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Electromagnetic field and dark dynamical scalars for spherical systems

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Abstract. This paper studies the effects of inhomogeneous fluid configurations on the evolution of relativistic spheres within the context of electromagnetic field and fourth order gravity. We assume a compact distribution of spherical structure that evolves due to the curvature influences coming from $f(R)$ gravity model and non-ideal matter source. After calculating the corresponding $f(R)$ equations of motion, an explicit formulation among Weyl scalar, dark dynamical scalars and material variables is developed. With $f(R)$ curvature and charged fluid terms, all the non-zero dynamical variables are presented from the orthogonal decomposition the Riemann curvature tensor. The Raychaudhuri, Weyl scalar and shearing equations are formulated through modified scalar functions with and without varying Ricci scalar for perfect and imperfect spherical systems.

1 Introduction

The theory of general relativity (GR) has been found to be one of the most significant gravitational theories, as it has unveiled many hidden aspects of our universe. Due to the advent of recent observational outcomes of modern cosmology, like Supernovae Type Ia, large scale structures, cosmic microwave background radiations, it is asserted that our cosmos is in the phase of accelerating expansion [1,2]. In this scenario, the dark energy (DE) has to be supplemented to understand the cosmological acceleration, while the contribution of baryonic matter may need to include through dark matter (DM) in order to provide acceptable representation of rotating galaxies' motion, orbital motion of galaxy clustering, large scale systems and the temperature scalar field of hot fluid in galaxies and their clusters. Recently, Qadir *et al.* [3] described that the consideration of modified versions of GR could be helpful to explore dark matter problem and quantum effects. Furthermore, the problems of nonrenormalizability and nonunitarity have been observed in the gravitational interaction provided by the GR action. These grounds have motivated researchers to modify GR at high energies and to use alternative cosmological approaches.

The "modified theories of gravity" (MTG) has become a standard terminology for the models explaining gravitational effects that differ from the most conventional general theory of relativity (GR). Among these MTGs, a great attention has been given to $f(R)$, $f(R,T)$ (T is the trace of energy momentum tensor) and $f(R,T,R_{\mu\nu}T^{\mu\nu})$ and $f(R,\Box R,T)$ gravity by theoretical astrophysicists. These theories could provide an effective tool to study accelerating expansion of the cosmos. The gravitational laws are being modified on large scales in MTGs, therefore, these could led some captivating observational signatures like, cosmic microwave background, modified spectrum in the cluster of galaxies, weak lensing, etc. (for details, please see [4–9]).

Nojiri and Odintsov [8] presented few MTG models and claimed that these would describe inflationary and late-time cosmic evolution in an appropriate way. The $f(R)$ gravity is a simplest, but fourth order gravity (FOG) MTG, in which a generic function of the Ricci scalar (R) is inserted in the usual GR action. The theoretical modeling of wormholes alongwith few exact cosmological solutions are being discussed in [10,11]. Herrera *et al.* explored the problem of cylindrical [12] and spherical [13] fate of collapse with the help of matching conditions. Buchdahl [14]

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discussed phenomenological aspects of this theory, whose results were then put forward by many peoples working in relativistic astrophysicists [15–19]. Bamba *et al.* [20,21] studied the effects of modified exponential model on the viability bounds of cosmic evolutions. conditions.

Oikonomou [22] analyzed few generic properties of exponential $f(R)$ model in an Einstein reference frame. He also related his results with that of Jordan frame. Bamba *et al.* [23] investigated the effects of few $f(R)$ models in order to describe early and late time cosmic evolution altogether. In the similar fashion Yousaf *et al.* studied another gravitational alternatives for the discussion of dark source terms in the relativistic system through $f(R,\Box R,T)$ [24] and $f(R,T,R_{\gamma\delta}T^{\gamma\delta})$ [25–27] gravity. Recently, Elizalde *et al.* [28] presented a viable exponential $f(R)$ version that would lead to sketch the clear picture of both the late and early time cosmic evolution in a unified way. The effects of the slight modification to this model has also being checked and is found to be non-singular and stable. Odintsov *et al.* [29] introduced logarithmic terms in the usual R squared gravity to analyze reheating cosmic eras. The discussion on the possible existence of observationally consistent solutions has also been made.

Extra curvature modified gravity terms in the GR gravitational interaction could provide significant results on the study of gravitational collapse. The system experiences the state of collapse, once it enters into the phase of inhomogeneous energy density (InED) over the universe. The theoretical examination in this direction has been performed by Penrose and Hawking [30] by making mathematical correspondence among Weyl tensor and fluid variables. Herrera *et al.* [31] studied the role of pressure anisotropy on the appearance of naked singularity (NS) and formulated the corresponding InED factors. Virbhadra *et al.* [32,33] proposed the a generic mathematical expression to study the physical features of NS and the black hole structures.

