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Exact solutions of semi-discrete sine-Gordon equation

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Abstract. In this paper, we study a semi-discrete sine-Gordon (sd-SG) equation and compute various types of solutions analytically. We apply Darboux transformation to the associated spectral problem and construct N-soliton solutions of sd-SG equation in terms of ratio of ordinary determinants. In addition, we also construct explicit expressions of discrete one-kink, two-kink, kink-antikink, breather and degenerate soliton solutions of sd-SG equation in zero background.

1 Introduction

The sine-Gordon (SG) equation is an important example of two-dimensional non-linear field theory (see, for example, [1–4]). The SG equation is an integrable equation and can be solved using inverse scattering transform (IST) method [5] and also owns all miraculous properties such as existence of soliton and breather solutions, Hamiltonian structure, existence of an infinite sequence of conserved quantities, etc. [6–11]. The SG equation is given by

$$s_{xt} = \sin s, \qquad s = s(x,t), \tag{1}$$

here subscripts denote partial derivative with respect to the indicated variable. The SG equation (1) can also be written as consistency condition of the following matrix-valued linear equation (also known as Lax pair):

$$\Theta_x = \Xi_1 \Theta, \qquad \Theta_t = \Xi_2 \Theta, \tag{2}$$

where the matrices Ξ_1 and Ξ_2 are given by

$$\Xi_1\,(x,t;\lambda) = \begin{pmatrix} \dfrac{is_x}{2} & \lambda \\ \lambda & -\dfrac{is_x}{2} \end{pmatrix}, \tag{3}$$

$$\Xi_2\,(x,t;\lambda) = \frac{1}{4\lambda}\begin{pmatrix} 0 & e^{is} \\ e^{-is} & 0 \end{pmatrix}. \tag{4}$$

The consistency condition (that is, $\Theta_{xt} = \Theta_{tx}$) of linear spectral problem (2) becomes a zero-curvature condition (ZCC), *i.e.*, $\Xi_{1t} - \Xi_{2x} + [\Xi_1, \Xi_2] = 0$, which gives (1).

The SG equation appeared in the study of differential geometry [12]. The SG equation has a wide history and many applications in different branches of sciences and engineering [13]. The SG equation describes oscillations of coupled pendulum [3], propagation of magnetic-flux on Josephson array [14], dynamics of crystal dislocation [15], motion of Bloch wall in magnetic crystals [16] and so on. The SG equation also describes many interesting phenomena of life sciences, for example, it describes the dynamics of DNA (deoxyribonucleic acid) double-helix molecule [17]. There has been an increasing interest in the study of SG equation due to its importance in different areas such as mathematics, sciences (physical and life) and engineering.

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In recent past, significant interest has been developed in the study of discrete integrable systems in the light of early fundamental work of Ablowitz and Ladik [18,19] and Hirota [20,21]. In nature there exists a wide range of physical phenomena that are governed by discrete equations or systems of equations [22,23]. The discrete systems have multiple applications in different fields, such as fluid mechanics, nonlinear optics, plasma physics, quantum gravity, field theories, mathematical biology and economics. The discrete integrable equations also share various interesting properties, such as exactly solvable by IST method, existence of conservation laws and multi-soliton solutions and so on [24–27]. A large number of discrete integrable equations have been solved by using IST method and have obtained discrete soliton solutions. The discrete soliton solution can also be computed by using Hirota bilinearization, dressing method, Bäcklund transformation, Darboux-Crum transformation [28–30] and generalized Cauchy matrix method [31, 32]. Integrability of discrete sine-Gordon equation has been studied [33–39].

All above-mentioned applications of SG equations motivated us to investigate different solutions like kink solutions, kink-kink interaction, kink-antikink interaction, breather solution and degenerate solutions of semi-discrete sine-Gordon (sd-SG) equation. Our work is organized as follows. In sect. 2, a semi-discrete sine-Gordon (sd-SG) equation and its linearization are presented. In sect. 3, we use Darboux transformation, and multi-soliton solutions have been constructed in terms of ratio of ordinary determinants. In sect. 4, we compute explicit expressions of one-kink, two-kink solution, kink-antikink interaction, breather solution and degenerate solutions of sd-SG equation. Finally we illustrate our results by plotting the obtained solution for different values of spectral parameters.

