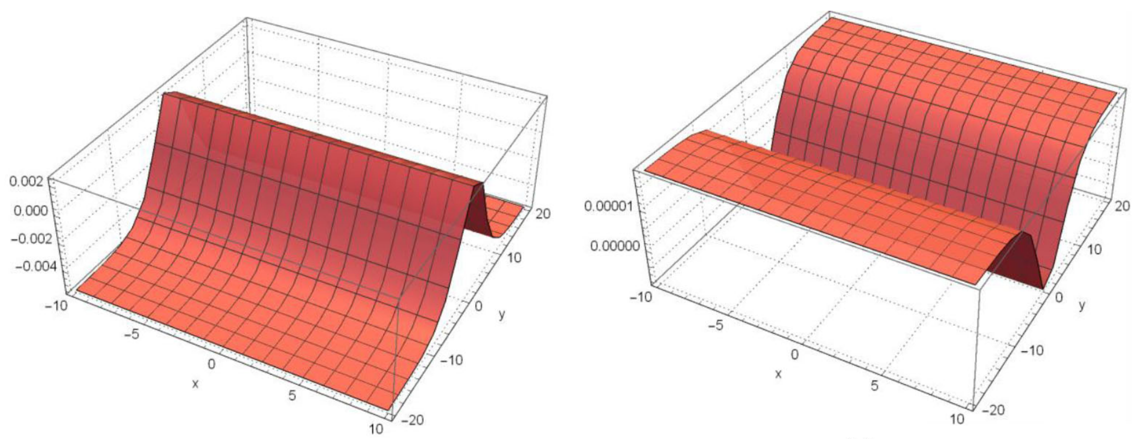


Fig. 6. The traveling waves to the ANNV equation in the case of $t = 1$, $\omega = 0.2$, $A_2 = -2$, $k = 0.5$ via the solution u_2 (a) and v_2 (b).



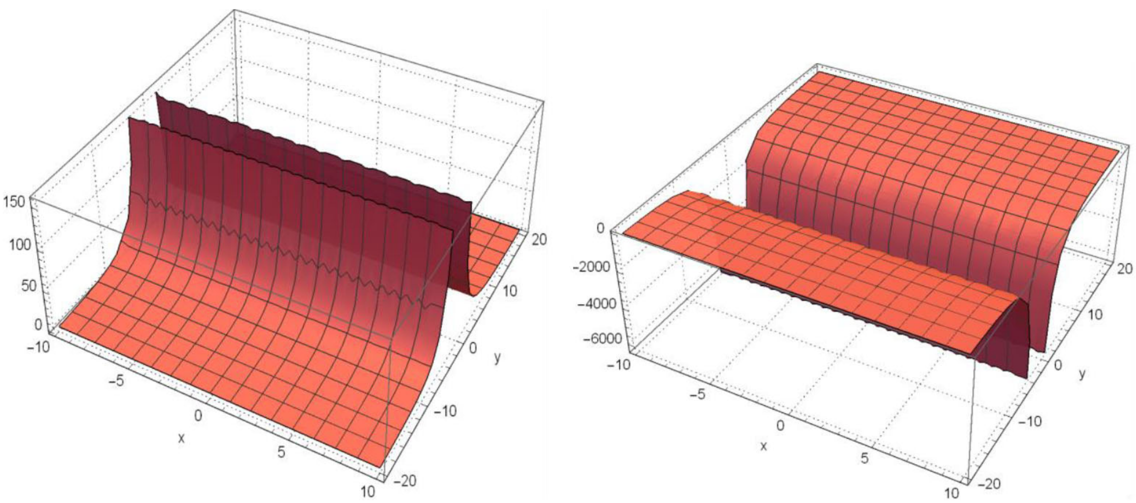
(a) (b)

Fig. 7. The traveling waves to the ANNV equation in the case of $t = 1$, $\omega = 3.2$, $A_2 = -1.2$, $k = 0.5$ via the solution u_4 (a) and v_4 (b).



(a) (b)

Fig. 8. The traveling waves to the ANNV equation in the case of $t = 1$, $\omega = -0.2$, $B_2 = -1.2$, $m = 0.4$ via the solution u_6 (a) and v_6 (b).



(a) (b)

Fig. 9. The traveling waves to the ANNV equation in the case of $t = 1$, $\omega = -0.01$, $B_2 = 1.2$, $m = 0.01$ via the solution u_8 (a) and v_8 (b).

4 Conclusions

In this work, we have presented a lot of exact wave solutions to the $(1+1)$-dimensional BBM and $(2+1)$-dimensional ANNV equations via the GERFM. The features and influences of free parameters α, β, γ, ω are elaborated. The extracted solutions exhibit abundant physical phenomena displayed as the traveling waves in figs. 1–9. The results show the GERFM is an effective and reliable mathematical tool for solving NLEEs in the related science. By applying the the proposed method, the variable-coefficient NLEEs can be solved straightforwardly and concisely. As a result, seeking exact variable-coefficient wave solutions can make great contributions to nonlinear science in real-world applications.

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