Eur. Phys. J. Plus (2019) **134**: 245
DOI 10.1140/epjp/i2019-12582-5

THE EUROPEAN
PHYSICAL JOURNAL PLUS

Regular Article

On the role of f(G, T) terms in structure scalars

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Received: 10 July 2018
Published online: 31 May 2019

Abstract. This work is devoted to exploring the effects of $f(G,T)$ terms on the study of structure scalars and their influence on the formulation of the Raychaudhuri, shear and Weyl scalar equations. For this purpose, we have assumed non-static spherically symmetric geometry coupled with shearing viscous locally anisotropic dissipative matter content. We have developed relations among Misner-Sharp mass, Weyl scalar, matter and structure variables. We have also formulated a set of $f(G,T)$ structure scalars after orthogonally breaking down the Riemann curvature tensor. The influences of these scalar functions on the modeling of relativistic radiating spheres are also studied. The factor involved in the emergence of inhomogeneities is also explored for the constant and varying modified curvature corrections. We inferred that $f(G,T)$ structure scalars could provide an effective tool to study the Penrose-Hawking singularity theorems and the Newman-Penrose formalism.

1 Introduction

Even after the introduction of one century, the well-known General Relativity (GR) is considered itself as the most effective relativistic theory to investigate the gravitational interactions in the sel-gravitating system and the related scenarios. However, the need for the exploration of its alternatives cannot be ignored that stem from the confrontation that GR presently experience [1–7]. The current burning issues of dark matter (DM) and accelerated expansion of our cosmos (along with the cosmological constant (Λ) problem) urged the need of hour towards GR modification. After performing a detailed analysis on the modified relativistic dynamics, Qadir *et al.* [8] suggested that GR may need to extend for the discussion of DM problem, quantum gravity effects and some other related burning issues. Many theoretical scheme have been seen in the literature with the aim to handle such problems.

The modified gravity theories (MGTs) are found to be one of the attractive mathematical tools to explore the dark sector of the universe. various DE theoretical models are being suggested by modifying the gravitational part of the usual Einstein-Hilbert (EH) action (for details, please see [9–12]). Nojiri and Odintsov [13] found some observationally well-consistent modified gravity models (MGM) for the study of dark cosmic aspect. They inferred that there could be few very interesting $f(R)$ models that can provide the dynamical study of relativistic systems compatible with the solar systems tests. Yousaf and Bhatti [14] claimed that some MGM prefer to host super-massive but with relatively smaller radii compact objects.

The notable $f(R)$ theory is found to be suffer from some scale dependence effects, which can not be ignored for the exploration of one post-Newtonian term alone. Furthermore, it could be interesting to include amalgams of curvature measuring mathematical quantities, like Riemann tensor ($R_{\alpha\beta\mu\nu}$), Ricci tensor ($R_{\mu\nu}$) and its scalar (R) in the EH action. This approach results with the $f(G)$ gravity theory, in which G is a topological invariant Gauss-Bonnet term and is defined as $G = R - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. This theory was first introduced in [15]. Such theories could help out to understand cosmic inflationary epochs and to see the transition phases of our cosmos in the accelerating phase from the corresponding deceleration era. It could be considered as a viable substitute to understand DE issues [12, 16, 17]. Recently, this theory has been modified further by including corrections from the trace of energy-momentum tensor (T) in the EH action. This gravitational model is known as $f(G,T)$ theory. Just like Harko *et al.* [18] modified $f(R)$ to $f(R,T)$ theory.

Houndjo [19] calculated few observationally viable mathematical models for $f(R,T)$ gravity and claimed that these could encode our cosmic dynamics associated with the matter dominated regime. Baffou *et al.* [20] applied

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perturbation theory to analyze the behavior of some cosmic mathematical models in the background of power law and de-Sitter spacetime. Bamba *et al.* [21] studied the influences of extra degrees of freedom mediated by modified gravity terms on the accelerating nature of our expanding universe. The phenomenon of gravitational instabilities for the self-gravitating stellar interiors were examined in the mode of $f(R)$ gravity by [22–24]. Yousaf and his collaborators studied the collapse rate of the compact bodies in account of various modified theories with the planar [25–27], spherical [28–33] and cylindrical [34, 35] environments. Ilyas *et al.* studied the impact of quadratic and exponential $f(R,T)$ models in the mathematical modeling and stability of observationally well-consistent spherical stars. Moraes *et al.* [36] explored hydrostatic state of strange stars in order to investigate their stable regimes with $f(R,T) = R + 2\lambda T$ model. Recently, Bhatti *et al.* [37] performed computational analysis to check the role of logarithmic $f(G,T)$ models on the existence of Her X-1 compact star.

Herrera *et al.* [38, 39] studied few dynamical features of cylindrical as well as spherical gravitational collapse after evaluating junction conditions. Tewari *et al.* [40] examined effects of locally anisotropic pressure on the spherical collapse. Sharif and Yousaf [41, 42] analyzed the problem of collapsing system by investigating the role of matter variables in modified gravity. Recently, Yousaf *et al.* [43, 44] smoothly matched non-static irrotational cylindrical spacetime with an exterior geometry of Einstein-Rosen bridge and examined the issue of dynamical instability. Sahoo *et al.* [45] considered the problem of cosmic evolution and discussed some kinematical features of the temporal varying deceleration parameters. Recently, Moraes *et al.* [46–48] studied various interesting cosmic and stellar issues in the field of $f(R,T)$ theory.