The very definition of structure scalars (SS) and their detailed analysis have been elaborated by Herrera *et al.* [34]. Such quantities are found to be very effective in the study of InED for relativistic systems. Herrera and his colleagues [35] explored InED factors and studied their contribution in the evolution of compact stars. They also studied the role of expansion scalar in the maintenance of anisotropic InED of the stellar bodies [36]. The conformally flat state of the system can be obtained by assigning null value to the Weyl scalar in the both shear-free [37,38] and expansion-free [39,40,6] self-gravitating systems. Bhatti and Yousaf [41,42] calculated few InED parameters in terms of structure scalars for the stellar system in modified gravity. Yousaf *et al.* [43] discussed the collapsing states of titled spherical expansion-free systems and found that modified gravity terms tends to induce stability in the anisotropic matter content. Herrera *et al.* [44] identified eight different scalar variables involved in the dynamical features of cylindrical systems and concluded that all the possible solutions can be well discussed through these variables. Later on, their results are being extended by including electric charge [45], $f(R)$ and $f(R)$-Maxwell theories of gravity [46]. Herrera *et al.* [47] explored the influences of cosmological constant on the maintenance of InED for the anisotropic spherical systems. Sharif and Yousaf [48,49] studied the effects of $R+\epsilon R^n$ terms on the definition of irregularity factor for the spherical systems. Recently, Capozziello *et al.* [50] considered generalized form of the n-dimensional universe model and analyzed the modeling of the $f(R)$ extra degrees of freedom in the gravitational sourcing of the equations of motion. They rigorously found very important result which assert that any $f(R)$ model can be recasted as an ideal cosmological fluid.

In this work, we have constructed structure scalar in the presence of $f(R)$ gravity models and electromagnetic field. We have also studied their importance in the Raychaudhuri, shear and Weyl differential equations. The paper is formatted as follows. The coming section will explain the basic equations of motion for $f(R)$-Maxwell gravity along with the spherical distribution of shearing locally anisotropic systems. The corresponding mass function and its relation with Weyl scalar are also mentioned. In sect. 3 will formulate structure scalars and their role in the modeling spherical systems with Maxwell-$f(R)$ corrections. Section 4 is devoted to describe dark dynamical variables with electric charge and current Ricci scalar background. Finally, the conclusions are mentioned in the last section.

2 Maxwell-f(R) gravity and spherical viscous system

The generalized Einstein Hilbert action for the case of Maxwell-$f(R)$ gravity can be mentioned as

$$S_{f(R)+M}=\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{f(R)}{\kappa}-\frac{\mathcal{F}}{2\pi}\right), \tag{1}$$

where $\mathcal{F}$, S_M, κ are the Maxwell scalar, matter action and coupling constant, respectively. Its variation with the metric tensor gives

$$f_RR_{\alpha\beta}-\frac{1}{2}fg_{\alpha\beta}-\nabla_\alpha\nabla_\beta f_R+g_{\alpha\beta}\Box f_R=\kappa(T_{\alpha\beta}+E_{\alpha\beta}), \tag{2}$$

in which subscript R indicates the partial differentiation of the corresponding quantity with respect to R, while the d'Alembertian operator $\Box$ through covariant derivation ∇_μ can be written as $\Box\equiv\nabla^\mu\nabla_\mu$. The matter Lagrangian,

$\mathcal{L}_M$ can be used to to define the usual energy momentum tensor $T^{(M)}_{\alpha\beta}$ as

$$T^{(M)}_{\alpha\beta}=-\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\alpha\beta}},$$

which, in the configurations of the Einstein tensor $G_{\mu\nu}$, can be rewritten as

$$G_{\alpha\beta}=\frac{\kappa}{f_R}\left(\overset{(D)}{T}_{\alpha\beta}+T^{(M)}_{\alpha\beta}+E_{\alpha\beta}\right), \tag{3}$$

where

$$\overset{(D)}{T}_{\alpha\beta}=\frac{1}{\kappa}\left\{\nabla_\alpha\nabla_\beta f_R+(f-Rf_R)\frac{g_{\alpha\beta}}{2}-\Box f_R g_{\alpha\beta}\right\}$$

and $E_{\alpha\beta}$ describes the electromagnetic stress energy tensor as

$$E_{\gamma\delta}=\frac{1}{4\pi}\left(F^{\mu}_{\gamma}F_{\delta\mu}-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}g_{\gamma\delta}\right), \tag{4}$$

where the Maxwell field tensor is represented by $F_{\gamma\delta}$ and is defined through four potential (ϕ_α) as $F_{\alpha\beta}=\Phi_{\beta,\alpha}-\Phi_{\alpha,\beta}$. The term $\overset{(D)}{T}_{\alpha\beta}$ will turn out to be zero under GR limit, *i.e.*, $f(R)\to R$.

We describe a relativistic spherically symmetric geometry with the help of a non-static diagonal spacetime as under

$$\mathrm{d}s^2=-B^2\mathrm{d}r^2-C^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+A^2\mathrm{d}t^2, \tag{5}$$

where the metric co-efficients of the above spacetime depend on r and t only. This geometry is assumed to be coupled with dissipative locally anisotropic matter content having the following energy momentum tensor:

$$T^{(M)}_{\alpha\beta}=(\mu+P_\perp)V_\alpha V_\beta+\varepsilon l_\alpha l_\beta+V_\alpha q_\beta+(P_r-P_\perp)\chi_\alpha\chi_\beta-P_\perp g_{\alpha\beta}+q_\alpha V_\beta-2\eta\sigma_{\alpha\beta}, \tag{6}$$

where $P_\perp$, P_r, are tangential and radial pressure components, μ is the energy density, q_β is a heat conducting vector, $\sigma_{\alpha\beta}$ is a shear tensor, while its coefficient is represented by η and radiation density of the fluid is expressed with the term ϵ. Under comoving coordinate system, the four vectors $V^\beta=\delta^\beta_0/A$, $\chi^\beta=\delta^\beta_1/B$, $l^\beta=\delta^\beta_0/A+\delta^\beta_1/B$ and conduction vector $q^\beta=q(t,r)\chi^\beta$ obey the relations

$$V^\alpha V_\alpha=-1,\qquad \chi^\alpha\chi_\alpha=1,\qquad \chi^\alpha V_\alpha=0,$$
$$V^\alpha q_\alpha=0,\qquad l^\alpha V_\alpha=-1,\qquad l^\alpha l_\alpha=0.$$

