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Proper treatment of scalar and vector exponential potentials in the Klein-Gordon equation: Scattering and bound states

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Abstract. We point out a misleading treatment in the literature regarding bound-state solutions for the s-wave Klein-Gordon equation with exponential scalar and vector potentials. Following the appropriate procedure for an arbitrary mixing of scalar and vector couplings, we generalize earlier works and present the correct solution to bound states and additionally we address the issue of scattering states. Moreover, we present a new effect related to the polarization of the charge density in the presence of weak short-range exponential scalar and vector potentials.

1 Introduction

The solution of the Klein-Gordon (KG) equation with the exponential potential may find applications in the study of pionic atoms, doped Mott insulators, doped semiconductors, interaction between ions, quantum dots surrounded by a dielectric or a conducting medium, protein structures, etc. Bound-state and scattering s-wave solutions of the three-dimensional KG equation for a minimally coupled exponential potential have already been analyzed in the literature [1]. The bound states were revisited in [2], and the continuum in [3]. The authors of [2], though, were not able to reproduce the numerical results found in [1], and the solution for continuum states found in [3] differs from that one found in [1]. Later, bound-state s-wave solutions received attention for a mixing of vector and scalar couplings with different magnitudes [4–6]. The authors of [4] and [5] used a quantization condition founded on a wrong boundary condition on the radial eigenfunction at the origin by considering the limit to infinity of a variable necessarily finite, so turning Kummer's function into a polynomial. In [6], a method to solve Kummer's equation was applied without paying attention to the proper behaviour of the radial eigenfunction at the origin, obtaining in that way a polynomial expression to Kummer's function. In ref. [7], the authors addressed the problem for arbitrary angular momentum in D dimensions with arbitrary scalar-vector mixing plus an exponential position dependent mass. A position dependent mass can be seen as an additional scalar potential in the KG theory. Manifestly, the eigenfunctions found in ref. [7] satisfy a wrong boundary condition at the origin. It is worthwhile to mention that bound-state solutions for the symmetric exponential potential in the one-dimensional case (sometimes called cusp potential or screened Coulomb potential) have also received attention for vector [8,9], scalar [10] and a general mixing of vector and scalar [11] couplings. Scattering in a repulsive exponential potential minimally coupled has been studied in [12]. For an enough deep and narrow vector potential it might appear additional antiparticle bound states in a potential attractive only for particles, the phenomenon called Schiff-Snyder-Weinberg (SSW) effect [13].

This work presents a detailed qualitative and quantitative analyses of continuum s-wave solutions of the KG equation for attractive or repulsive exponential potentials with arbitrary mixing of vector and scalar couplings in a three-dimensional space. Quantization condition and constraints on the potential parameters for bound states are identified by two different processes: vanishing of the radial eigenfunction at infinity and poles of the scattering amplitude. With this systematic plan of action we not only generalize previous approaches but also elucidate some

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important obscure points referred to in the previous paragraph. A particular case of our results for the continuum gives support to that found in [3]. Unquestionable bound states satisfying proper boundary conditions at the origin and at infinity are obtained from the zeros of Bessel's function of the first kind in the case of vector and scalar couplings with equal magnitudes, or from the zeros of Kummer's function (confluent hypergeometric function) in the case of vector and scalar couplings with unequal magnitudes. There is no need for breaking off the series defining Bessel's function or Kummer's function.

2 KG equation with vector and scalar interactions

The time-independent KG equation for a spinless particle with rest mass m and energy E under the influence of external scalar, S, and vector, V, interactions reads ($\hbar = c = 1$)

$$\left[\nabla^2 + (E - V)^2 - (m + S)^2\right] \phi = 0, \tag{1}$$

with charge density and charge current density expressed as

$$\varrho = \frac{E - V}{m}|\phi|^2, \qquad \boldsymbol{J} = \frac{i}{2m}\left(\phi\vec{\nabla}\phi^* - \phi^*\vec{\nabla}\phi\right). \tag{2}$$