The self-gravitating relativistic system would undergo the collapsing phase, once it experiences inhomogeneities in its energy density. This has urged many researchers to explore those factors that are involved in the emergence of irregularities in the initially homogeneous celestial object. Penrose and Hawking [49] discussed the causes of inhomogeneous energy density (IED) with the help of tidal forces producing tensor called the Weyl tensor for the relativistic sphere. Herrera *et al.* [50] explored few parameters involved in the maintenance of IED in an environment of anisotropic spheres and inferred that effects of anisotropy in the stellar pressure could to the formations of naked singularity (NS). In the mathematical viewpoint, Virbhadra *et al.* [51–53] provided formula for helping the relativistic to analyze the formation of NS and black holes during the evolutionary phases of stars.

Herrera *et al.* [54] studied the gravitational time arrow in the Einstein gravity for the radiating compact stars and presented a relation among the tidal forces, IED and locally anisotropic pressure. Herrera *et al.* [55] explored the variations of expansion scalar in the maintenance of IED foe the viscous charged spherical objects. Yousaf *et al.* [56] extended these results for the case of radiating relativistic sphere with extra degrees of freedom coming from modified gravity. They concluded that modified gravity terms has greatly influence the role of Weyl scalar in the maintenance of IED. Bhatti and his collaborators [57, 58] examined some characteristics of collapsing spheres and explored the corresponding IED factors in modified gravity. Herrera *et al.* [59] and Herrera [60] evaluated transport equations for the spherically symmetric matter distributions which undergoes in the collapsing state as seen by a tilted observer. Yousaf *et al.* [61] modified these results and looked into the effects of Palatini $f(R)$ terms in the rate of gravitational collapse. Most recently, Herrera [62] investigated that why observations of non-comoving congruences observe dissipation from the celestial objects which seem to be isentropic for moving observers.

Here, we have extended the work of Herrera *et al.* [55] with the aim to analyze the influences of $f(G,T) = \alpha G^n(\beta G^m + 1) + \lambda T$ model in the mathematical modeling of structure scalars, Weyl, expansion and shear equations. We shall outline this paper as follows. The coming section will describe some equations required to describe $f(G,T)$ gravity and spherical dissipative viscous matter configurations. Section 3 is devoted to evaluating modified $f(G,T)$ structure scalars obtained from the orthogonal breaking down of the Riemann curvature tensor with power law Gauss Bonnet and linear T terms. The role of differential equation corresponding to Weyl, shear and expansion scalars are also discussed in this gravity. In sect. 4, we demonstrate the working of $f(G,T)$ scalar functions in the evolution of IED over the surface of the initially smooth dust ball with constant G and T terms. Finally, we describe our conclusion in the last section.

2 Spherical viscous spherical system and f(G, T) gravity

The action function for the theory of $f(G,T)$ gravity can be written as

$$S = \frac{1}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2} + f(G,T)\right] + S_M\left(g^{\mu\nu},\psi\right), \tag{1}$$

where κ^2 is a coupling constant which is taken to be unity here. In the above equation, GT is the Gauss-Bonnet term whose expression with the help of the Ricci tensor ($R_{\mu\nu}$), scalar (R) and the Riemann curvature tensors ($R_{\mu\nu\alpha\beta}$) can be given as

$$G = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R.$$

Further, g and S_M are the metric tensor determinant and the matter action, respectively. The quantity T is the trace of the following energy momentum tensor:

$$T_{\gamma\delta}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\gamma\delta}}\,. \tag{2}$$

This equation after some manipulations and assumptions, can be rewritten as

$$T_{\gamma\delta}=g_{\gamma\delta}L_m-\frac{2\partial L_m}{\partial g_{\gamma\delta}}\,, \tag{3}$$

whose δ variations gives

$$\frac{\delta T_{\gamma\delta}}{\delta g^{\mu\nu}}=\frac{\delta g_{\gamma\delta}}{\delta g^{\mu\nu}}L_m-\frac{2\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\gamma\delta}}+g_{\gamma\delta}\frac{\partial L_m}{\partial g^{\mu\nu}}\,. \tag{4}$$

