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# Analytical solutions of the Klein-Gordon equation for the deformed generalized Deng-Fan potential plus deformed Eckart potential

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**Abstract.** In this paper, we study approximate analytical solutions of the Klein-Gordon equation with arbitrary $l$ state for the deformed generalized Deng-Fan potential plus deformed Eckart potential using the Nikiforov-Uvarov method and employing the approximation scheme for the centrifugal term. We obtain the energy eigenvalue equation and corresponding wave functions. Also, we discuss nonrelativistic limit of the energy equation. Finally, some numerical results are presented and show that these results are in good agreement with those obtained previously by other methods.

## 1 Introduction

The solutions of the wave equations for both relativistic and nonrelativistic cases have become an important area of research in different branches of physics. Thus, in recent years, there has been increased attention to investigate the analytical solutions of the wave equations with exactly solvable potentials such as the Woods-Saxon potential [1,2], the Eckart potential [3–5], the Pöschl-Teller potential [6,7], The Deng-Fan potential [8], The Manning-Rosen potential [9–12], Hulthén potential [13,14], the Scarf potential [15,16], the Morse potential [17–19], etc. So far, many methods have been developed to obtain the exact and approximate solutions of the quantum mechanics systems. These methods include the asymptotic iteration method (AIM) [20,21], supersymmetry quantum mechanics [22,23], Nikiforov-Uvarov (NU) method [24–27], group theoretical approach [28,29], Laplace transformation [30], quantization rules [31] and others. The Deng-Fan potential [32] has been one of the most useful and convenient potential models to study diatomic molecular energy spectra and electromagnetic transitions [33]. This potential model can be used to describe the motion of the nucleons in the mean field produced by the interactions between nuclei [34]. It has been widely used in the chemical physics, molecular spectroscopy, molecular physics, and related fields. The Eckart potential [35], introduced by Eckart in 1930, is a diatomic molecular potential model. Due to its importance in physics and chemical physics, the bound state solutions of the wave equations for this potential have been carried out [3–5]. Thus, it is worth to investigate the solution of the Klein-Gordon equation for the deformed generalized Deng-Fan potential plus deformed Eckart potential. This potential was studied in [36] for the Schrödinger equation. The deformed generalized Deng-Fan potential plus deformed Eckart potential can be given as

$$V(r) = V_0\left(c - \frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2 - \frac{V_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + \frac{V_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2}\,, \tag{1}$$

with

$$b = e^{-\alpha r_e} - 1,$$

where the parameters $V_0$, $V_1$ and $V_2$ are potential depths and $q$ is the deformation parameter. $c$, $\alpha$ and $b$ are adjustable constant, range of the potential and the position of the minimum $r_e$ ($r_e$ is equilibrium inter-nuclear distance), respectively. When $q = c = 1$ and $V_1 = V_2 = 0$, the above potential reduce to the Deng-Fan potential and with $V_0 = 0$ we obtain the Eckart potential. For most potentials, the analytical solutions of the wave equations are possible only in

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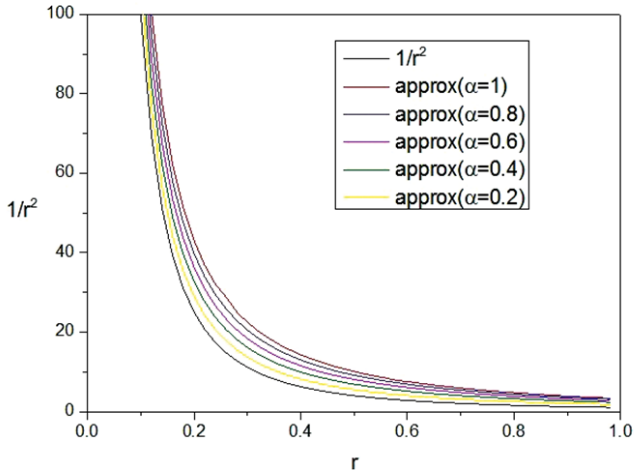


**Fig. 1.** Comparison between $1/r^2$ and the approximation scheme as functions of $r$ for $\omega = 5.00$, $q = 1$ and various values of $\alpha = 1.0$, 0.8, 0.6, 0.4 and $0.2\,\mathrm{fm}^{-1}$.