2 Semi-discrete sine-Gordon equation

A semi-discrete sine-Gordon (sd-SG) equation is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(s_{n+1}-s_n\right)=4\gamma\sin\frac{1}{2}\left(s_{n+1}+s_n\right), \tag{5}$$

where γ be a real constant. The sd-SG equations (5) can also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}U_n+U_nV_n-V_{n+1}U_n=O, \tag{6}$$

which can be considered as a consistency condition of following linear difference-differential equations for an auxiliary function $\Theta_n(t;\lambda)$:

$$L\Theta_n=\Theta_{n+1}\equiv U_n\Theta_n=(\lambda J+P_n)\Theta_n, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_n\equiv V_n\Theta_n=\lambda^{-1}Q_n\Theta_n, \tag{8}$$

where L denotes forward shift operator (that is $Lf_n=f_{n+1}$). The coefficient matrices J, P_n, Q_n and the column vector Θ_n are given as

$$J=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad P_n=\begin{pmatrix}\left(\dfrac{\omega_n}{\omega_{n+1}}\right)^{1/2}&0\\0&\left(\dfrac{\omega_n}{\omega_{n+1}}\right)^{-1/2}\end{pmatrix},$$

$$Q_n=\gamma\begin{pmatrix}0&\omega_n^{-1}\\\omega_n&0\end{pmatrix},\qquad \Theta_n=\begin{pmatrix}\psi_n\\\phi_n\end{pmatrix}, \tag{9}$$

here $\omega_n=\exp(is_n)$. The linear difference-differential equations (7)-(8) are known as semi-discrete (sd) Lax pair [34], whereas the condition (6) is known as semi-discrete zero-curvature condition (sd-ZCC).

3 Darboux transformation

The Darboux transformation (DT) is one of the renowned solution-generating technique among different techniques, *e.g.*, Hirota's bilinear method, inverse scattering transform method, dressing method and Bäcklund transformation. The Darboux transformation is a type of gauge transformation that maps a known (trivial) solution to a linear equation into a non-trivial solution.

The Lax pair (7)-(8) can also be written as

$$\psi_{n+1} = \left(\frac{\omega_n}{\omega_{n+1}}\right)^{1/2} \psi_n + \lambda\phi_n, \qquad \phi_{n+1} = \lambda\psi_n + \left(\frac{\omega_n}{\omega_{n+1}}\right)^{-1/2} \phi_n, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n = \lambda^{-1}\gamma\omega_n^{-1}\phi_n, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi_n = \lambda^{-1}\gamma\omega_n\psi_n. \tag{11}$$

Let us define Darboux transformation as

$$\psi_n[1] = \lambda\phi_n - \frac{\lambda_1\phi_n^{(1)}}{\psi_n^{(1)}}\psi_n, \tag{12}$$

$$\phi_n[1] = \lambda\psi_n - \frac{\lambda_1\psi_n^{(1)}}{\phi_n^{(1)}}\phi_n, \tag{13}$$

where $\psi_n^{(1)}$, $\phi_n^{(1)}$ denote particular solution set to the Lax pair (10)-(11) at $\lambda = \lambda_1$. The linear system (10)-(11) is covariant under the action of Darboux transformation (12)-(13), that is

$$\psi_{n+1}[1] = \left(\frac{\omega_n[1]}{\omega_{n+1}[1]}\right)^{1/2} \psi_n[1] + \lambda\phi_n[1],$$

$$\phi_{n+1}[1] = \lambda\psi_n[1] + \left(\frac{\omega_n[1]}{\omega_{n+1}[1]}\right)^{-1/2} \phi_n[1], \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n[1] = \lambda^{-1}\gamma\omega_n^{-1}[1]\phi_n[1],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_n[1] = \lambda^{-1}\gamma\omega_n[1]\psi_n[1]. \tag{15}$$