Let our system evolve in the presence of electromagnetic field, then the corresponding electric current ($\mathcal{J}^\beta$) and its potential (Φ^β) in the four vector formalism are defined by

$$\mathcal{J}^\beta=\xi V^\beta,$$
$$\Phi^\beta=\Phi\delta^\beta_0,$$

in which ξ and Φ are the density of the charge and scalar potential, respectively. Both of these are the functions of t and r. The equation of motion for the charged medium can be given as

$$F^{\alpha\beta}_{\ ;\beta}=\mu_0\mathcal{J}^\alpha,$$
$$F_{[\alpha\beta;\gamma]}=0, \tag{7}$$

where μ_0 is controlling the effects of magnetic permeability. Equation (7) after using eq. (5) turns out to be

$$\Phi''-(ACB'+BCA'-2ABC')\frac{\Phi'}{ABC}=4\pi\mu_0B^2A, \tag{8}$$

$$\dot{\Phi}'-(AC\dot{B}B+BC\dot{A}A-2AB\dot{C}C)\frac{\Phi'}{ABC}=0, \tag{9}$$

where $' \equiv \partial/\partial r$ and $\cdot \equiv \partial/\partial t$. The solution of the differential equation (11) has been found to be

$$\Phi' = \frac{AsB}{C^2}\,,$$

in which

$$s(r) = 2\pi \int_0^r \mu_0 B C^2 \mathrm{d}r \tag{10}$$

is the amount of charge within the relativistic sphere of radius r. The corresponding electric field intensity can be defined as follows:

$$E(t,r) = \frac{s}{4\pi C^2}\,. \tag{11}$$

The general formula to calculate the shear tensor for the locally anisotropic fluid distribution can be written as

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} - \frac{\Theta}{3}\left(g_{\alpha\beta} + V_\alpha V_\beta\right) + a_{(\alpha}V_{\beta)}, \tag{12}$$

where the covariant derivative of the four velocity gives expansion scalar with the definition

$$\Theta = V^{\alpha}_{;\alpha}, \tag{13}$$

and its product with four velocity gives four acceleration as $a_\alpha = V_{\alpha;\beta}V^\beta$. Equation (12) has three non-zero components that can be expressed through shear scalar σ as

$$\sigma_{11} = \frac{2\sigma}{3}B^2\,,$$
$$\sigma_{22} = -\frac{\sigma}{3}C^2 = \frac{\sigma_{33}}{\sin^2\theta}\,,$$
$$\sigma^2 = \frac{3}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta}.$$

The quantity Θ has been found for the spherical systems as

$$\Theta = \frac{1}{A}\left(\frac{2\dot{C}}{C} + \frac{\dot{B}}{B}\right).$$

Depending upon the selection on the formulations of $f(R)$ models, one could consider this gravity as a viable and observationally well-consistent gravitational theory. We consider a specific combination of $f(R)$ gravity as follows [51]

$$f(R) = R - \beta R_E\left(1 - e^{-R/R_E}\right), \tag{14}$$

where R_E and β are constants. The above model can be written alternatively as

$$l(R,b) = 1 - e^{R/\Lambda b}, \tag{15}$$

where Λ is a cosmological constant and $l(R,b)$ controls the impact of deviations from GR. The influence of $f(R)$ modification can be examined with the help of some suitable distortion in the parametric value of b. The results of eq. (14) coincides with that of ΛCDM under the constraints $R_E \to 0$, $\beta \to \infty$ with $R_E\beta \to 2\Lambda$.

The non-zero components of $f(R)$-Maxwell equations of motion for the radiating and dissipative spherical are

$$G_{00}=\frac{\kappa}{1-\beta e^{-R/R_E}}\left[\mu A^2+\varepsilon A^2+\frac{s^2A^2}{8\pi C^4}+\frac{A^2}{\kappa}\left\{-\frac{\beta}{2}R_E+\frac{\beta}{2}e^{-R/R_E}(R+R_E)\right.\right.$$
$$-\left(\frac{\dot B}{B}+\frac{2\dot C}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot R-\left(\frac{B'}{B}-\frac{2C'}{C}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}R'+\frac{\beta}{B^2R_E}e^{-R/R_E}$$
$$\left.\left.\times\left(R''-\frac{R'^2}{R_E}\right)\right\}\right], \tag{16}$$

$$G_{01}=\frac{\kappa}{1-\beta e^{-R/R_E}}\left[qBA+\varepsilon BA-\frac{1}{\kappa}\left\{\frac{\beta}{R_E}e^{-R/R_E}\left(\dot R'-\frac{\dot RR'}{R_E}\right)-\frac{\beta}{R_E}e^{-R/R_E}\right.\right.$$
$$\left.\left.\times\frac{\dot BR'}{B}-\frac{\beta}{R_E}e^{-R/R_E}\frac{A'\dot R}{A}\right\}\right], \tag{17}$$