Note that if ϕ is a solution for a particle (antiparticle) with energy E for the potentials V and S, then $\pm\phi^*$ is a solution for a antiparticle (particle) with energy $-E$ and for the potentials $-V$ and S. It is also valuable to note that one finds the nonrelativistic regime governed by Schrödinger equation

$$\left[\nabla^2 + 2m\left(\pm E - m - S \mp V\right)\right]\phi = 0, \tag{3}$$

for weak couplings and $E \simeq \pm m$. In the nonrelativistic regime one finds $\varrho \simeq \pm|\phi|^2$ for $E \simeq \pm m$. However, the charge density has not a definite sign for strong vector couplings. Of course, the resulting binding force depends on the average charge closer to the center of force. Therefore, not only for strong couplings, intrinsically relativistic effects can also be related to short-range vector potentials due the polarization of the charge density.

For spherically symmetric interactions, *i.e.* $S(\vec{r}) = S(r)$ and $V(\vec{r}) = V(r)$, the wave function can be factorized as

$$\phi_{\mu l m_l}(\vec{r}) = \frac{u_\mu(r)}{r}Y_{lm_l}(\theta, \varphi), \tag{4}$$

where $Y_{lm_l}(\theta, \varphi)$ is the usual spherical harmonic, with $l = 0, 1, 2, \ldots$, $m_l = -l, -l+1, \ldots, l$ and μ denotes the principal quantum number plus other possible quantum numbers which may be necessary to characterize ϕ.

The radial function $u(r)$ obeys the radial equation (for $l = 0$, s-wave)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[k^2 + V^2 - S^2 - 2(EV + mS)\right]u = 0, \tag{5}$$

with $k = \sqrt{E^2 - m^2}$. Equation (5) is effectively the time-independent Schrödinger equation with the effective potential $S^2 - V^2 + 2(EV + mS)$. One can see that the effective potential tends to $S^2 - V^2$ for potentials which tend to infinity at large distances so that the KG equation furnishes a purely discrete (continuous) spectrum for $|S| > |V|$ ($|S| < |V|$). On the other hand, if the potentials vanish at large distances the continuum spectrum is omnipresent but the necessary conditions for the existence of a discrete spectrum is not an easy task for general functional forms. Assuming that $r^2 S(r)$ and $r^2 V(r)$ go to zero as $r \to 0$, one must impose the homogeneous Dirichlet condition $u(0) = 0$ (see, *e.g.* [14]). On the other hand, if both potentials vanish at large distances the solution u behaves like $e^{\pm ikr}$ as $r \to \infty$.

For scattering states in spherically symmetric scatterers, the scattering amplitude can be written as a partial wave series (see, *e.g.*, [14])

$$f_k(\theta) = \sum_{l=0}^{\infty}(2l + 1)f_l(k)P_l(\cos\theta), \tag{6}$$

where θ is the angle of scattering, P_l is the Legendre polynomial of order l and the partial scattering amplitude is

$$f_l(k) = \left[e^{2i\delta_l(k)} - 1\right]/(2ik). \tag{7}$$

For elastic scattering the phase shift $\delta_l(k)$ is a real number in such a way that at large distances

$$u(r) \sim e^{-ikr} + (-1)^{l+1}e^{2i\delta_l(k)}e^{+ikr}. \tag{8}$$

Information about the energies of the bound-state solutions can be obtained from poles of the partial scattering amplitude when one considers k imaginary, but it carries an important caveat: not all the poles correspond to bound states. For potentials with range a one finds $ka \ll l$ (see, *e.g.*, [14]). Hence, for short-range potentials and low enough momentum the partial wave series converges rapidly and the contribution is predominantly s-wave, *i.e.* $f_k(\theta) \approx f_0(k)$, which is of great importance for what follows.