On substitution of scalar functions $\psi_n[1]$ and $\phi_n[1]$ from (12)-(13) in (14)-(15), we obtain

$$\omega_n[1] = \omega_n\left(\frac{\psi_n^{(1)}}{\phi_n^{(1)}}\right)^2. \tag{16}$$

Using $\omega_n = \exp(is_n)$ in eq. (16), we get a transformed solution to the sd-SG equation (5), that is

$$s_n[1] = s_n + 2i\ln\left(\frac{\phi_n^{(1)}}{\psi_n^{(1)}}\right). \tag{17}$$

Similarly, one can define two-fold Darboux transformation on scalar functions ψ_n, ϕ_n as

$$\psi_n[2] = \lambda\phi_n[1] - \lambda_2\frac{\phi_n^{(2)}[1]}{\psi_n^{(2)}[1]}\psi_n[1], \tag{18}$$

$$\phi_n[2] = \lambda\psi_n[1] - \lambda_2\frac{\psi_n^{(2)}[1]}{\phi_n^{(2)}[1]}\phi_n[1], \tag{19}$$

along with

$$\psi_n^{(2)}[1] = \lambda_2\phi_n^{(2)} - \lambda_1\frac{\phi_n^{(1)}}{\psi_n^{(1)}}\psi_n^{(2)},$$

$$\phi_n^{(2)}[1] = \lambda_2\psi_n^{(2)} - \lambda_1\frac{\psi_n^{(1)}}{\phi_n^{(1)}}\phi_n^{(2)}, \tag{20}$$

where $\psi_n^{(2)}$, $\phi_n^{(2)}$ represent particular solutions to (10)-(11) at $\lambda = \lambda_2$. We can reexpress equations (18) and (19) as ratio of ordinary determinants:

$$\psi_n[2] = \frac{\Delta[3]}{\Delta[2]} = \frac{\begin{vmatrix} \psi_n & \lambda\phi_n & \lambda^2\psi_n \\ \psi_n^{(1)} & \lambda_1\phi_n^{(1)} & \lambda_1^2\psi_n^{(1)} \\ \psi_n^{(2)} & \lambda_2\phi_n^{(2)} & \lambda_2^2\psi_n^{(2)} \end{vmatrix}}{\begin{vmatrix} \psi_n^{(1)} & \lambda_1\phi_n^{(1)} \\ \psi_n^{(2)} & \lambda_2\phi_n^{(2)} \end{vmatrix}}, \tag{21}$$

$$\phi_n[2] = \frac{\Omega[3]}{\Omega[2]} = \frac{\begin{vmatrix} \phi_n & \lambda\psi_n & \lambda^2\phi_n \\ \phi_n^{(1)} & \lambda_1\psi_n^{(1)} & \lambda_1^2\phi_n^{(1)} \\ \phi_n^{(2)} & \lambda_2\psi_n^{(2)} & \lambda_2^2\phi_n^{(2)} \end{vmatrix}}{\begin{vmatrix} \phi_n^{(1)} & \lambda_1\psi_n^{(1)} \\ \phi_n^{(2)} & \lambda_2\psi_n^{(2)} \end{vmatrix}}. \tag{22}$$

Similarly, we obtain two-fold transformations as

$$\begin{aligned} \omega_n[2] &= \omega_n[1]\left(\frac{\psi_n^{(2)}[1]}{\phi_n^{(2)}[1]}\right)^2, \\ &= \omega_n\left(\frac{\lambda_2\psi_n^{(1)}\phi_n^{(2)} - \lambda_1\psi_n^{(2)}\phi_n^{(1)}}{\lambda_2\psi_n^{(2)}\phi_n^{(1)} - \lambda_1\psi_n^{(1)}\phi_n^{(2)}}\right)^2, \\ &= \omega_n\left(\frac{\Delta[2]}{\Omega[2]}\right)^2. \end{aligned} \tag{23}$$