$$G_{11}=\frac{\kappa}{1-\beta e^{-R/R_E}}\left[P_rB^2-\frac{4}{3}\eta\sigma B^2+\frac{s^2B^2}{8\pi C^4}+\varepsilon B^2-\frac{B^2}{\kappa}\left\{-\frac{\beta}{2}R_E+\frac{\beta}{2}\right.\right.$$
$$\times e^{-R/R_E}(R+R_E)+\left(\frac{\dot A}{A}-\frac{2\dot C}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot R+\left(\frac{A'}{A}+\frac{2C'}{C}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}$$
$$\left.\left.\times R'-\frac{\beta}{A^2R_E}e^{-R/R_E}\left(\ddot R-\frac{\dot R^2}{R_E}\right)\right\}\right], \tag{18}$$

$$G_{22}=\frac{\kappa}{1-\beta e^{-R/R_E}}\left[P_\perp C^2+\frac{2}{3}\eta\sigma C^2+\frac{s^2}{8\pi C^2}-\frac{C^2}{\kappa}\left\{-\frac{\beta}{2}R_E+\frac{\beta}{2}e^{-R/R_E}(R\right.\right.$$
$$+R_E)-\frac{\beta}{A^2R_E}e^{-R/R_E}\left(\ddot R-\frac{\dot R^2}{R_E}\right)+\left(\frac{\dot A}{A}-\frac{\dot B}{B}+\frac{\dot C}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot R$$
$$\left.\left.+\frac{\beta}{R_E}e^{-R/R_E}\left(R''-\frac{R'^2}{R_E}\right)+\left(\frac{C'}{C}-\frac{B'}{B}+\frac{A'}{A}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}R'\right\}\right], \tag{19}$$

where G_{ii} are the components of the Einstein tensor. Misner and Sharp [52] proposed a formula to calculate the total mass of the sphere. This formula in an environment of electromagnetic field for the system equation (5) provides

$$m(t,r)=\frac{C}{2}(1-g^{\alpha\beta}C_{,\alpha}C_{,\beta})=\left\{1+\left(\frac{\dot C}{A}\right)^2-\left(\frac{C'}{B}\right)^2\right\}\frac{C}{2}+\frac{s^2}{2C}, \tag{20}$$

which can be re-expressed as

$$E\equiv\frac{C'}{B}=\left[1+U^2+\frac{s^2}{C^2}-\frac{2m(t,r)}{C}\right]^{1/2}. \tag{21}$$

Here, we have used U as the velocity of the fluid, which through the operator $D_T\equiv A^{-1}\frac{\partial}{\partial t}$ has the form

$$U=D_TC=\frac{\dot C}{A}. \tag{22}$$

For the case of a collapsing star, U would be less than zero. The variations in the spherical mass can be obtained from eqs. (16), (17) and (20) as

$$D_Cm=\frac{\kappa}{2f_R}\left\{\hat\mu+\frac{s^2}{8\pi C^4}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)+\frac{\psi_\mu}{A^2}+\left(\hat q-\frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2$$
$$+2\pi s\left[\frac{8\pi}{C}\left(s'-\frac{2sC'}{C}\right)+\frac{3sC'}{4\pi C^2}\right], \tag{23}$$

where

$$\frac{\psi_\mu}{A^2}=\frac{1}{\kappa}\left\{-\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot{R}-\left(\frac{B'}{B}-\frac{2C'}{C}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}R'\right.$$
$$\left.+\frac{\beta}{B^2R_E}e^{-R/R_E}\left(R''-\frac{R'^2}{R_E}\right)\right\}, \tag{24}$$

$$\frac{\psi_q}{AB}=-\frac{1}{\kappa}\left\{\frac{\beta}{R_E}e^{-R/R_E}\left(\dot{R}'-\frac{\dot{R}R'}{R_E}\right)-\frac{\beta}{R_E}e^{-R/R_E}\frac{\dot{B}R'}{B}-\frac{\beta}{R_E}e^{-R/R_E}\frac{A'\dot{R}}{A}\right\}, \tag{25}$$

and $D_C=\frac{1}{C'}\frac{\partial}{\partial r}$ and over hat means the combination of the corresponding quantity with ε. Equation (23) yields

$$\frac{3m}{C^3}=\frac{3\kappa}{2C^3}\int_0^r\left[\frac{1}{(1-\beta e^{-R/R_E})}\left\{\hat{\mu}+\frac{s^2}{8\pi C^4}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)\right.\right.$$
$$\left.\left.+\frac{\psi_\mu}{A^2}+\left(\hat{q}-\frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2C'+\frac{3s^2}{2C^4}\right]\mathrm{d}r. \tag{26}$$

This relation has developed a correspondence among mass function with the other structural variables of the spherical system. Further one can see the role of FOG gravity and dissipative terms in the presence of electric charge terms on the formulations of mass function. It is pertinent to mention that the tensorial quantity involved in the study of irregular phases of initially homogeneous spheres is the Weyl tensor. This tensor can be decomposed after using comoving orthogonal reference coordinates into its two parts. These can be conventionally called its electric and magnetic parts. The former can be expressed via four vectors as

$$E_{\alpha\beta}=\mathcal{E}\left[\chi_\alpha\chi_\beta-\frac{1}{3}(g_{\alpha\beta}+V_\alpha V_\beta)\right],$$

in which

$$\mathcal{E}=\left[\frac{\ddot{C}}{C}+\left(\frac{\dot{C}}{C}+\frac{\dot{A}}{A}\right)\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)-\frac{\ddot{B}}{B}\right]\frac{1}{2A^2}-\frac{1}{2C^2}$$
$$-\left[\frac{C''}{C}-\left(\frac{C'}{C}+\frac{B'}{B}\right)\left(\frac{A'}{A}-\frac{C'}{C}\right)-\frac{A''}{A}\right]\frac{1}{2B^2} \tag{27}$$