3 Exponential potentials

Let us consider scalar and vector exponential interactions in the form

$$S(r) = -S_0 e^{-\alpha r}, \qquad V(r) = -V_0 e^{-\alpha r}, \tag{9}$$

where α is a positive constant. Substituting (9) into (5) we get

$$\frac{d^2 u}{dr^2} + \left(k^2 - V_1 e^{-\alpha r} - V_2 e^{-2\alpha r} \right) u = 0, \tag{10}$$

where

$$V_1 = -2(EV_0 + mS_0), \qquad V_2 = S_0^2 - V_0^2. \tag{11}$$

Equation (10) is effectively a time-independent Schrödinger equation for the exponential potential, when $|S_0| = |V_0|$, and for the generalized Morse potential, when $|S_0| \neq |V_0|$. These effective potentials have well structures, when $V_1 < 0$ and $V_2 > 0$, or $V_1 < -V_2$ and $V_2 < 0$. Bound states are expected for $|E| < m$. By the way, the positive(negative)-energy solutions are not to be promptly identified with the bound states for particles (antiparticles). Rather, whether it is positive or negative, an eigenenergy can be unambiguously identified with a bound-state solution for a particle (antiparticle) only by observing if the energy level emerges from the upper (lower) continuum.

When $|S_0| = |V_0|$, no bound state is expected when $S_0 < 0$. Nevertheless, when $S_0 > 0$ and $V_0 = +S_0$ ($V_0 = -S_0$) the well potential is deeper (shallower) for positive-energy levels than that one for negative-energy levels, and bound states with $E \simeq -m$ ($E \simeq +m$) can only be found asymptotically as S_0 increases. In this particular case, one can asseverate that the discrete spectrum consists only of particle (antiparticle) energy levels with no chance for pair production associate with Klein's paradox.

When $|S_0| \neq |V_0|$ the possible existence of bound-state solutions permits us to distinguish two subclasses: a) $V_1 < 0$ and $V_2 > 0$, corresponding to $S_0 + V_0 E/m > 0$ with $S_0 > |V_0|$, allowing the presence of energy levels with $E \simeq \pm m$; b) $V_1 < -V_2$ and $V_2 < 0$, with positive(negative)-energy levels occurring exclusively for $V_0 > 0$ ($V_0 < 0$) with $-|V_0| < S_0 < m - \sqrt{m^2 + V_0^2}$. More than this, there may appear energy levels for $V_0 \gtrless 0$ with $E \simeq \pm m$ when $|V_0| < m$ and $|S_0| < |V_0|$, and $E \simeq \mp m$ when $|V_0| > m$ and $-|V_0| + 2m < S_0 < |V_0|$. In this last case, the spectrum including $E \simeq \mp m$ for a strong pure vector coupling with $V_0 \gtrless \pm 2m$ may be related either to the SSW effect or to energy levels of particles (antiparticles) diving into the continuum of antiparticles (particles). Because the scalar coupling does not contribute to the polarization of the charge density its addition contributes to lower the threshold of this peculiar effect: $V_0 = \pm m$ when $S_0 = m$.

Now we move to consider a quantitative treatment of our problem by considering the two distinct classes of effective potentials.

4 The effective exponential potential ($|\mathbf{S_0}| = |\mathbf{V_0}|$)

With the change of variable

$$y = y_0 e^{-\alpha r/2} \tag{12}$$

and the definitions

$$y_0 = \frac{2i\sqrt{V_1}}{\alpha}, \qquad \nu = \frac{2ik}{\alpha}, \tag{13}$$

eq. (10) becomes Bessel's equation of order ν

$$y^2 \frac{d^2 u}{dy^2} + y \frac{du}{dy} + \left(y^2 - \nu^2 \right) u = 0. \tag{14}$$

One solution of this equation is Bessel's function of the first kind of order ν [15]

$$J_\nu(y) = \sum_{j=0}^{\infty} (-1)^j \frac{(y/2)^{\nu+2j}}{j! \Gamma(\nu+j+1)}, \tag{15}$$

where $\Gamma(z)$ denotes the meromorphic gamma function with no zeros, and with simple poles $z = 0, -1, -2, \ldots$ Bessel's function $J_\nu(z)$ is an analytic function, except for a branch point at $z = 0$. The principal branch of $J_\nu(z)$ is analytic in the z-plane cut along the interval $(-\infty, 0]$. For $z \neq 0$ each branch of $J_\nu(y)$ is entire in ν. Bessel's function of real order has an infinite number of positive zeros $j_{\nu n}$, where n designates the n-th zero, and all of these zeros are simple. The zeros obey the inequalities $j_{\nu,n} < j_{\nu+1,n}\ j_{\nu,n+1}$, when $\nu \geqslant 0$ [15].