Upon varying eq. (1) with the metric tenor, we obtain the following $f(G,T)$ equations of motion:

$$G_{\gamma\delta}=T^{\rm eff}_{\gamma\delta}, \tag{5}$$

where $G_{\gamma\delta}=R_{\gamma\delta}-\frac{1}{2}Rg_{\gamma\delta}$ and $T^{\rm eff}_{\gamma\delta}$ is

$$\begin{aligned}T^{\rm eff}_{\gamma\delta}&=\kappa^2T_{\gamma\delta}-(T_{\gamma\delta}+\Theta_{\gamma\delta})f_T(G,T)+\frac{1}{2}g_{\gamma\delta}f(G,T)-(2RR_{\gamma\delta}-4R^{\varepsilon}_{\gamma}R_{\varepsilon\beta}-4R_{\gamma\varepsilon\delta\eta}R^{\varepsilon\eta}+2R^{\varepsilon\eta\mu}_{\gamma}R_{\delta\varepsilon\eta\mu})f_G(G,T)\\&-(2Rg_{\gamma\delta}\nabla^2-2R\nabla_\gamma\nabla_\delta-4R_{\gamma\delta}\nabla^2-4g_{\gamma\delta}R^{\varepsilon\eta}\nabla_\varepsilon\nabla_\eta+4R^{\varepsilon}_{\gamma}\nabla_\delta\nabla_\varepsilon+4R^{\varepsilon}_{\delta}\nabla_\gamma\nabla_\varepsilon+4R_{\gamma\varepsilon\delta\eta}\nabla^\varepsilon\nabla^\eta)f_G(G,T),\end{aligned}$$

where ∇_γ is an operator for covariant derivations and $\nabla^2\equiv\nabla_\gamma\nabla^\gamma$ and the subscript in the above terms describe the respective partial differentiations. Moreover, the expression for $\Theta_{\gamma\delta}$ is given by

$$\Theta_{\gamma\delta}=g^{\mu\nu}\frac{\delta T_{\mu\nu}}{\delta g_{\gamma\delta}}\,. \tag{6}$$

After using eq. (4), eq. (6) turns out to be

$$\Theta_{\gamma\delta}=g_{\gamma\delta}L_m-2T_{\gamma\delta}-2g^{\mu\nu}\frac{\partial^2 L_m}{\partial g^{\gamma\delta}\partial g^{\mu\nu}}\,.$$

The trace of eq. (5) is given by

$$T+R-(\Theta+T)f_T+2Gf_G+2f-2R\nabla^2 f_G+4R_{\gamma\delta}\nabla^\gamma\nabla^\delta f_G=0.$$

One of the aims of this work is to analyze the role of heat dissipation (q_γ), radiation density (ε) and pressure anisotropicity $\Pi\equiv P_r-P_\perp$ in the definitions of modified scalars functions, the scalars whose expressions can be achieved by the orthogonal breaking down of the Riemann tensor. For this purpose, we assume the following form of the stress-energy tensor:

$$T_{\lambda\nu}=\mu V_\lambda V_\nu+P_\perp h_{\lambda\nu}+\Pi\chi_\lambda\chi_\nu-2\eta\sigma_{\lambda\nu}+\varepsilon l_\lambda l_\nu+q(\chi_\nu V_\lambda+\chi_\lambda V_\nu), \tag{7}$$

where η describes the magnitude of the shear tensor $\sigma_{\gamma\delta}$ and $h_{\gamma\delta}$ is the projection tensor that can be given via four vector V_γ as $h_{\gamma\delta}=g_{\gamma\delta}+V_\gamma V_\delta$. Further, l^β and χ^β are the null and radial four vectors, respectively.

The extra curvature corrections of the $f(G,T)$ gravity can be invoked by considering separate formulations for the functions of G and T. Therefore, we choose an $f(G,T)$ model of the following type:

$$f(G,T)=f(G)+g(T). \tag{8}$$

Models of such type could be regarded as the possible corrections in the well-known $f(G)$ gravity. Nojiri and Odintsov firstly introduced $f(G)$ gravity in [15]. The choices of models given above could be considered as a possible toy models for the understanding of the dark sector of the universe. Here, we use linear $g(T)$ with the aim to see some striking consequences on the dynamics of spherical stars on the basis of extra curvature terms stem from the $f(G)$ gravity. Therefore, we have

$$f(G,T)=f(G)+\lambda T,$$

in which λ is a constant number. To include Gauss-Bonnet corrections, we consider $f(G)$ model containing three different constants α, β and m, given as follows [63]:

$$f(G) = \alpha G^n \left(\beta G^m + 1\right), \tag{9}$$

where $n > 0$. This model was introduced to understand the finite time future singularities.

Now, we consider a diagonally symmetric non-static general form of spherical spacetime as

$$\mathrm{d}s^2 = B^2(t,r)\mathrm{d}r^2 - A^2(t,r)\mathrm{d}t^2 + C^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{10}$$

where the scale factors A, B and C are considered to be positive. We assume that our relativistic sphere have an anisotropic shearing viscous and radiating interior whose mathematical form is mentioned in eq. (7). In the non-tilted frame of reference, the four vectors appearing in the formulation of usual energy momentum tensor (7) are obeying the relations

$$\chi^\nu\chi_\nu = 1, \qquad V^\nu V_\nu = -1, \qquad \chi^\nu V_\nu = 0,$$
$$l^\nu V_\nu = -1, \qquad V^\nu q_\nu = 0, \qquad l^\nu l_\nu = 0,$$

along with their definitions

$$V^\nu = \frac{1}{A}\delta^\nu_0, \qquad \chi^\nu = \frac{1}{C}\delta^\nu_1, \qquad l^\nu = \frac{1}{A}\delta^\nu_0 + \frac{1}{B}\delta^\nu_1, \qquad q^\nu = q(t,r)\chi^\nu.$$