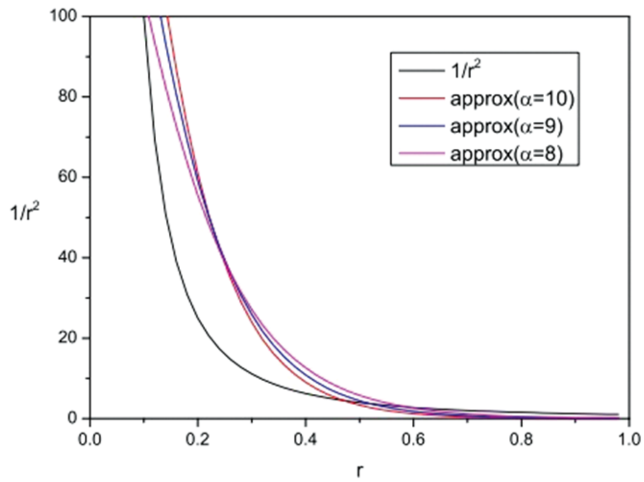


**Fig. 2.** Comparison between $1/r^2$ and the approximation scheme as functions of $r$ for $\omega = 5.00$, $q = -1$ and various values of $\alpha = 10$, 9 and $8\,\mathrm{fm}^{-1}$.

the $s$-wave case with the angular momentum $l = 0$. However, when $l \neq 0$, we have to use a suitable approximation scheme for the centrifugal term. Here, the $q$-deformed version of the approximation scheme proposed in [37] will be used to the centrifugal term, that is,

$$\frac{1}{r^2} \approx \alpha^2 \left[ \frac{\omega e^{-\alpha r}}{1 - q e^{-\alpha r}} + \frac{e^{-2\alpha r}}{(1 - q e^{-\alpha r})^2} \right]. \tag{2}$$

In order to show the validity of such an approximation scheme, the plots of the centrifugal term $1/r^2$ and the approximation scheme to it (eq. (2)), as functions of the variable $r$ with different potential range parameter $\alpha$ are displayed in figs. 1 and 2, for $q = 1, -1$, respectively.

This paper is organized as follows: In sect. 2, we give a brief review of the NU method. In sect. 3, the $l$-wave bound state solutions and the radial wave functions the Klein-Gordon equation for the deformed generalized Deng-Fan potential plus deformed Eckart potential are derived by apply the NU method and also, we derive the nonrelativistic energy eigenvalue equation from the relativistic energy eigenvalue equation. In sect. 4, we study several special cases of this model. We have found that our results for the energy spectra of these special cases are in complete agreement with those obtained previously. In sect. 5 some numerical results are presented. Finally a brief conclusion is given in sect. 6.

## 2 Review of Nikiforov-Uvarov method

The differential equations whose solutions are special functions of hypergeometric type can be studied by using the Nikiforov-Uvarov (NU) method which has been developed by Nikiforov and Uvarov [24]. In this method, the Schrödinger and Schrödinger-like equations are transformed into a second-order differential equation with an appropriate coordinate transformation, $s = s(r)$, of the form

$$\frac{\mathrm{d}^2\psi(s)}{\mathrm{d}s^2} + \frac{\tilde{\tau}(s)}{\sigma(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{3}$$

where $\sigma(s)$, $\tilde{\sigma}(s)$ are at most second-degree polynomials and $\tilde{\tau}(s)$ is a first-degree polynomial. In order to find the exact solution of eq. (3), we set the wave function as

$$\psi(s) = \varphi(s)y_n(s). \tag{4}$$

So, eq. (3) reduces to an equation of hypergeometric type,

$$\sigma(s)\frac{\mathrm{d}^2y_n}{\mathrm{d}s^2} + \tau(s)\frac{\mathrm{d}y_n}{\mathrm{d}s} + \lambda y_n = 0, \tag{5}$$

and the wave function $\varphi(s)$ is defined as

$$\frac{\varphi(s)}{\varphi'(s)} = \frac{\sigma(s)}{\pi(s)}. \tag{6}$$

Also, the hypergeometric function $y_n(s)$ has polynomial solutions given by the Rodrigues relation,

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{7}$$

where $B_n$ being the normalization constant and $\rho(s)$ is the weight function, which should satisfy the following condition:

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s), \tag{8}$$

with

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tag{9}$$

where the derivative of $\tau(s)$ with respect to $s$ should be negative.