is a Weyl scalar. Equation (27) is the geometrical expression of the Weyl scalar. Now, this can be expressed through matter variables, charged and dark source terms as under

$$\mathcal{E}=\frac{\kappa}{2f_R}\left(\hat{\mu}-\hat{\Pi}+\frac{3s^2}{8\pi C^4}+2\eta\sigma-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)+\frac{\psi_\mu}{A^2}-\frac{\psi_{P_r}}{B^2}+\frac{\psi_{P_\perp}}{C^2}\right)$$
$$-\frac{3\kappa}{2C^3}\int_0^r\left[\frac{1}{f_R}\left\{\hat{\mu}+\frac{s^2}{8\pi C^4}\frac{\psi_\mu}{A^2}+\left(\hat{q}-\frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2C'-\frac{3s^2}{2C^4}\right]\mathrm{d}r, \tag{28}$$

where $\hat{\Pi}$ can be obtained by subtracting $P_\perp$ from $\hat{P}_r$ and where ψ_{P_r} and $\psi_{P_\perp}$ indicate the dark source terms corresponding to 11 and 22 components of field equations. Their values are

$$\frac{\psi_{P_r}}{B^2}=-\frac{1}{\kappa}\left\{\left(\frac{\dot{A}}{A}-\frac{2\dot{C}}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot{R}+\left(\frac{A'}{A}+\frac{2C'}{C}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}\right.$$
$$\left.\times R'-\frac{\beta}{A^2R_E}e^{-R/R_E}\left(\ddot{R}-\frac{\dot{R}^2}{R_E}\right)\right\}, \tag{29}$$

$$\frac{\psi_{P_\perp}}{C^2}=-\frac{1}{\kappa}\left\{-\frac{\beta}{A^2R_E}e^{-R/R_E}\left(\ddot{R}-\frac{\dot{R}^2}{R_E}\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\frac{\beta}{A^2R_E}e^{-R/R_E}\dot{R}\right.$$
$$\left.+\frac{\beta}{R_E}e^{-R/R_E}\left(R''-\frac{R'^2}{R_E}\right)+\left(\frac{C'}{C}-\frac{B'}{B}+\frac{A'}{A}\right)\frac{\beta}{B^2R_E}e^{-R/R_E}R'\right\}. \tag{30}$$

Equation (28) is an explicit expression of the Weyl scalar in terms of extra curvature $f(R)$-Maxwell corrections and radiating shear terms.

3 FOG model and dark dynamical variables

This section is devoted to study the effects of MTGs terms and electric charge on the formulation of SS for the anisotropic spherical system. We shall also check their roles on the Raychaudhuri equation (RE), shear differential equation (SDE) and Weyl scalar equation (WSE). Like the decomposition of Weyl tensor, the Riemann tensor can be disintegrated with the aid of comoving coordinate system. It is worthy to note that this comoving coordinates system exists orthogonally to each other, therefore one can call this decomposition of Riemann tensor to be orthogonal. One can get SS from this splitting by evaluating the trace and trace-free of the corresponding equations. Thus, SS are being found to be very interesting tool to understand various geometric behavior of collapsing and non-collapsing stellar systems in gravitational physics [53,54]. Our work is aimed to develop such relations in the environment of a specific $f(R)$ model and electromagnetic field. Following the technique of Herrera [34], the Riemann tensor gives

$$X_{\alpha\beta} = {}^{\dagger}R^{\dagger}_{\alpha\gamma\beta\delta}V^{\gamma}V^{\delta} = \frac{1}{2}\eta^{\varepsilon\rho}{}_{\alpha\gamma}R^{\dagger}_{\epsilon\rho\beta\delta}V^{\gamma}V^{\delta}, \qquad Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}V^{\gamma}V^{\delta}, \tag{31}$$

where the dagger sign at the right, left and both sides of the term indicates the right, left and second dual of the tensor with their definitions as follows:

$$ {}^{\dagger}R_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\eta_{\alpha\beta\varepsilon\rho}R^{\varepsilon\rho}{}_{\gamma\delta}, \qquad R^{\dagger}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\eta_{\varepsilon\rho\gamma\delta}R^{\epsilon\rho}{}_{\alpha\beta}, \qquad {}^{\dagger}R^{\dagger}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\eta_{\alpha\beta}{}^{\varepsilon\rho}R^{\dagger}_{\varepsilon\rho\gamma\delta}.$$

The tensors mentioned in eq. (31) can be rewritten as follows:

$$X_{\alpha\beta} = \frac{1}{3}X_T h_{\alpha\beta} + X_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{32}$$

$$Y_{\alpha\beta} = \frac{1}{3}Y_T h_{\alpha\beta} + Y_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{33}$$

in which $Y_{\alpha\beta}$ encapsulates the effects of Weyl tensor electric part, while $X_{\alpha\beta}$ comes from the second dual of $R_{\alpha\beta\gamma\delta}$ and $h_{\gamma\delta} \equiv g_{\gamma\delta} + V_\gamma V_\delta$. The subscripts T and TF stand for trace and trace-free components of the subsequent quantities. By using $f(R)$-Maxwell equations, (32) and (33), the SS can be given through charged fluid profiles as