The corresponding expansion scalar and shear tensor are given by

$$\sigma A = \left(\frac{\dot{H}}{H} - \frac{\dot{C}}{C}\right), \qquad \Theta_1 A = \left(\frac{\dot{H}}{H} + \frac{2\dot{C}}{C}\right),$$

where dot notation represents the $\frac{\partial}{\partial t}$ operator. The $f(G,T)$ field equations for eqs. (7)–(10) are

$$G_{00} = A^2\left[\mu + \varepsilon + \varepsilon\lambda - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\varphi_{00}}{A^2}\right], \tag{11}$$

$$G_{01} = BA\left[-(1+\lambda)(q+\varepsilon) + \frac{\varphi_{01}}{BA}\right], \tag{12}$$

$$G_{11} = B^2\left[\mu\lambda + (1+\lambda)\left(P_r + \varepsilon - \frac{4}{3}\eta\sigma\right) + \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n + \frac{\lambda T}{2} + \frac{\varphi_{11}}{H^2}\right], \tag{13}$$

$$G_{22} = C^2\left[(1+\lambda)\left(P_\perp + \frac{2}{3}\eta\sigma\right) + \mu\lambda + \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n + \frac{\lambda T}{2} + \frac{\varphi_{22}}{C^2}\right], \tag{14}$$

where the expressions of $G_{\gamma\delta}$ can be found from [55]. Here, the notation prime indicates radial partial differentiation. Now, we consider the definitions of the fluid 4-velocity as

$$U = D_T C = \frac{\dot{C}}{A}. \tag{15}$$

The mass $\mathfrak{m}$ for the spherical structures can be calculated via Misner-Sharp directions as [64]

$$\mathfrak{m}(t,r) = \frac{C}{2}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{H^2}\right). \tag{16}$$

The variations in the physical quantity $\mathfrak{m}$ can found through eqs. (11)–(13) and (15) as

$$D_T\mathfrak{m} = \frac{-1}{2}\left[U\left\{(1+\lambda)\left(\bar{P}_r - \frac{4}{3}\eta\sigma\right) + \lambda\mu + \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n + \frac{\lambda T}{2} + \frac{\varphi_{11}}{H^2}\right\} + E\left\{(1+\lambda)\bar{q} - \frac{\varphi_{01}}{BA}\right\}\right], \tag{17}$$

$$D_C\mathfrak{m} = \frac{C^2}{2}\left[\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\varphi_{00}}{A^2} - \frac{U}{E}\left\{\frac{\varphi_{01}}{AH} - (1+\lambda)\bar{q}\right\}\right], \tag{18}$$

where $\bar{H} = h + \varepsilon$, while $D_T = \frac{1}{A}\frac{\partial}{\partial t}$. Equation (18) can be remanipulated through $E \equiv \frac{C'}{H}$ as

$$\mathfrak{m} = \frac{1}{2}\int_0^C C^2\left[\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\varphi_{00}}{A^2} + \frac{U}{E}\left\{\frac{\varphi_{01}}{BA} + (1+\lambda)\bar{q}\right\}\right]\mathrm{d}C. \tag{19}$$

The influences of tidal forces can also be described as

$$E \equiv \frac{C'}{H} = \left[1 + U^2 - \frac{2\mathfrak{m}(t,r)}{C}\right]^{1/2}. \tag{20}$$

Equations (17)–(20) yield

$$\frac{3\mathfrak{m}}{C^3} = \frac{3\kappa}{2C^3}\int_0^r \left[\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\varphi_{00}}{A^2} + \frac{U}{E}\left\{(1+\lambda)\bar{q} + \frac{\varphi_{01}}{BA}\right\}C^2C'\right]\mathrm{d}r. \tag{21}$$

This expression connects $f(G,T)$ dark source corrections with the dissipative fluid variables, like spherical mass, energy density, heat conduction. The decomposition of the Weyl tensor provides us with two major parts, namely, magnetic and electric denoted, respectively, by $H_{\alpha\beta}$ and $E_{\alpha\beta}$. These two are defined, respectively, as

$$H_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\gamma\eta\delta}C^{\eta\delta}{}_{\beta\rho}V^\gamma V^\rho = \tilde{C}_{\alpha\gamma\beta\delta}V^\gamma V^\delta =, \qquad E_{\alpha\beta} = C_{\alpha\phi\beta\varphi}V^\phi V^\varphi,$$

where $\epsilon_{\lambda\mu\nu\omega} \equiv \sqrt{-g}\eta_{\lambda\mu\nu\omega}$ with $\eta_{\lambda\mu\nu\omega}$ as a Levi-Civita symbol. The quantity $E_{\lambda\nu}$ can be given via V_γ as

$$E_{\lambda\nu} = \left[\chi_\lambda\chi_\nu - \frac{g_{\lambda\nu}}{3} - \frac{1}{3}V_\lambda V_\nu\right]\mathcal{E},$$

where $\mathcal{E}$ is a Weyl scalar whose expression can be given alternatively as

$$\mathcal{E} = \left[\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) - \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right]\frac{1}{2A^2} - \frac{1}{2C^2} - \left[-\left(\frac{A'}{A} - \frac{C'}{C}\right)\left(\frac{C'}{C} + \frac{B'}{B}\right) + \frac{C''}{C} - \frac{A''}{A}\right]\frac{1}{2B^2}. \tag{22}$$