Again using $\omega_n = \exp(is_n)$ in eq. (23), we can have

$$\begin{aligned} s_n[2] &= s_n + 2i\ln\left(\frac{\lambda_2\psi_n^{(2)}\phi_n^{(1)} - \lambda_1\psi_n^{(1)}\phi_n^{(2)}}{\lambda_2\psi_n^{(1)}\phi_n^{(2)} - \lambda_1\psi_n^{(2)}\phi_n^{(1)}}\right), \\ &= s_n + 2i\ln\left(\frac{\Omega[2]}{\Delta[2]}\right). \end{aligned} \tag{24}$$

The N-fold Darboux transformations are expressed as

$$\psi_n[N] = \frac{\Delta[N+1]}{\Delta[N]}, \tag{25}$$

$$\phi_n[N] = \frac{\Omega[N+1]}{\Omega[N]}, \tag{26}$$

$$\omega_n[N] = \omega_n\left(\frac{\Delta[N]}{\Omega[N]}\right)^2, \tag{27}$$

$$s_n[2] = s_n + 2i\ln\left(\frac{\Omega[N]}{\Delta[N]}\right), \tag{28}$$

with

$$\Delta[N+1] = \begin{vmatrix} \psi_n & \lambda\phi_n & \lambda^2\psi_n & \cdots & \lambda^{N-1}\phi_n & \lambda^N X_n^k \\ \psi_n^{(1)} & \lambda_1\phi_n^{(1)} & \lambda_1^2\psi_n^{(1)} & \cdots & \lambda_1^{N-1}\phi_n^{(1)} & \lambda_1^N X_n^{(1),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_n^{(N)} & \lambda_N\phi_n^{(N)} & \lambda_N^2\psi_n^{(N)} & \cdots & \lambda_N^{N-1}\phi_n^{(N)} & \lambda_N^N X_n^{(N),k} \end{vmatrix}; \qquad \begin{aligned} X_n^{(N),2l-1} &= \psi_n^{(N)}, \\ X_n^{(N),2l} &= \phi_n^{(N)}, \end{aligned}$$

$$\Omega[N+1] = \begin{vmatrix} \phi_n & \lambda\psi_n & \lambda^2\phi_n & \cdots & \lambda^{N-1}\psi_n & \lambda^N Y_n^k \\ \phi_n^{(1)} & \lambda_1\psi_n^{(1)} & \lambda_1^2\phi_n^{(1)} & \cdots & \lambda_1^{N-1}\psi_n^{(1)} & \lambda_1^N Y_n^{(1),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n^{(N)} & \lambda_N\psi_n^{(N)} & \lambda_N^2\phi_n^{(N)} & \cdots & \lambda_N^{N-1}\psi_n^{(N)} & \lambda_N^N Y_n^{(N)} \end{vmatrix}; \qquad \begin{aligned} Y_n^{(N),2l-1} &= \phi_n^{(N)}, \\ Y_n^{(N),2l} &= \psi_n^{(N)}. \end{aligned}$$

and

$$\Delta[N] = \begin{vmatrix} \psi_n^{(1)} & \lambda_1\phi_n^{(1)} & \lambda_1^2\psi_n^{(1)} & \cdots & \lambda_1^{N-1}X_n^{(1),k} \\ \psi_n^{(2)} & \lambda_2\phi_n^{(2)} & \lambda_2^2\psi_n^{(2)} & \cdots & \lambda_2^{N-1}X_n^{(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_n^{(N)} & \lambda_N\phi_n^{(N)} & \lambda_N^2\psi_n^{(N)} & \cdots & \lambda_N^{N-1}X_n^{(N),k} \end{vmatrix}; \qquad \begin{array}{c} X_n^{(N),2l-1} = \psi_n^{(N)}, \\ X_n^{(N),2l} = \phi_n^{(N)}, \end{array}$$