$$X_T = \frac{\kappa}{1-\beta e^{-R/R_E}}\left(\hat{\mu} + \frac{s^2}{8\pi C^4} - \frac{\beta}{2\kappa}R_E + \frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E) + \frac{\psi_\mu}{A^2}\right), \tag{34}$$

$$X_{TF} = -\mathcal{E} - \frac{\kappa}{1-\beta e^{-R/R_E}}\left(\hat{\Pi} - 2\sigma\eta - \frac{s^2}{4\pi C^4} + \frac{\psi_{P_r}}{B^2} - \frac{\psi_{P_\perp}}{C^2}\right), \tag{35}$$

$$Y_T = \frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\hat{\mu} + \frac{s^2}{4\pi C^4} + \frac{\psi_{P_r}}{B^2} + \frac{\beta}{\kappa}R_E - \frac{\beta}{\kappa}e^{-R/R_E}(R+R_E) + \frac{2\psi_{P_\perp}}{C^2} + 3\hat{P}_r + \frac{\psi_\mu}{A^2} - 2\hat{\Pi}\right), \tag{36}$$

$$Y_{TF} = \mathcal{E} - \frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\hat{\Pi} - \frac{s^2}{4\pi C^4} - 2\eta\sigma + \frac{\psi_{P_r}}{B^2} - \frac{\psi_{P_\perp}}{C^2}\right). \tag{37}$$

Another form of the trace-free electric part of the curvature tensor can be expressed after using eq. (37) as

$$\begin{aligned} Y_{TF} = {} & \frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\hat{\mu} + \frac{5s^2}{8\pi C^4} - 2\hat{\Pi} + 4\eta\sigma + \frac{\psi_\mu}{A^2} - \frac{2\psi_{P_r}}{B^2} + \frac{2\psi_{P_\perp}}{C^2}\right) \\ & - \frac{3\kappa}{2C^3}\int_0^r \left[\frac{1}{1-\beta e^{-R/R_E}}\left\{\hat{\mu} + \frac{s^2}{8\pi C^4} + \frac{\psi_\mu}{A^2} + \left(\hat{q} - \frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2C'\right]\mathrm{d}r. \end{aligned} \tag{38}$$

One can see the clear aspects of dark dynamical SS, even in the charged and dark medium, if we have some effective configurations of fluid contents. Therefore, we define them as under

$$\begin{aligned} &\mu_{eff} = \hat{\mu} + \frac{s^2}{8\pi C^4} + \frac{\psi_\mu}{A^2}, \qquad P_r^{eff} = \hat{P}_r - \frac{s^2}{8\pi C^4} - \frac{4}{3}\eta\sigma + \frac{\psi_{P_r}}{B^2}, \\ &P_\perp^{eff} = P_\perp + \frac{s^2}{8\pi C^4} + \frac{2}{3}\eta\sigma + \frac{2\psi_{P_\perp}}{C^2}, \\ &\Pi^{eff} = P_r^{eff} - P_\perp^{eff} = \Pi - 2\eta\sigma + \frac{\psi_{P_r}}{B^2} - \frac{\psi_{P_\perp}}{C^2} - \frac{s^2}{4\pi C^4}. \end{aligned}$$

These effective variables has the combined effects of viscosity, electromagnetic field, radiations as well as FOG corrections. In the form of above relations, the dark dynamical SS can be written as

$$X_{TF}=\frac{3\kappa}{2C^3}\int_0^r\left[\frac{1}{1-\beta e^{-R/R_E}}\left\{\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)+\left(\hat{q}-\frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2C'\right]\mathrm{d}r$$
$$-\frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)\right)+\frac{3s^2}{2C^4}\,, \tag{39}$$

$$Y_{TF}=\frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\mu_{eff}-2\Pi^{eff}+\frac{3\beta}{2\kappa}R_E-\frac{3\beta}{2\kappa}e^{-R/R_E}(R+R_E)+\frac{4\psi_{P_r}}{B^2}\right)$$
$$-\frac{3\kappa}{2C^3}\int_0^r\left[\frac{1}{1-\beta e^{-R/R_E}}\left\{\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)+\left(\hat{q}-\frac{\psi_q}{AB}\right)\frac{U}{E}\right\}C^2C'\right]\mathrm{d}r-\frac{3s^2}{2C^4}, \tag{40}$$

$$Y_T=\frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\mu_{eff}+3P_r^{eff}-2\Pi^{eff}+\frac{\beta}{\kappa}R_E-\frac{\beta}{\kappa}e^{-R/R_E}(R+R_E)\right), \tag{41}$$

$$X_T=\frac{\kappa}{(1-\beta e^{-R/R_E})}\left(\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)\right). \tag{42}$$

These relations describe the role of fluid parameters, tidal forces, electromagnetic field and FOG terms in the mathematical modeling of SS. The usual GR scalars can be recovered from the above relations by using $f(R)\to R$ limit. Next, we shall describe the importance of these scalars in the field of relativity and gravitational physics. Its have been noted from the literature that above mentioned scalar variables have major roles in the study of dynamical properties of collapsing bodies. The stability regimes of spherical, planar and axially symmetric geometries [55,56,49] can be well discussed with the aid of above SS. The well-known RE occupied fundamental importance in finding the exact solutions of differentia equations. We have seen, after making some mathematical manipulations, that such kind of equation can be expressed through one of the charged $f(R)$ SS as

$$-Y_T=V^\alpha\Theta_{;\alpha}+\frac{2}{3}\sigma^{\alpha\beta}\sigma_{\alpha\beta}+\frac{\Theta^2}{3}-a^\alpha_{;\alpha}. \tag{43}$$