The Weyl scalar $\mathcal{E}$ with extra degrees of freedom mediated from $f(G,T)$ gravity is given by

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2}\left[\bar{\mu} + \lambda\varepsilon - (1+\lambda)(\bar{\Pi} - 2\eta\sigma) - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} - \frac{\varphi_{00}}{A^2} + \frac{\varphi_{11}}{B^2} - \frac{\varphi_{22}}{C^2}\right] \\ & - \frac{3}{2C^3}\int_0^r C^2\left[\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\varphi_{00}}{A^2} + \frac{U}{E}\left\{(1+\lambda)\bar{q} - \frac{\varphi_{01}}{BA}\right\}\right]C'\mathrm{d}r,\end{aligned} \tag{23}$$

where $\bar{\Pi} \equiv \bar{P}_r - P_\perp$.

3 Structure scalars and f(G, T) gravity

This section discusses the analytical computation of extended forms of structure scalars for the spherical relativistic interiors framed with $f(G) + \lambda T$ gravity. In this background, we define couple of tensorial expressions, $X_{\alpha\beta}$ and $Y_{\alpha\beta}$, that were presented firstly by Herrera *et al.* [54, 55, 65]. They not only presented the way to evaluate such scalar (orthogonal splitting of Riemann tensor) but also used these variables in the modeling of the many stellar objects in the realm of GR. These are

$$X_{\alpha\beta} = {}^*R^*_{\alpha\gamma\beta\delta}V^\gamma V^\delta = \frac{1}{2}\eta^{\varepsilon\rho}{}_{\alpha\gamma}R^*_{\epsilon\rho\beta\delta}V^\gamma V^\delta, \qquad Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}V^\gamma V^\delta, \tag{24}$$

where the notation steric on the both, left- and right-sides of the tensor indicate double, left and right duals of the subsequent entities, respectively,

$$\begin{aligned}X_{\gamma\delta} = X^{(m)}_{\gamma\delta} + X^{(D)}_{\gamma\delta} = {} & \frac{1}{3}\left[\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \frac{\psi_{00}}{A^2}\right]h_{\gamma\delta} \\ & - \frac{1}{2}\left[(1+\lambda)(\bar{\Pi} - 2\eta\sigma) - \frac{\psi_{11}}{B^2} + \frac{\psi_{22}}{C^2}\right]\left(\chi_\gamma\chi_\delta - \frac{1}{3}h_{\gamma\delta}\right) - E_{\gamma\delta},\end{aligned} \tag{25}$$

$$\begin{aligned}Y_{\gamma\delta} = Y^{(m)}_{\gamma\delta} + Y^{(D)}_{\gamma\delta} = {} & \frac{1}{6}\left[\bar{\mu} + \lambda\varepsilon + 3\mu\lambda + (1+\lambda)(3P_r - 2\bar{\Pi}) - \frac{\psi_{00}}{A^2} - \frac{\psi_{11}}{B^2} + \frac{2\psi_{22}}{C^2} + \frac{\alpha}{2}\right. \\ & \left.\times\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n + \frac{\lambda T}{2}\right]h_{\gamma\delta} + \frac{1}{2f_R}\left[(1+\lambda)(\bar{\Pi} - 2\eta\sigma) - \frac{\psi_{11}}{B^2} + \frac{\psi_{22}}{C^2}\right] \\ & \times\left(\chi_\gamma\chi_\delta - \frac{1}{3}h_{\gamma\delta}\right) - E_{\gamma\delta}.\end{aligned} \tag{26}$$

Such tensorial forms can be given in another way via their trace (denoted with subscript T) and trace-less (labeled with subscript TF) parts as

$$X_{\gamma\delta} = \frac{1}{3}\,\mathrm{Tr}\,X h_{\gamma\delta} + X_{\langle\gamma\delta\rangle}, \tag{27}$$

$$Y_{\gamma\delta} = \frac{1}{3}\,\mathrm{Tr}\,Y h_{\gamma\delta} + Y_{\langle\gamma\delta\rangle}, \tag{28}$$

where

$$X_{\langle\gamma\delta\rangle} = h^{\nu}_{\gamma} h^{\mu}_{\delta}\left(X_{\nu\mu} - \frac{1}{3}\,\mathrm{Tr}\,X h_{\nu\mu}\right), \tag{29}$$

$$Y_{\langle\gamma\delta\rangle} = h^{\nu}_{\gamma} h^{\mu}_{\delta}\left(Y_{\nu\mu} - \frac{1}{3}\,\mathrm{Tr}\,Y h_{\nu\mu}\right). \tag{30}$$