$$\Omega[N] = \begin{vmatrix} \phi_n^{(1)} & \lambda_1\psi_n^{(1)} & \lambda_1^2\phi_n^{(1)} & \cdots & \lambda_1^{N-1}Y_n^{(1),k} \\ \phi_n^{(2)} & \lambda_2\psi_n^{(2)} & \lambda_2^2\phi_n^{(2)} & \cdots & \lambda_2^{N-1}Y_{n,l}^{(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n^{(N)} & \lambda_N\psi_n^{(N)} & \lambda_N^2\phi_n^{(N)} & \cdots & \lambda_N^{N-1}Y_n^{(N),k} \end{vmatrix}; \qquad \begin{array}{c} Y_n^{(N),2l-1} = \phi_n^{(N)}, \\ Y_n^{(N),2l} = \psi_n^{(N)}. \end{array}$$

Formulas (25)-(28) allow us to compute multi-soliton solution of sd-SG equation (5). In the next section, we will calculate explicit expressions of single- and double-soliton solutions.

4 Explicit solution in zero background

In this section, we shall compute explicit soliton solutions for the sd-SG equation by using Darboux transformation. For this, we take a simple trivial solution to the solution to sd-SG equation (5), *i.e.* $s_n = 0$, the linear system of difference-differential equations (10)-(11) becomes

$$\psi_{n+1} = \lambda\phi_n + \psi_n, \qquad \phi_{n+1} = \lambda\psi_n + \phi_n, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n = \lambda^{-1}\gamma\phi_n, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi_n = \lambda^{-1}\gamma\psi_n. \tag{30}$$

The solution to the linear difference-differential equations (29)-(30) is given by

$$\psi_n = A\,(1+\lambda)^n \exp\left(\frac{\gamma}{\lambda}t\right) + B\,(1-\lambda)^n \exp\left(-\frac{\gamma}{\lambda}t\right), \tag{31}$$

$$\phi_n = A\,(1+\lambda)^n \exp\left(\frac{\gamma}{\lambda}t\right) - B\,(1-\lambda)^n \exp\left(-\frac{\gamma}{\lambda}t\right), \tag{32}$$

where A and B are constants.

In order to obtain an explicit expression of one-soliton, we have

$$\psi_n^{(1)} = A_1\,(1+\lambda_1)^n \exp\left(\frac{\gamma}{\lambda_1}t\right) + B_1\,(1-\lambda_1)^n \exp\left(-\frac{\gamma}{\lambda_1}t\right), \tag{33}$$

$$\phi_n^{(1)} = A_1\,(1+\lambda_1)^n \exp\left(\frac{\gamma}{\lambda_1}t\right) - B_1\,(1-\lambda_1)^n \exp\left(-\frac{\gamma}{\lambda_1}t\right), \tag{34}$$

substituting $\psi_n^{(1)}$ and $\phi_n^{(1)}$ from eqs. (33)-(34) in expression (17), we obtain

$$s_n[1] = 2i\ln\frac{A_1(1+\lambda_1)^n - B_1(1-\lambda_1)^n\exp(\frac{-2\gamma}{\lambda_1}t)}{A_1(1+\lambda_1)^n + B_1(1-\lambda_1)^n\exp(\frac{-2\gamma}{\lambda_1}t)}. \tag{35}$$

For the choice $A_1 = 1$, $B_1 = i$, above expression reduces to [39]

$$s_n[1] = 2i\ln\frac{\psi_n^{(1)*}}{\psi_n^{(1)}} = 4\tan^{-1}\left[\left(\frac{1+\lambda_1}{1-\lambda_1}\right)^n\exp\left(\frac{2\gamma}{\lambda_1}t\right)\right], \tag{36}$$

which represents a single-kink solution and its profile is presented in fig. 1 for $\lambda_1 = 0.2$ and $\gamma = 0.25$. In what follows, we would like to study dynamics of one-kink soliton solution of sd-SG equation. The one-kink soliton (36) has wave span of 2π as shown in fig. 1 with moving inflexion point trace given by