In order to obtain the eigenfunctions and the eigenvalues, we need to define the function $\pi(s)$ and the parameter $\lambda$ as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + t\sigma} \tag{10}$$

and

$$\lambda = t + \pi'(s), \tag{11}$$

respectively. On the other hand, the value of $t$ is simply defined by setting the discriminant of the square root equal to zero. The values of $t$ can be used for calculation of energy eigenvalues using the following equation:

$$\lambda = t + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s). \tag{12}$$

## 3 Bound-state solutions

As is well known, the Klein-Gordon equation is the equation of motion of a quantum scalar or pseudoscalar field. For a spinless relativistic quantum particle under a scalar and a vector potential $S(r)$ and $V(r)$, respectively, the time-independent Klein-Gordon equation is given by

$$\left[-\nabla^2 + (M + S(r))^2\right]\Psi_{nlm}(r,\theta,\varphi) = (E - V(r))^2\,\Psi_{nlm}(r,\theta,\varphi), \tag{13}$$

where $E$ is the energy and $\Psi$ is the wavefunction of the particle, and $M$ denotes the mass of the particle (we shall assume $\hbar = c = 1$ throughout this paper).

Defining $\Psi_{nlm}(r,\theta,\varphi) = r^{-1}U_{nl}(r)Y_{lm}(\theta,\varphi)$, the radial Klein-Gordon equation is given by

$$\frac{\mathrm{d}^2U_{nl}(r)}{\mathrm{d}r^2} + \left[E^2 - M^2 - 2\left(MS(r) + EV(r)\right) + V^2(r) - S^2(r) - \frac{l(l+1)}{r^2}\right]U_{nl}(r) = 0, \tag{14}$$

where $n$ and $l$ are the radial quantum number and the orbital angular momentum quantum number, respectively.

Obviously, eq. (14) cannot be solved analytically with $l \neq 0$, due to the centrifugal term. In order to solve the above equation with nonzero angular momentum, we evaluate the centrifugal term in this equation using the mentioned approximation scheme in eq. (2). Hence, in the case of equal scalar and vector deformed generalized Deng-Fan potential plus deformed Eckart potential equation (14) becomes

$$\frac{\mathrm{d}^2 U_{nl}(r)}{\mathrm{d}r^2} + \Bigg[E^2 - M^2 - 2(M+E)\left(V_0\left(c - \frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2 - \frac{V_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + \frac{V_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2}\right) - l(l+1)\alpha^2\left[\frac{\omega e^{-\alpha r}}{1-qe^{-\alpha r}} + \frac{e^{-2\alpha r}}{(1-qe^{-\alpha r})^2}\right]\Bigg] U_{nl}(r) = 0. \tag{15}$$

By using the following coordinate transformation,

$$s = e^{-\alpha r}, \tag{16}$$

we obtain the following equation:

$$\frac{\mathrm{d}^2 U_{nl}}{\mathrm{d}s^2} + \frac{1-qs}{s(1-qs)}\frac{\mathrm{d}U_{nl}}{\mathrm{d}s} + \frac{1}{s^2(1-qs)^2}\left[\left(-\varepsilon^2 q^2 - \gamma - \nu q\right)s^2 + (2\varepsilon^2 q - \beta + \nu)s - \varepsilon^2\right] U_{nl}(s) = 0, \tag{17}$$

where

$$\begin{aligned}
-\varepsilon^2 &= \frac{E^2 - M^2 - 2(M+E)V_0 c^2}{\alpha^2},\\
\gamma &= \frac{2(M+E)V_0 b^2}{\alpha^2} + l(l+1),\\
\beta &= \frac{2(M+E)V_2}{\alpha^2},\\
\nu &= \frac{2(M+E)(2V_0 bc + V_1)}{\alpha^2} - l(l+1)\omega.
\end{aligned} \tag{18}$$

In this form of the above equation, we can use the Nikiforov-Uvarov method to evaluate the solution of the relevant second-order differential equation. Thus, we compare the above equation with the generalized hypergeometric type, eq. (3), we have,

$$\tilde{\tau}(s) = 1 - qs, \qquad \sigma(s) = s(1-qs),$$
$$\tilde{\sigma}(s) = -\left(\varepsilon^2 q^2 + \gamma + \nu q\right)s^2 + \left(2\varepsilon^2 q - \beta + \nu\right)s - \varepsilon^2.$$

When these polynomials are substituted into eq. (10), we get the $\pi(s)$ as follows:

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2}\left[\left(q^2 + 4q^2\varepsilon^2 + 4q\nu + 4\gamma - 4qt\right)s^2 + 4(t - 2q\varepsilon^2 + \beta - \nu)s + 4\varepsilon^2\right]^{\frac{1}{2}}. \tag{19}$$

Further, according to this method, the discriminant of the square root has to be zero and, due to $\pi(s)$, is at most a first-degree polynomial. Thus, the constant $t$ can be determined as

$$t = \nu - \beta \pm \varepsilon\sqrt{q^2 + 4\beta q + 4\gamma}, \tag{20}$$

which yields

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2}\left[\left(-2q\varepsilon \pm \sqrt{q^2 + 4\beta q + 4\gamma}\right)s + 2\varepsilon\right]. \tag{21}$$

In the NU method, the derivative of the polynomial $\tau(s)$ with respect to $s$ must be negative. To do this, we can take the $t = \nu - \beta - \varepsilon\sqrt{q^2 + 4\beta q + 4\gamma}$ and, then, we have

$$\pi(s) = -\frac{qs}{2} + \frac{1}{2}\left[\left(-2q\varepsilon - \sqrt{q^2 + 4\beta q + 4\gamma}\right)s + 2\varepsilon\right]. \tag{22}$$

$\tau(s)$ is obtained as

$$\tau = 1 - 2qs - \left[\left(2q\varepsilon + \sqrt{q^2 + 4\beta q + 4\gamma}\right)s + 2\varepsilon\right]. \tag{23}$$

Using eq. (12) we can consequently find the following energy eigenvalue equation:

$$\varepsilon_{nl}^{q}=\frac{\nu-\beta-n^2q-n\sqrt{q^2+4\beta q+4\gamma}}{q(2n+1)+\sqrt{q^2+4\beta q+4\gamma}}-\frac{1}{2}\,. \tag{24}$$

Let us now find the corresponding wave functions.

By substituting $\pi(s)$, $\sigma(s)$ into eq. (6), and solving the first-order differential equation we obtain

$$\varphi=s^{\varepsilon}(1-qs)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}}. \tag{25}$$

Considering eq. (8), $\rho(s)$ is obtained as

$$\rho=s^{2\varepsilon}(1-qs)^{\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}}, \tag{26}$$

and, then, we get $y_n$ given by the Rodrigues relation, eq. (7), as

$$y_n=B_n s^{-2\varepsilon}(1-qs)^{-\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[s^{2\varepsilon+n}(1-qs)^{\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}+n}\right]. \tag{27}$$

On the other hand, the Jacobi polynomials are defined [38] as

$$P_n^{(a,b)}(s)=\frac{(-1)^n}{n!2^n(1-s)^a(1+s)^b}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[(1-s)^{a+n}(1+s)^{b+n}\right]$$

and

$$P_n^{(a,b)}(1-2s)=\binom{n+a}{n}{}_2F_1\left(-n,a+b+n+1;a+1,s\right).$$

Therefore, $y_n(s)$ can be expressed in terms of the hypergeometric function as follows:

$$y_n=N_n\,{}_2F_1\left(-n,2\varepsilon+\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}+n+1;2\varepsilon+1,qs\right), \tag{28}$$

where $N_n$ is a new normalization constant. Finally, the corresponding radial wave function is obtained as

$$U_{nl}(r)=N_n\,e^{-\alpha r\varepsilon}(1-qe^{-\alpha r})^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}}\,{}_2F_1\left(-n,2\varepsilon+\sqrt{1+\frac{4\beta}{q}+\frac{4\gamma}{q^2}}+n+1;2\varepsilon+1,qe^{-\alpha r}\right). \tag{29}$$

The above results are new and, to our knowledge, have not been reported in the literature.