From eqs. (23)–(26), we found

$$\mathrm{Tr}\,X \equiv X_T = \left\{\bar{\mu} + \lambda\varepsilon - \frac{\alpha}{2}\left\{\beta(1-n-m)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} - \frac{\lambda}{2}T + \frac{\hat{\psi}_{00}}{A^2}\right\}, \tag{31}$$

$$\mathrm{Tr}\,Y \equiv Y_T = \left\{\bar{\mu} + \lambda\varepsilon + 3\mu\lambda + 3(1+\lambda)\bar{P}_r - \frac{\hat{\psi}_{11}}{B^2} + 2\alpha(1-n)R^n - \frac{\hat{\psi}_{00}}{A^2}\right.$$
$$\left. -2(1+\lambda)\bar{\mathit{\Pi}} - \frac{2\hat{\psi}_{22}}{C^2} + 2\beta(3-n)R^{2-n} - 2\lambda T\right\}, \tag{32}$$

where X_{TF} and Y_{TF} stand for the trace-free components of the tensors $X_{\alpha\beta}$ and $Y_{\alpha\beta}$, respectively (for details, please see [66–74]). We can also write $X_{\langle\alpha\beta\rangle}$ and $Y_{\langle\alpha\beta\rangle}$ in an alternate form:

$$X_{\langle\gamma\delta\rangle} = X_{TF}\left(\chi_\gamma\chi_\delta - \frac{1}{3}h_{\gamma\delta}\right), \tag{33}$$

$$Y_{\langle\gamma\delta\rangle} = Y_{TF}\left(\chi_\gamma\chi_\delta - \frac{1}{3}h_{\gamma\delta}\right). \tag{34}$$

Using eqs. (11)–(15), (28) and (29), we obtain

$$X_{TF} = -\mathcal{E} - \frac{1}{2}\left\{(\lambda+1)(-2\sigma\eta + \bar{\mathit{\Pi}}) + \frac{\varphi_{22}}{C^2} - \frac{\varphi_{11}}{H^2}\right\}, \tag{35}$$

$$Y_{TF} = \mathcal{E} - \frac{1}{2}\left\{(\bar{\mathit{\Pi}} - 2\eta\sigma)(\lambda+1) + \frac{\varphi_{22}}{C^2} - \frac{\varphi_{11}}{H^2}\right\}. \tag{36}$$

The value of Y_{TF} can follow from eqs. (23) and (36):

$$Y_{TF} = \frac{1}{2}\left(\bar{\mu} + \varepsilon\lambda - \frac{\alpha}{2}\left\{\beta(1-m-n)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} - 2(1+\lambda)(\bar{\mathit{\Pi}} - 4\eta\sigma) - \frac{\varphi_{00}}{A^2} + \frac{2\varphi_{11}}{H^2} - \frac{2\varphi_{22}}{C^2}\right)$$
$$-\frac{3}{2C^3}\int_0^r \frac{C^2}{1+2R\lambda T^2}\left[\bar{\mu} - \frac{\alpha}{2}\left\{\beta(1-m-n)G^m + (1-n)\right\}G^n - \frac{\lambda T}{2} + \varepsilon\lambda + \frac{\varphi_{00}}{A^2} + \frac{U}{E}\left\{(1+\lambda)\bar{q} + \frac{\varphi_{01}}{AB}\right\}C^2C'\right]\mathrm{d}r. \tag{37}$$

It can be interesting to visualize our set of equations in terms of some dagger variables which are defined as follows:

$$\mu^{\dagger} \equiv \bar{\mu} - \frac{\varphi_{00}}{A^2}, \qquad P_r^{\dagger} \equiv \bar{P}_r - \frac{\varphi_{11}}{H^2} - \frac{4}{3}\eta\sigma,$$
$$P_\perp^{\dagger} \equiv P_\perp - \frac{\varphi_{22}}{C^2} + \frac{2}{3}\eta\sigma,$$
$$\mathit{\Pi}^{\dagger} \equiv P_r^{\dagger} - P_\perp^{\dagger} = \mathit{\Pi} - 2\eta\sigma + \frac{\varphi_{22}}{C^2} - \frac{\varphi_{11}}{B^2}.$$

Then, eqs. (30)–(36) give

$$X_{TF}=\frac{3}{2C^3}\int_0^r\left[\left\{\mu^\dagger-\frac{\alpha}{2}\left\{G^m\beta(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}+\lambda\varepsilon+\left(\hat{q}(\lambda+1)+\frac{\varphi_q}{BA}\right)\times\frac{U}{E}\right\}C^2C'\right]\mathrm{d}r$$
$$-\frac{1}{2}\left[\mu^\dagger-\frac{\alpha}{2}\left\{G^m\beta(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}+\lambda\varepsilon\right], \quad (38)$$