This equation is apparently same as obtained by Herrera *et al.* in the realm of GR. However, if we did deep analysis, one can say that this equation contains $f(R)$ dark source and charged terms. Equation (43) is equal to the SS Y_T and the expression of Y_T can be seen from eq. (41). So, by making comparison between eqs. (41) and (43), many attractive phenomenon of the Raychaudhuri can be unveiled. Thus, the physics of the Penrose-Hawking singularity theorems can be dealt with the help of Y_T. In other words, we say that the contribution of RE can be seen by taking a deep analysis of Y_T. Using eqs. (12), (13) and (16)–(19), we have

$$Y_{TF}=a^2+\chi^\alpha a_{;\alpha}-\frac{aC'}{BC}-\frac{2}{3}\Theta\sigma-V^\alpha\sigma_{;\alpha}-\frac{1}{3}\sigma^2. \tag{44}$$

This is a well-known SDE. The implosion rate of shearing viscous massive objects can be seen from Y_{TF}. The simultaneous use of mass variation equation, Weyl scalar, field equation would provide Ellis equations. These Ellis equations after few manipulations gives a peculiar link among Weyl scalar and charged material variables in the form of a WDE as under

$$\left[X_{TF}+\frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)\right)\right]'=$$
$$-X_{TF}\frac{3C'}{C}+\frac{\kappa(\Theta-\sigma)}{2(1-\beta e^{-R/R_E})}\left(\hat{q}B-\frac{\psi_q}{A}\right). \tag{45}$$

This equation could help to describe the emergence of InED for the radiating spherical self-gravitating systems in the presence of electromagnetic field. In the above WDE, the X_{TF} is an InED factor. One can easily notice this, by considering following scenarios:

1) If a system evolve in such a way that the effects of expansion (or contraction) of the spacetime are equal to the shearing motion of the spherical structure. In that case, we take $\Theta=\sigma$, then the above equation boils down to

$$\left[X_{TF}+\frac{\kappa}{2(1-\beta e^{-R/R_E})}\left(\mu_{eff}-\frac{\beta}{2\kappa}R_E+\frac{\beta}{2\kappa}e^{-R/R_E}(R+R_E)\right)\right]'=-X_{TF}\frac{3C'}{C}\,, \tag{46}$$

which shows that the system will be in the conformally flat state if and only if one of the charged modified SS X_{TF} has zero value along with $\beta=0$. In that state, the spherical structure will be in regular mode having uniform distribution of energy density.

2) There have been many fascinating outcomes from the expansion-free state of the collapsing system. It was seen by Skripkin [57] that zero expansion condition in the dynamical equations compels the system to develop a Minkowskian core at radius $r = 0$. Then there will be two different types of hypersurfaces on the evolution of collapsing stars. The inner one will separate the vacuum core from the outer shearing viscous anisotropic fluid. The formation of such cores has been seen in the formation of cosmological voids. Voids are cosmological stellar structures that have long and thin ends. For the expansion-free systems, the factors that halt the working of X_{TF} are shear scalar and dark source terms coming from $f(R)$ gravity (as seen from eq. (45)). Thus, our results indicate that X_{TF} will be a inhomogeneity factor for those expansion-free systems (possibly voids) that experience shear-free motion.
3) If the contribution of the heat radiations are equal to the tr diagonal dark sources terms coming from the $f(R)$ models, then again the system will be in the homogeneous phase (if $X_{TF} = 0 = \beta$). This can only be possible if the repulsive effects of the dark source are counterbalanced by the heat radiations of the relativistic spherical structures. Thus, there could be some states of stellar bodies under which $\hat{q} = \psi_q$, then again the system will have homogeneous distribution of energy density if and only if $X_{TF} = 0 = \beta$.
4) It could be interesting to study scenario of dust cloud ball with the present values of Ricci scalar terms. Such conditions will nullify all types of radiations with $R = \tilde{R}$ (tilde shows the present value of the Ricci scalar) in our subsequent equations. This selection could be helpful to study the role of X_{TF}. In the following, we shall study this case.

4 Non-interacting particles and present Ricci scalar

In this section, we shall consider the dynamics of the dust cloud in the presence of constant Ricci scalar. The corresponding mass function in terms of energy density and dark source terms becomes

$$m = \frac{\kappa}{2(1-\beta e^{-\tilde{R}/R_E})}\int_0^r \mu C^2 C' \mathrm{d}r - \frac{1}{4(1-\beta e^{-\tilde{R}/R_E})}\left(-\frac{\beta}{2\kappa}R_E + \frac{\beta}{2\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right). \tag{47}$$

Equations (26) and (28) provide

$$\frac{3m}{C^3} = \frac{\kappa}{2(1-\beta e^{-\tilde{R}/R_E})}\left(\mu - \frac{1}{C^3}\int_0^r \mu' C^3 \mathrm{d}r\right) - \frac{1}{4(1-\beta e^{-\tilde{R}/R_E})}\left(-\frac{\beta}{2\kappa}R_E + \frac{\beta}{2\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right), \tag{48}$$

$$\mathcal{E} = \frac{\kappa}{2C^3(1-\beta e^{-\tilde{R}/R_E})}\int_0^r \mu' C^3 \mathrm{d}r, \tag{49}$$

whereas the SS (in $f(R)$ gravity) for the collection of non-interacting particles ar found as follows:

$$\tilde{Y}_T = \frac{\kappa}{2(1-\beta e^{-\tilde{R}/R_E})}\left(\mu + \frac{\beta}{\kappa}R_E - \frac{\beta}{\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right), \qquad -\tilde{X}_{TF} = \tilde{Y}_{TF} = \mathcal{E}, \tag{50}$$

$$\tilde{X}_T = \frac{\kappa}{(1-\beta e^{-\tilde{R}/R_E})}\left(\mu - \frac{\beta}{2\kappa}R_E + \frac{\beta}{2\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right). \tag{51}$$