The general solution of eq. (14) can be expressed as

$$u(y) = AJ_{+\nu}(y) + BJ_{-\nu}(y), \quad \nu \neq \text{integer}. \tag{16}$$

The condition $u|_{r=0} = 0$ makes

$$u(y) = \begin{cases} AJ_{-\nu}(y), & \text{for } J_{-\nu}(y_0) = 0, \\ A\left[J_{+\nu}(y) - \dfrac{J_{+\nu}(y_0)}{J_{-\nu}(y_0)}J_{-\nu}(y)\right], & \text{for } J_{-\nu}(y_0) \neq 0, \end{cases} \tag{17}$$

and the limiting form for small argument of Bessel' function prescribes that u behaves for large r as

$$u(r) \sim \begin{cases} \dfrac{(y_0/2)^{-\nu}}{\Gamma(1-\nu)}e^{+\nu\alpha r/2}, & \text{for } J_{-\nu}(y_0) = 0, \\ \dfrac{(y_0/2)^{\nu}}{\Gamma(1+\nu)}e^{-\nu\alpha r/2} - \dfrac{(y_0/2)^{-\nu}}{\Gamma(1-\nu)}\dfrac{J_{+\nu}(y_0)}{J_{-\nu}(y_0)}e^{+\nu\alpha r/2}, & \text{for } J_{-\nu}(y_0) \neq 0. \end{cases} \tag{18}$$

For $J_{-\nu}(y_0) = 0$, the asymptotic behaviour only suggests that bound states might exist if $\nu < 0$. As for $J_{-\nu}(y_0) \neq 0$, the asymptotic behaviour suggests that bound states might exist if $\nu > 0$ and $J_{+\nu}(y_0) = 0$, and scattering states requires that ν is an imaginary.

4.1 Bound states

In this case k is an imaginary number. This means that $|E| < m$. Regardless the sign of ν and explicitly using the fact that ν is not an integer, the condition determining bound-state solutions takes the concise form

$$J_{2|k|/\alpha}(y_0) = 0, \tag{19}$$

with corresponding eigenfunction expressed as

$$u(r) = AJ_{2|k|/\alpha}\left(y_0 e^{-\alpha r/2}\right). \tag{20}$$

Because of the way the zeros of Bessel's function of positive order interlace, one conclude that the s-wave spectrum is nondegenerate. The order of Bessel's function in eq. (19) is a positive number so that $y_0 > 0$ and the effective exponential potential has a well structure when $S_0 > 0$. As a result, from eqs. (11) and (13), S_0 must be strong enough to make the existence of bound states possible. In fact, eq. (19) has at least one solution, when

$$S_0 > \frac{(\alpha j_{2|k|/\alpha,1})^2}{8(m \pm E)}, \qquad V_0 = \pm S_0. \tag{21}$$

Therefore, one conclude that a solution with $E \simeq \pm m$ appears if $S_0 \gtrsim (\alpha j_{0,1})^2/(16m)$. Consequently, bound states in a weak potential are only allowed if the range of the potential is enough large. On the other hand, solutions with $E \simeq \mp m$ might appear for very large S_0. Bound-state solutions in a short-range potential need strong couplings.

4.2 Scattering states

As for the continuous spectrum, k is a real number so that $|E| > m$. From the second line of eq. (18), the asymptotic form of u for large r clearly shows incoming and outgoing partial s-waves with amplitudes differing by factors related to the phase shift,

$$u(r) \sim \frac{(y_0/2)^{2ik/\alpha}}{\Gamma(1+2ik/\alpha)}e^{-ikr} - \frac{(y_0/2)^{-2ik/\alpha}}{\Gamma(1-2ik/\alpha)}\frac{J_{+2ik/\alpha}(y_0)}{J_{-2ik/\alpha}(y_0)}e^{+ikr}, \tag{22}$$

in such a way that

$$e^{2i\delta_0} = \left(\frac{2}{y_0}\right)^{4ik/\alpha}\frac{\Gamma(1+2ik/\alpha)}{\Gamma(1-2ik/\alpha)}\frac{J_{+2ik/\alpha}(y_0)}{J_{-2ik/\alpha}(y_0)}. \tag{23}$$