$$Y_{TF}=\frac{1}{2}\left[\mu^\dagger-\frac{\alpha}{2}\left\{G^m\beta(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}+\varepsilon\lambda-2\lambda\right.$$
$$\left.\times\left(\frac{\varphi_{11}}{H^2}-\frac{\varphi_{22}}{C^2}\right)\right]-\frac{3}{2C^3}\int_0^r\left[\left\{\mu^\dagger-\frac{\alpha}{2}\left\{G^m\beta(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}+\lambda\varepsilon\right.\right.$$
$$\left.\left.+\left(\hat{q}(\lambda+1)+\frac{\varphi_q}{BA}\right)\frac{U}{E}\right\}C^2C'\right]\mathrm{d}r, \quad (39)$$

$$Y_T=\frac{1}{2}\left[(1+3\lambda)\mu^\dagger-2\lambda\varepsilon+3(1+\lambda)P_r^\dagger-2\Pi^\dagger(1+\lambda)+\frac{\alpha}{2}\left\{\beta(1-m-n)G^m+(1-n)\right\}G^n\right.$$
$$\left.+\lambda\left(2\frac{\varphi_{22}}{C^2}+\frac{\varphi_{11}}{B^2}+3\frac{\varphi_{00}}{A^2}\right)+\frac{\lambda T}{2}\right], \quad (40)$$

$$X_T=\mu^\dagger-\frac{\alpha}{2}\left\{\beta G^m(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}+\varepsilon\lambda. \quad (41)$$

These are $f(G,T)$ structure scalars which are four in number. These scalars occupy very important role in the dynamical properties of the self-gravitating structures, for example, irregular energy density, mass function, tidal forces, curvature of spacetime, etc. The well-known equations like, shear evolution equation (SEE), expansion evolution equation (EEE) also known as Raychaudhuri equation and the Weyl differential equation (WDE). The so-called Raychaudhuri equation was also calculated autonomously by Landau [75]. Through $f(G,T)$ scalar variables, the one of the structure scalar Y_T can be recasted as

$$-(Y_T)=\frac{1}{3}\left(2\sigma^{\alpha\beta}\sigma_{\alpha\beta}+\Theta^2\right)+V^\alpha\Theta_{;\alpha}-a^\alpha_{;\alpha}. \quad (42)$$

It can be checked from the above expression that the notable EEE can be well-written through one of the $f(G,T)$ matter scalar. Similarly, the SEE can be manipulated through Y_{TF} as follows:

$$\mathcal{E}-\frac{1}{2}\left\{(\bar{\Pi}-2\eta\sigma)(\lambda+1)+\frac{\varphi_{22}}{C^2}-\frac{\varphi_{11}}{H^2}\right\}=Y_{TF}=a^2+\chi^\alpha a_{;\alpha}-\frac{aC'}{BC}-\frac{1}{3}\left(2\Theta\sigma+\sigma^2\right)-V^\alpha\sigma_{;\alpha}. \quad (43)$$

Equations (21)–(21) provide WDE for the shearing viscous spherical matter distribution as

$$\left[X_{TF}+\frac{1}{2}\left(\mu^\dagger-\frac{\alpha}{2}\left\{\beta(1-n-m)G^m+(1-n)\right\}G^n-\frac{\lambda T}{2}\right)\right]'=-X_{TF}\frac{3C'}{C}+\frac{1}{2}(\Theta-\sigma),$$
$$\times\left(qB(\lambda+1)+\frac{\varphi_q}{A}\right). \quad (44)$$

It can be seen from the above equation that in the configurations of WDE, the $f(G,T)$ structure scalar, X_{TF}, plays a pivotal role. The solution of the above equation would present X_{TF} as a factor of controlling inhomogeneous matter density in the background of relativistic spheres in $f(G,T)$ gravity.

4 Dust ball with constant G and T

This section is devoted to examine the influences of $f(G)+\lambda T$ MGT on the formulations of SEE, WDE and EEE for the pressure less non-interacting particles with constant G and T terms. We shall use the notation tilde to represent that the corresponding terms are compute with constant choices of G and T. Thus, the mass function in the context turns out to be

$$\mathfrak{m}=\frac{1}{2}\int_0^r\mu C^2\mathrm{d}C-\frac{\lambda R^2T^2}{2\{1+2R\lambda T^2\}}\int_0^r C^2C'\mathrm{d}r. \quad (45)$$

The value of the Weyl scalar in an environment of relativistic dust ball is found to be

$$\mathcal{E}=\frac{1}{2C^3}\int_0^r\mu'C^3\mathrm{d}r-\frac{\alpha}{2}\left\{\beta(1-n-m)G^m+(1-n)\right\}G^n-\frac{\lambda T}{2}, \quad (46)$$

while the spherical dust ball mass function is

$$\frac{3\mathfrak{m}}{C^3}=\frac{1}{2}\left[\mu-\frac{1}{C^3}\int_0^r\mu' C^3\mathrm{d}r\right]-\frac{\alpha}{2}\left\{\beta G^m(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}\,. \tag{47}$$

The set of scalar functions for the case of $f(G,T)$ gravity turns out to be

$$\tilde{X}_T=\mu-\frac{\alpha}{2}\left\{\beta G^m(1-m-n)+(1-n)\right\}G^n-\frac{\lambda T}{2}, \tag{48}$$

$$\tilde{Y}_{TF}=-\tilde{X}_{TF}=\mathcal{E}, \tag{49}$$

$$\tilde{Y}_T=\frac{1}{2}\left[\mu+\alpha\left\{\beta(1-m-n)G^m+(1-n)\right\}G^n+\lambda T\right]. \tag{50}$$