Setting $V(r)\to\frac{V(r)}{2}$, $E+M\to\frac{2\mu}{\hbar^2}$, and $E-M\to E$, we can obtain the energy eigenvalue equation for the deformed generalized Deng-Fan potential plus deformed Eckart potential in the nonrelativistic limit. In this conditions, the nonrelativistic limit of our result in eq. (24) reduces to

$$E=-\frac{\hbar^2\alpha^2}{2\mu q^2(2n+1+\eta)^2}\left[\frac{2\mu(2V_0bc+V_1-V_2)}{\hbar^2\alpha^2}-q\left(n^2+n+\frac{1}{2}\right)-l(l+1)\omega-q\left(n+\frac{1}{2}\right)\eta\right]^2+V_0c^2, \tag{30}$$

with

$$\eta=\sqrt{1+\frac{8\mu}{q^2\hbar^2\alpha^2}\left(V_0b^2+V_2q\right)+\frac{4l(l+1)}{q^2}}\,.$$

By making the parameter replacements used by Awoga *et al.* [36] as follows:

$$\varphi=\frac{2\mu b^2V_0}{\hbar^2\alpha^2}\,,\qquad \beta=\frac{2\mu V_2}{\hbar^2\alpha^2}\,,\qquad \sigma_q=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{q^2}(\varphi+q\beta)}\,,$$

eq. (30), for $l=0$, is identical to eq. (53) of [36].

## 4 Special cases

In this section, we consider some special cases of eq. (1), like that of Awoga *et al.* [36]. Indeed, by an appropriate adjustment of constants which appear in the difinition of the potential in eq. (1), one could hold the following cases.

### 4.1 Woods-Saxon potential

The potential given in eq. (1) can be converted to the Woods-Saxon potential by replacing $V_0 = V_2 = 0$ and $q = -1$. Under these conditions, we obtain the Woods-Saxon potential as

$$V(r) = -\frac{V_1 e^{-\alpha r}}{1 + e^{-\alpha r}}\,. \tag{31}$$

Hence, we can find the eigenvalue equation for the Woods-Saxon potential from eq. (24) as

$$\varepsilon_{nl} = \frac{2(M+E)V_1/\alpha^2 - l(l+1)\omega + n^2 - n\sqrt{1+4l(l+1)}}{-(2n+1) + \sqrt{1+4l(l+1)}} - \frac{1}{2}\,, \tag{32}$$

which is in full agreement with the result obtained by Hamzavi *et al.* [39]. To see whether this agrees with relation (26) in [39], we use the following transformations of parameters in [39]:

$$S_0 = V_0 \to V_1, \qquad a^2 \to \frac{1}{\alpha^2}\,, \qquad \frac{a^2}{{R_0}^2} \to 1$$

and, also,

$$D_0 = 0, \qquad D_1 = \omega, \qquad D_2 = 1.$$

### 4.2 Hulthén potential

If $V_0 = V_2 = 0$ and $q = 1$, the potential in eq. (1) turns to the Hulthén potential,

$$V(r) = -\frac{V_1 e^{-\alpha r}}{1 - e^{-\alpha r}}\,. \tag{33}$$

Then, we obtain the corresponding eigenvalue equation from eq. (24) as

$$\varepsilon_{nl} = \frac{2(M+E)V_1/\alpha^2 - l(l+1)\omega - n^2 - n\sqrt{1+4l(l+1)}}{2n+1 + \sqrt{1+4l(l+1)}} - \frac{1}{2}\,. \tag{34}$$

When we make the following transformations of parameters, we note that the above equation agrees with the bound state solutions of the $s$-wave Klein-Gordon equation with equally mixed scalar and vector $S(r) = V(r)$ for Hulthén potential derived in [40],

$$\alpha \to \frac{1}{a}\,, \qquad V_1 \to V_0.$$

### 4.3 Manning-Rosen potential

Setting $V_1 = V_2 = 0$, $c = 0$ and $q = 1$ in eq. (1), we get the Manning-Rosen potential,

$$V(r) = b^2 V_0 \left(\frac{e^{-\alpha r}}{1 - e^{-\alpha r}}\right)^2. \tag{35}$$

In this case, Immediately the energy eigenvalue equation for the Manning-Rosen potential is obtain from eq. (24) as

$$\varepsilon_{nl} = \frac{-l(l+1)\omega - n^2 - n\sqrt{1+8(M+E)V_0 b^2/\alpha^2 + 4l(l+1)}}{2n+1 + \sqrt{1+8(M+E)V_0 b^2/\alpha^2 + 4l(l+1)}} - \frac{1}{2}\,. \tag{36}$$

This expression just agrees with the relation (24) in [41], provided we take the following transformation [41]:

$$A \to 0, \qquad \frac{1}{\beta} \to \alpha, \qquad \frac{\alpha(\alpha-1)}{2M\beta^2} \to V_0 b^2\,, \qquad \lambda \to \varepsilon.$$