Therefore, if one considers the analytic continuation for the entire complex k-plane $J_{+2ik/\alpha}(y_0)$ is an analytic function of k, and the same happens with $\Gamma(1+2ik/\alpha)$ except for $k = i\alpha n/2$ with $n = 1, 2, 3\ldots$ Hence, the partial scattering

amplitude for s-waves is analytical in the entire complex k-plane, except for isolated singularities related either to the poles of the gamma function or to the zeros of $J_{-2ik/\alpha}(y_0)$. It is true that poles of the partial scattering amplitude for s-waves make

$$u(r) \sim e^{-|k|r}, \quad kr \gg 1, \tag{24}$$

so that they could be related to bound states. However, poles of the gamma function do not furnish licit bound states because they make $\nu = -1, -2, -3, \ldots$, values already excluded from the general solution expressed by eq. (16). There remains

$$J_{-2ik/\alpha}(y_0) = 0. \tag{25}$$

As seen before, only the solution with $k = +i|k|$ correspond to bound states.

5 The effective generalized Morse potential ($|S_0| \neq |V_0|$)

With the changes

$$y = y_0 e^{-\alpha r}, \qquad u(y) = y^{-1/2} w(y) \tag{26}$$

and the definitions

$$y_0 = \frac{2\sqrt{V_2}}{\alpha}, \qquad \kappa = -\frac{V_1}{\alpha^2 y_0}, \qquad \nu = \frac{ik}{\alpha}, \tag{27}$$

eq. (10) becomes Whittaker's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \left(-\frac{1}{4} + \frac{\kappa}{y} + \frac{1/4 - \nu^2}{y^2} \right) w = 0. \tag{28}$$

The general solution of (28) can be expressed in terms of Kummer's function [15]

$$M(a_1, b_1, z) = \frac{\Gamma(b_1)}{\Gamma(a_1)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + n)}{\Gamma(b_1 + n)} \frac{z^n}{n!}, \quad b_1 \neq 0, -1, -2, \ldots \tag{29}$$

Kummer's function $M(a_1, b_1, z)$ is entire in z and a_1, and is a meromorphic function of b_1 with simple poles at $b_1 = 0, -1, -2, \ldots$ It converges to $e^z z^{a_1 - b_1}/\Gamma(a_1)$ as $z \to \infty$ and has an infinite set of complex zeros when a_1 and $b_1 - a_1$ are different from $0, -1, -2, \ldots$ The number of real zeros is finite when both a_1 and b_1 are real. For $b_1 \geq 0$, Kummer's function has no zeros when $a_1 \geqslant 0$, and a number of positive zeros given by the ceiling of $-a_1$ when $a_1 < 0$ [15].

Now, y lies in the interval $(0, y_0]$ and the general solution of eq. (28) is expressed as

$$w(y) = y^{1/2} e^{-y/2} \left[A y^\nu M^{(+)}(y) + B y^{-\nu} M^{(-)}(y) \right], \tag{30}$$

with

$$M^{(\pm)}(y) = M(1/2 - \kappa \pm \nu, 1 \pm 2\nu, y). \tag{31}$$

Therefore,

$$u(y) = e^{-y/2} \left[A y^\nu M^{(+)}(y) + B y^{-\nu} M^{(-)}(y) \right]. \tag{32}$$

The condition $u|_{r=0} = 0$ enforces

$$u(y) = \begin{cases} A e^{-y/2} y^{-\nu} M^{(-)}(y), & \text{for } M^{(-)}(y_0) = 0, \\ A e^{-y/2} \left[\left(\dfrac{y}{y_0} \right)^\nu M^{(+)}(y) - \dfrac{M^{(+)}(y_0)}{M^{(-)}(y_0)} \left(\dfrac{y}{y_0} \right)^{-\nu} M^{(-)}(y) \right], & \text{for } M^{(-)}(y_0) \neq 0. \end{cases} \tag{33}$$