These expressions show that the effects of the scalars, *i.e.*, Y_T, X_T and Y_{TF}, X_{TF} are controlled by matter variables, like $f(G,T)$ terms, tidal forces and fluid energy density, in the subsequent evolution of the star. The causes for the emergence of IED can be explored for the dust matter through the following WDE:

$$\left[\frac{\mu}{2}+\frac{\alpha}{2}\left\{\beta G^m(1-m-n)+(1-n)\right\}G^n+\frac{\lambda T}{2}+\tilde{X}_{TF}\right]'=-\frac{3}{C}\tilde{X}_{TF}C'. \tag{51}$$

This points out that the $f(G,T)$ scalar, *i.e.* $\tilde{X}_{TF}$, is a factor for producing and reducing inhomogeneities over the surface of the initially homogeneous dust ball. One can analyze that $\mu=\mu(t)\Leftrightarrow\tilde{X}_{TF}=0=\alpha=\lambda$. This asserts that dark source terms coming from $f(G,T)$ gravity are trying to produce resistance against the fluctuations of IED. Furthermore, the EEE and SEE boil down to

$$V^\alpha\Theta_{;\alpha}+\frac{2}{3}\sigma^2+\frac{\Theta^2}{3}-a^\alpha_{;\alpha}=\frac{1}{2}\left[\mu+\alpha\left\{\beta(1-n-m)G^m+(1-n)\right\}G^n+\lambda T\right]$$
$$=-\tilde{Y}_T, \tag{52}$$

$$V^\alpha\sigma_{;\alpha}+\frac{\sigma^2}{3}+\frac{2}{3}\sigma\Theta=-\mathcal{E}=-\tilde{Y}_{TF}, \tag{53}$$

which shows the importance of $\tilde{Y}_{TF}$ and $\tilde{Y}_T$ in the definitions of EEE and SEE.

5 Conclusions

In this paper, we have investigated the influences of $f(G,T)$ corrections on some dynamical properties of evolving stellar bodies. For this purpose, we have considered spherically symmetric geometry which is assumed to coupled with anisotropic radiating matter contents. We assumed that the relativistic fluid distribution has a shearing viscous property and is emitting radiations in the free streaming and diffusion approximations. The dissipation is carrying out without scattering. We have then assumed extra degrees of freedom mediated by $f(G,T)$ gravity. We have then calculated the corresponding field equations and dynamical equation. The source of producing tidal forces, *i.e.*, Weyl scalar has been calculated and then related it with the matter and metric variables along with dark sources terms coming from polynomial $f(G,T)$ gravity. With the help of notable Misner-Sharp mass function, we have calculated quantity of matter content within the spherical geometry. We then expressed such a relation with the previously calculated Weyl scalar equation. This relation has peculiar importance in the modeling of stellar structures.

We have then broke down the expression of the Riemann curvature tensor by applying the orthogonal decomposition technique. The technique that was firstly developed by Herrera *et al.* [65]. We have applied this technique for our stellar model with one of the modified gravity theories, *i.e.*, $f(G,T)$ gravity. With this, we have developed relations of two tensor namely, $X_{\mu\nu}$ and $Y_{\mu\nu}$. The trace and trace-free parts of these tensorial quantities are computed, which have utmost relevance in the study of gravitational collapse, stellar evolution, etc. These trace and trace-less parts are called here as $f(G,T)$ structure scalars. We have then evaluated a well-known Raychaudhuri equation and expressed it in terms of these structure scalars. It is worthy to mention that this equation plays a vital role in the discussion of the Penrose-Hawking singularity theorems. This equation has also been used to find many exact solutions of gravitational field equations in the literature. Thus, our one of the structure scalar Y_T occupy fundamental importance in understanding the scenario under which gravitational effects could lead to singularities. Thus, Y_T could be helpful to understand the singularity appearances in various black holes, like Schwarzschild, Kerr, the Reissner-Nordström and the Kerr-Newman metrics, etc.

After that, we have evaluated shear and Weyl scalar evolution equations through $f(G,T)$ structure scalars. The Weyl scalar equation that is expressed through X_{TF} describes the propagation of tidal forces in the modeling of radiating spherical stars in modified gravity. This scalar has utmost relevance in the study of Newman-Penrose formalism along with degrees of freedom mediated by $f(G,T)$ gravity. The scalar X_{TF} which is related with Weyl scalar could be fruitful to examine the outgoing gravitational radiation from the asymptotically flat geometry. Finally, we have encode all of our results for the case of non-dissipative dust ball with constant values of G and T. In this scenario, we have shown that two $f(G,T)$ scalar functions are directly related with Weyl scalar and thus tidal forces, Further, the $f(G,T)$ scalar X_{TF} is controlling the appearance of inhomogeneities in the initially regular compact body.

This work has been supported financially by National Research Project for Universities (NRPU) under research project No. 8754.

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