**Table 1.** The relativistic energy eigenvalues in unit of $fm^{-1}$ of the deformed generalized Deng-Fan potential plus deformed Eckart potential as a function of parameter $\alpha$ for several states.

| States | $\alpha$ | $E$ |
|---|---|---|
| 2p | 0.05 | 4.41272 |
| | 0.075 | 4.32729 |
| | 0.1 | 4.31632 |
| | 0.15 | 4.34548 |
| | 0.2 | 4.39297 |
| 3p | 0.05 | 8.69997 |
| | 0.075 | 8.64742 |
| | 0.1 | 8.66453 |
| | 0.15 | 8.74691 |
| | 0.2 | 8.84638 |
| 3d | 0.05 | 5.9705 |
| | 0.075 | 5.89972 |
| | 0.1 | 5.89388 |
| | 0.15 | 5.92674 |
| | 0.2 | 5.97535 |
| 4p | 0.05 | 11.7698 |
| | 0.075 | 11.7515 |
| | 0.1 | 11.7931 |
| | 0.15 | 11.9177 |
| | 0.2 | 12.0566 |
| 4d | 0.005 | 9.60322 |
| | 0.075 | 9.56137 |
| | 0.1 | 9.58179 |
| | 0.15 | 9.66552 |
| | 0.2 | 9.76426 |
| 4f | 0.05 | 7.60609 |
| | 0.075 | 7.54961 |
| | 0.1 | 7.55032 |
| | 0.15 | 7.59081 |
| | 0.2 | 7.6448 |

## 5 Numerical results

Some approximate numerical results of energy levels for various values of $n$, $l$ and $\alpha$ are shown in table 1, for the Klein-Gordon equation with deformed generalized Deng-Fan potential plus deformed Eckart potential, when the potential parameters are chosen as $V_0 = 15$, $V_1 = \alpha$, $V_2 = 0.0001$, $r_e = 0.4$ and $q, M, \omega = 1$.

Table 2 presents the results for eq. (30), that is, the nonrelativistic limit of eq. (24), for a set of selected values parameter in table 1.

To check our analytical expressions, in tables 3 and 4, we list some energy eigenvalues in nonrelativistic limit for the Deng-Fan potential and Eckart potential, respectively, and the comparison of calculation results with those of other methods, has proven the success of the formalism.

**Table 2.** The nonrelativistic energy eigenvalues of the deformed generalized Deng-Fan potential plus deformed Eckart potential as a function of parameter $\alpha$ for several states in atomic units $\hbar = \mu = 1$.

| States | $\alpha$ | $E$ |
|---|---|---|
| 2p | 0.025 | 6.88007 |
| | 0.05 | 6.88007 |
| | 0.075 | 6.7104 |
| | 0.1 | 6.74792 |
| | 0.15 | 6.84231 |
| 3p | 0.025 | 10.3374 |
| | 0.05 | 10.3267 |
| | 0.075 | 10.3982 |
| | 0.1 | 10.4814 |
| | 0.15 | 10.6542 |
| 3d | 0.025 | 9.44239 |
| | 0.05 | 9.41246 |
| | 0.075 | 9.46784 |
| | 0.1 | 9.53554 |
| | 0.15 | 9.67775 |
| 4p | 0.025 | 12.0073 |
| | 0.05 | 12.065 |
| | 0.075 | 12.1646 |
| | 0.1 | 12.2687 |
| | 0.15 | 12.4756 |
| 4d | 0.025 | 11.5463 |
| | 0.05 | 11.5968 |
| | 0.075 | 11.6888 |
| | 0.1 | 11.7857 |
| | 0.15 | 11.979 |
| 4f | 0.025 | 11.2213 |
| | 0.05 | 11.2697 |
| | 0.075 | 11.3562 |
| | 0.1 | 11.4471 |
| | 0.15 | 11.6286 |

## 6 Conclusion

In this work, we investigated the approximate bound state solutions of the Klein-Gordon equation with deformed generalized Deng-Fan potential plus deformed Eckart potential within the framework of Nikiforov-Uvarov method by approximating the centrifugal term. We obtained explicity the energy equation and the corresponding wave function. By choosing appropriate parameters in the deformed generalized Deng-Fan potential plus deformed Eckart potential, it can yield some special cases, such as the Woods-Saxon potential, Hulthén potential and Manning-Rosen potential.