From eq. (29), $M(a_1, b_1, 0) = 1$, hence one gets the following asymptotic expression for u at large distance:

$$u(r) \sim \begin{cases} y_0^{-\nu} e^{+\nu \alpha r}, & \text{for } M^{(-)}(y_0) = 0, \\ e^{-\nu \alpha r} - \dfrac{M^{(+)}(y_0)}{M^{(-)}(y_0)} e^{+\nu \alpha r}, & \text{for } M^{(-)}(y_0) \neq 0. \end{cases} \tag{34}$$

For $M^{(-)}(y_0) = 0$, the asymptotic behaviour only suggests that bound states might exist if $\nu < 0$. As for $M^{(-)}(y_0) \neq 0$, the asymptotic behaviour suggests that bound states might exist if $\nu > 0$ and $M^{(+)}(y_0) = 0$, and scattering states requires that ν is an imaginary number.

5.1 Bound states

In this case k is imaginary ($|E| < m$). The quantization condition takes the concise form

$$M(1/2 - \kappa + |k|/\alpha, 1 + 2|k|/\alpha, y_0) = 0, \tag{35}$$

with the corresponding eigenfunction expressed as

$$u(r) = A y_0^{|k|/\alpha} e^{-|k|r - y_0 e^{-\alpha r/2}} M(1/2 - \kappa + |k|/\alpha, 1 + 2|k|/\alpha, y_0 e^{-\alpha r}). \tag{36}$$

$|S_0| > |V_0|$ makes $V_2 > 0$, and the relation mentioned before involving the number of positive zeros and the parameters of Kummer's function requires $V_1 < 0$. This is actually when the effective generalized Morse potential has a well structure. Furthermore, $S_0 > |V_0|$, and

$$|V_1|/\sqrt{V_2} \gtrsim \alpha. \tag{37}$$

This last condition let us to get some conclusions regarding the range of the potential. One finds $E \simeq \pm m$ for

$$S_0 \gtrsim \mp V_0 \frac{1+\beta}{1-\beta}, \quad \beta = \left(\frac{\alpha}{2m}\right)^2. \tag{38}$$

Hence, solutions with $E \simeq \pm m$ ($E \simeq \mp m$) and $V_0 \gtrless 0$ for large(short)-range potentials just demand $S_0 > |V_0|$, whereas solutions with $E \simeq \mp m$ ($E \simeq \pm m$) demand a stronger scalar coupling $S_0 \gtrsim |V_0|(1 + 2\beta)$ ($S_0 \gtrsim |V_0|(1 + 2/\beta)$). Even for weak potentials, the absence of solutions with $E \simeq \pm m$ for $|V_0| < S_0 \lesssim |V_0|(1 + 2/\beta)$ for short-range potentials is a genuine relativistic quantum effect. Due to the polarization of the charge density, an attractive (repulsive) vector coupling for particles (antiparticles) for a large-range potential undergoes reversion of its effects as the range of the potential decreases. Because the stronger scalar coupling is always attractive, the final outcome for this sort of mixing of couplings is that the large(short)-range potential is more attractive for particles (antiparticles).

As for $|S_0| < |V_0|$, one obtains the set of imaginary zeros. Unfortunately, due to the lack of necessary information about the set of imaginary zeros of Kummer's function, we can not say more than we have already said before.

It does not take long to convince oneself that one can find bound-state solutions for $|S_0| < |V_0|$ as well as for $|S_0| > |V_0|$, at least for short-range strong potentials. In this case, with S_0 and V_0 proportional to α, the quantization condition (35) can be approximate by $M(1/2, 1, y_0) = 0$. Relations between Kummer's function with $a_1 = 1/2$ and $b_1 = 1$ and Bessel's function, *viz.* $M(1/2, 1, 2z) = e^z J_0(z)$ and $M(1/2, 1, 2iz) = e^{iz} J_0(z)$ (see, *e.g.*, [15]), make eq. (35) equivalent to $J_0(|y_0|/2) = 0$ in such way that one finds at least one solution when $|S_0^2 - V_0^2| \gtrsim (\alpha j_{0,1})^2$.