Furthermore, the RE, SDE are found to be related with SS as

$$V^\alpha \Theta_{;\alpha} + \frac{2}{3}\sigma^2 + \frac{\Theta^2}{3} - a^\alpha_{;\alpha} = \left(\mu + \frac{\beta}{\kappa}R_E - \frac{\beta}{\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right) = -\tilde{Y}_T, \tag{52}$$

$$V^\alpha \sigma_{;\alpha} + \frac{\sigma^2}{3} + \frac{2}{3}\sigma\Theta = -\mathcal{E} = -\tilde{Y}_{TF}. \tag{53}$$

Next, we have evaluated Ellis equation for the dust cloud and constant Ricci scalar correction. We have then used eqs. (27) and (50) to produce WDE as follows:

$$\left[\tilde{X}_{TF} + \frac{\kappa}{2(1-\beta e^{-\tilde{R}/R_E})}\left(\mu - \frac{\beta}{2\kappa}R_E + \frac{\beta}{2\kappa}e^{-\tilde{R}/R_E}(\tilde{R}+R_E)\right)\right]' = -\tilde{X}_{TF}\frac{3C'}{C}. \tag{54}$$

This equation clearly indicates that $\tilde{X}_{TF}$ is factor that control the appearance of irregularities over the homogenous surface of dust cloud. Thus the system will be conformally flat when, $\tilde{X}_{TF} = 0$.

5 Summary

In this paper, we examined the influence of extra curvature terms of $f(R)$ gravity as well as the electromagnetic effects on the dynamical features of celestial objects. In this scenario, we have selected the relativistic spherical geometry coupled with dissipative anisotropic viscous fluid configurations. We have considered one of the viable models of $f(R)$ gravity that could provide results compatible with ΛCDM model, under some constraints. The modified field equations with electromagnetic field have been explored. The mass function is also calculated with the help of Misner Sharp formalism. A relation among metric, matter, charged and dark source variables is developed after using Weyl scalar. We have then express this equation in terms of mass function. Such kind of result is then used to calculate Ellis equations.

By adopting the procedure developed by Herrera *et al.* [34], we have constructed modified form of SS for the spherically symmetric spacetime. We have splitted orthogonally the Riemann tensor into two tensors. These tensors are mentioned at eq. (31). These tensors are then expressed in terms of their trace and trace less parts as seen by eqs. (32) and (33). After using field equations, the relations of SS are being written in term of matter variables. In view of exploring their applications, effective variables are introduced and the previous relations are expressed through these relations. These relations contain dark source terms coming from exponential $f(R)$ model. Three well known differential equations, named as RE, SDE and WDE are being expressed through $f(R)$-Maxwell SS. It seen that the influence of RE is fully controlled by SS Y_T, whereas physics of SDE can be unveiled through the SS Y_{TF}. In other words, the rate of expansion/contraction of the congruences, exact solutions of field equations, etc cab be dealt with Y_T and the shearing viscous evolution of the spherical celestial object can be well-discussed via Y_{TF}.

The inhomogeneity factor has also been examined for the non-adiabatic spherical geometry. We have concluded that dark source terms coming from exponential $f(R)$ model tend to reduce the rate of appearance of InED as seen from eq. (45). We have also studied the working of X_{TF} in the presence and absence of heat radiation, expansion and shear scalar. It is seen that expansion-free system (like voids) has regular energy density (if $\beta = 0$), either they have shear free system or heat radiations are equal to the quantity ψ_q. Furthermore, if a system evolves in such a way that it has expansion to be proportional to shear, then X_{TF} will be inhomogeneity factor if and only if the effects of $f(R)$ models are negligible. Thus, the $f(R)$ model is freezing the effects of X_{TF} in controlling regular distributions of energy density.

We have also studied these results for the dust cloud with constant Ricci scalar condition. and found the following results:

– The SS, X_T is directly governing the role of effective distributions of energy density for the collection of non-interacting particles.
– The SS Y_{TF} and X_{TF} are directly related in producing tidal forces on the regular environment of spherical spacetime, as seen by eq. (50).
– The expansion motion of the non-interacting ball is influenced by the exponential Ricci scalar terms coming in RE as described by eq. (43).
– The results of ref. [45] can be recovered, on putting $\beta = 0$.

When the lifespan of a stellar structure ends after experiencing the phenomenon of gravitational collapse, the black hole, white dwarf or neutron star may be formed depending upon the initial mass of the self-gravitating system. This phenomenon often happens when the gravitational pull overtakes the thermal pressure. Black hole acts as a black body regardless of the fact that they would emit Hawking radiations. This interesting phenomenon can only happen when the surface of the relativistic self-gravitating object undergoes in the irregular/inhomogeneous state. To understand the different astrophysical aspects of the compact objects, the obtained InED factors, *i.e.*, X_{TF} and $\tilde{X}_{TF}$ would play a significant role. The InED factor is a character of the splitting of Riemann tensor due to which the system would lie in the stable/unstable regimes.

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