$$n(t) = \frac{\gamma t}{\lambda_1\tanh^{-1}(\lambda_1)}. \tag{37}$$

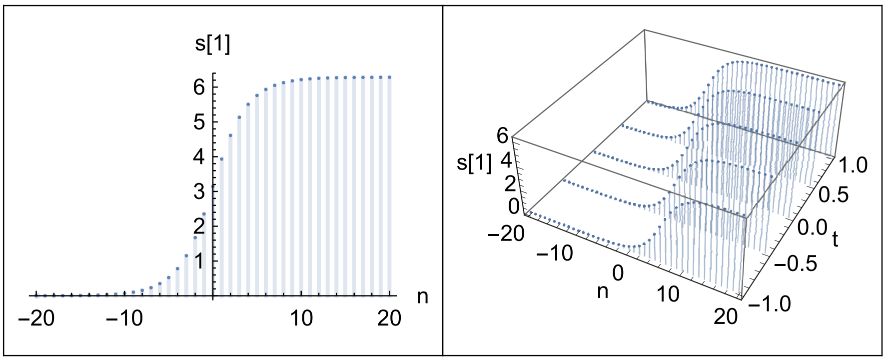

Fig. 1. The snapshot (at $t = 0$) and time-evolution of one-kink solution (35) is shown on left- and right-hand side of figure, respectively.

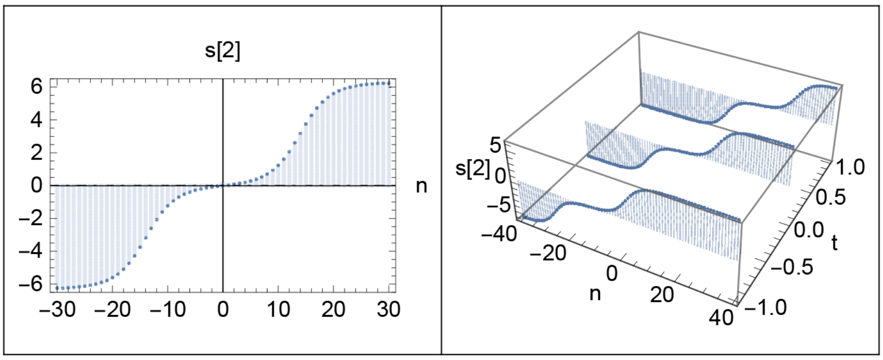

Fig. 2. The snapshot (at $t = 0$) and time-evolution of two-kink solution.

The value of $s_n[1]$ and slope at inflexion point are π and $4\arctan(\frac{1+\lambda_1}{1-\lambda_1})$ respectively. For further details on dynamics of such soliton solutions see refs. [39,40].

In order to obtain an explicit expression of two-soliton solution, we recall eq. (24)

$$s_n[2] \equiv 2i\ln\left(\frac{\det\begin{pmatrix}\phi_n^{(1)} & \lambda_1\psi_n^{(1)}\\ \phi_n^{(2)} & \lambda_2\psi_n^{(2)}\end{pmatrix}}{\det\begin{pmatrix}\psi_n^{(1)} & \lambda_1\phi_n^{(1)}\\ \psi_n^{(2)} & \lambda_2\phi_n^{(2)}\end{pmatrix}}\right), \tag{38}$$

along with

$$\psi_n^{(k)} = A_k\,(1+\lambda_k)^n \exp\left(\frac{\gamma}{\lambda_k}t\right) + B_k\,(1-\lambda_k)^n \exp\left(-\frac{\gamma}{\lambda_k}t\right),$$
$$\phi_n^{(k)} = A_k\,(1+\lambda_k)^n \exp\left(\frac{\gamma}{\lambda_k}t\right) - B_k\,(1-\lambda_k)^n \exp\left(-\frac{\gamma}{\lambda_k}t\right),$$

where $k = 1, 2$. Furthermore, if we take $A_1 = A_2 = 1$ and $B_1 = B_2 = i$ the expression of two-soliton solution (38) reduces to