5.2 Scattering states

As for the continuous spectrum, k is a real number so that $|E| > m$. From the second line of eq. (34), the asymptotic form of u for large r clearly shows incoming and outgoing partial s-waves with amplitudes differing by factors related to the phase shift:

$$u(r) \sim e^{-ikr} - \frac{M^{(+)}(y_0)}{M^{(-)}(y_0)} e^{+ikr}, \tag{39}$$

in such a way that

$$e^{2i\delta_0} = \frac{M(1/2 - \kappa + ik/\alpha, 1 + 2ik/\alpha, y_0)}{M(1/2 - \kappa - ik/\alpha, 1 - 2ik/\alpha, y_0)}. \tag{40}$$

Therefore, if one considers the analytic continuation for the entire complex k-plane $M(1/2 - \kappa + ik/\alpha, 1 + 2ik/\alpha, y_0)$ is an analytic function of k, except for $k = i\alpha n/2$ where n is a nonnegative integer. Hence, the partial scattering amplitude for s-waves is analytical in the entire complex k-plane, except for isolated singularities related either to the poles of $M(1/2 - \kappa + ik/\alpha, 1 + 2ik/\alpha, y_0)$ or to the zeros of $M(1/2 - \kappa - ik/\alpha, 1 - 2ik/\alpha, y_0)$. It is true that poles of the partial scattering amplitude for s-waves give

$$u(r) \sim e^{-|k|r}, \quad kr \gg 1, \tag{41}$$

so that they could be related to bound states. However, poles of $M(1/2 - \kappa + ik/\alpha, 1 + 2ik/\alpha, y_0)$ do not furnish licit bound states because they make $\nu = -1, -2, -3, \ldots$, values already excluded from the general solution expressed by eq. (16). There remains

$$M(1/2 - \kappa - ik/\alpha, 1 - 2ik/\alpha, y_0) = 0. \tag{42}$$

6 Final remarks

In this work, we pointed out a misleading treatment in the literature regarding to bound-state solutions for the s-wave KG equation with exponential scalar and vector potentials in a three-dimensional space. We showed a detailed qualitative and quantitative analyses of continuum s-wave solutions of the KG equation for attractive or repulsive exponential potentials with arbitrary mixing of vector and scalar couplings. The care needed in applying the proper boundary conditions was emphasized. Using the proper boundary conditions at the origin and at infinity, we found the quantization condition and constraints on the potential parameters for bound states. We obtained the possible energy levels from the zeros of Bessel's function of the first kind in the case of vector and scalar couplings with equal magnitudes, or from the zeros of Kummer's function in the case of vector and scalar couplings with unequal magnitudes. Although the solutions in refs. [4–7] are licit when one considers the limiting form of Kummer's function as $y_0 \to \infty$ (either for a scalar coupling much stronger than a vector coupling or for large-range potentials), we showed that there is no need for breaking off the series defining Kummer's function in a more general circumstance. Never mentioned in the literature, we showed that an effect related to the polarization of the charge density in the presence of short-range exponential scalar and vector potentials manifests when $S_0 > |V_0|$ even in the case of weak couplings.

We would like to point out that our results for s-wave bound-state solutions (eigenvalues and also eigenfunctions) are exact, whereas the scattering amplitude is approximate. The poles of the scattering amplitude furnish results coincident with the exact bound-state solutions, as it should be in a proper treatment. If one wants to treat the case of arbitrary l one should appeal to approximation methods that is out of the scope of this article. It is important to notice that our results have nothing to do with the class of multiparameter exponential-type potential studied in the Schrödinger equation in [16] and in the KG equation with equal vector and scalar couplings in [17,18]. This is so because the class of multiparameter exponential-type potential studied in [16] and [17] does not reduce to a pure exponential potential as that one studied in this paper, and that one studied in [18] does so only in an approximation scheme.

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