$$s_n[2] = 4\cot^{-1}\left(\frac{(\lambda_1-\lambda_2)\left(\beta_n^+ e^{\gamma\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)t} + \beta_n^- e^{-\gamma\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)t}\right)}{(\lambda_1+\lambda_2)\left(\delta_n^+ e^{\gamma\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)t} - \delta_n^- e^{-\gamma\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)t}\right)}\right), \tag{39}$$

along with $\beta_n^\pm = (1\pm\lambda_1)^n(1\pm\lambda_2)^n$, $\delta_n^\pm = (1\pm\lambda_1)^n(1\mp\lambda_2)^n$. The dynamics of two-soliton solutions is presented in fig. 2 for $\lambda_1 = 0.14$, $\lambda_2 = -0.15$ and $\gamma = 0.25$. Similarly kink and antikink interaction is shown in fig. 3 for $\lambda_1 = 0.14$, $\lambda_2 = 0.15$.

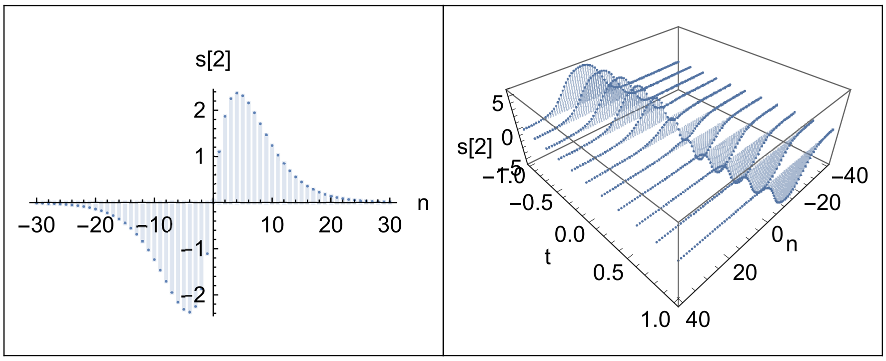

Fig. 3. The snapshot (at $t = 0$) and time-evolution of interaction of kink and antikink solution.

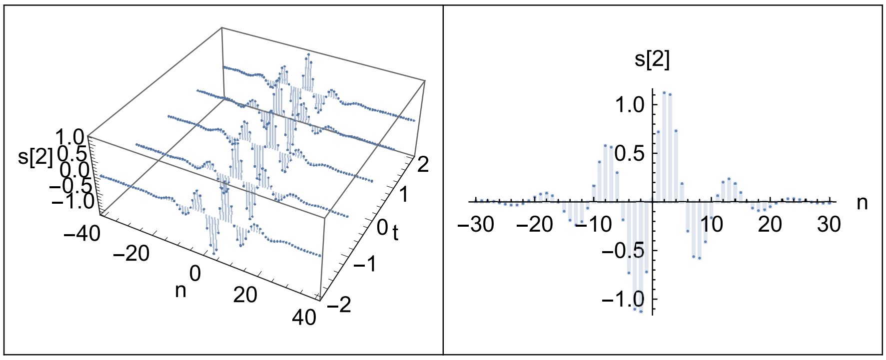

Fig. 4. A breather soliton solution for sd-SG equation.

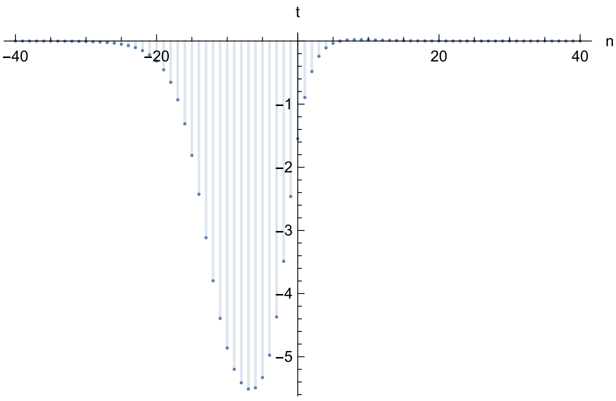

Fig. 5. Degenerate soliton solution (41).

One can easily obtain explicit expression of breather soliton solution for sd-SG equation, for this take $\lambda_2 = \lambda_1^*$, we obtain

$$s_n[2] = 4i\coth^{-1}\left(\frac{\Im(\lambda_1)\left(\varepsilon_n^+ e^{\left(\frac{2\gamma\Re(\lambda_1)}{|\lambda_1|^2}\right)t} + \varepsilon_n^- e^{\left(\frac{-2\gamma\Re(\lambda_1)}{|\lambda_1|^2}\right)t}\right)}{\Re(\lambda_1)\left(\rho_n^+ e^{\left(\frac{2i\gamma\Im(\lambda_1)}{|\lambda_1|^2}\right)t} - \rho_n^- e^{\left(\frac{-2i\gamma\Im(\lambda_1)}{|\lambda_1|^2}\right)t}\right)}\right), \tag{40}$$

where $\varepsilon_n^\pm = (1 \pm 2\Re(\lambda_1) + |\lambda_1|^2)^n$, $\rho_n^\pm = (1 \mp 2i\Im(\lambda_1) - |\lambda_1|^2)^n$. A snapshot (at $t = 0$) and space-time evolution of breather solution (40) are shown in fig. 4 for $\lambda_1 = 0.1 + 0.3i$. Under the limit, that is, $\lambda_2 \longrightarrow \lambda_1$, eq. (39) yields

$$s_n[2] = 4\tanh^{-1}\left(\frac{4(1-\lambda_1^2)^{n-1}(\lambda_1^2 n - \gamma t(1-\lambda_1^2))}{\lambda_1\left((1+\lambda_1)^{2n}e^{\frac{2\gamma t}{\lambda_1}} + (1-\lambda_1)^{2n}e^{\frac{-2\gamma t}{\lambda_1}}\right)}\right). \tag{41}$$

The above expression is known as degenerate soliton solution [41,42] and depicted in fig. 5 for $\lambda_1 = 0.2$ and $t = 1$.

In a similar manner we can also compute higher-order degenerate soliton solutions. Similarly, if we take limit $\lambda_2 \longrightarrow 1/\lambda_1$ in eq. (40), we obtain another representation of breather solution:

$$s_n[2] = -4\cot^{-1}\left(\left(\frac{\lambda_1^2-1}{\lambda_1^2+1}\right)\frac{\left((1+\lambda_1)^{2n}e^{\gamma\left(\frac{1+\lambda_1^2}{\lambda_1}\right)t}+(-1)^n(1-\lambda_1)^{2n}e^{-\gamma\left(\frac{1+\lambda_1^2}{\lambda_1}\right)t}\right)}{\left((1-\lambda_1^2)^n e^{-\gamma\left(\frac{1-\lambda_1^2}{\lambda_1}\right)t}-(-1)^n(1-\lambda_1^2)^n e^{\gamma\left(\frac{1-\lambda_1^2}{\lambda_1}\right)t}\right)}\right). \quad (42)$$

Similarly, one can construct two-breather solution and two-degenerate-breather solutions of sd-SG equation. All expressions of explicit solutions for sd-SG equation reduce to the already known solutions of continuous SG equation under continuum limit.

5 Conclusions

In this paper we have investigated the construction of multi-soliton solutions of sd-SG equation by using Darboux transformation. We obtained a determinant representation of multi-soliton solution of sd-SG equation. We derived explicit expressions of one- and two-soliton solutions by expanding the determinant formula. We also computed explicit expressions of discrete kink, two-kink, breather and degenerate solutions of sd-SG equation. Finally we illustrated results obtained in this article by plotting our results. The results obtained in this paper may be useful in other fields like string theory, theory of condensed-matter physics and biological science. For future work we would like to study Darboux transformation for fully discrete sine-Gordon equation.

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