**Table 3.** The nonrelativistic energy eigenvalues of the Deng-Fan potential as a function of parameter $\alpha$ for several states in atomic units $\hbar = \mu = 1$.

| States | $\alpha$ | $E^{(a)}$ | $E^{(b)}$ | $E^{(c)}$ |
|---|---|---|---|---|
| 2p | 0.05 | 7.8606 | 7.8606 | 7.8628 |
| | 0.1 | 7.95247 | 7.95247 | 7.95537 |
| | 0.15 | 8.04322 | 8.04322 | 8.04724 |
| | 0.2 | 8.13287 | 8.13287 | 8.13842 |
| 3p | 0.05 | 10.9976 | 10.9976 | 10.9998 |
| | 0.1 | 11.1617 | 11.1617 | 11.1647 |
| | 0.15 | 11.3224 | 11.3224 | 11.32647 |
| | 0.2 | 11.4795 | 11.4795 | 11.48513 |
| 3d | 0.05 | 10.2154 | 10.2154 | 10.21651 |
| | 0.1 | 10.351 | 10.351 | 10.35409 |
| | 0.15 | 10.4837 | 10.4837 | 10.48992 |
| | 0.2 | 10.6135 | 10.6135 | 10.62403 |
| 4p | 0.05 | 12.4974 | 12.4974 | 12.4992 |
| | 0.1 | 12.696 | 12.696 | 12.69851 |
| | 0.15 | 12.8865 | 12.8865 | 12.8901 |
| | 0.2 | 13.0689 | 13.0689 | 13.074 |
| 4d | 0.05 | 12.0977 | 12.0977 | 12.0989 |
| | 0.1 | 12.2825 | 12.2825 | 12.2857 |
| | 0.15 | 12.4608 | 12.4608 | 12.46715 |
| | 0.2 | 12.6326 | 12.6326 | 12.64324 |
| 4f | 0.05 | 11.8195 | 11.8195 | 11.8209 |
| | 0.1 | 11.993 | 11.993 | 11.9981 |
| | 0.15 | 12.1604 | 12.1604 | 12.1718 |
| | 0.2 | 12.3221 | 12.3221 | 12.3421 |

(a) Our results. (b) Results obtained in [42]. (c) Results obtained in [43].

**Table 4.** The nonrelativistic energy eigenvalues of the Eckart potential as a function of parameter $\alpha$ for several states in atomic units $\hbar = \mu = 1$.

| States | $\alpha$ | $E^{(a)}$ | $E^{(b)}$ | $E^{(c)}$ |
|---|---|---|---|---|
| 2p | 0.025 | 0.100888 | 0.1008358 | 0.1015944 |
| | 0.05 | 0.0980434 | 0.0978358 | 0.098298 |
| | 0.075 | 0.0888831 | 0.0884183 | 0.0885875 |
| | 0.1 | 0.079206 | 0.0783854 | 0.0784035 |
| | 0.15 | 0.0609141 | 0.0591059 | 0.059287 |
| 3p | 0.025 | 0.0401768 | 0.040125 | 0.0403106 |
| | 0.05 | 0.0324516 | 0.0322482 | 0.0323958 |
| | 0.075 | 0.0239989 | 0.0235553 | 0.0237732 |
| | 0.1 | 0.0166099 | 0.0158588 | 0.0162724 |
| | 0.15 | 0.00580856 | 0.0044091 | 0.005434 |
| 3d | 0.025 | 0.0415198 | 0.0413642 | 0.041479 |
| | 0.05 | 0.032812 | 0.0321973 | 0.0321085 |
| | 0.075 | 0.0241516 | 0.0227991 | 0.0229644 |
| | 0.1 | 0.0166878 | 0.0143675 | 0.0152256 |
| | 0.15 | 0.00583229 | 0.001365 | 0.0044524 |
| 4p | 0.025 | 0.0185142 | 0.0184632 | 0.0185468 |
| | 0.05 | 0.0109073 | 0.0107159 | 0.0108554 |
| | 0.075 | 0.00488286 | 0.00445059 | 0.004792 |
| | 0.1 | 0.0012129 | 0.0007212 | 0.00114 |
| 4d | 0.025 | 0.0190752 | 0.0189216 | 0.0189774 |
| | 0.05 | 0.0110422 | 0.0104603 | 0.0106863 |
| | 0.075 | 0.00492932 | 0.0037658 | 0.0045048 |
| 4f | 0.025 | 0.0193308 | 0.019022 | 0.0189461 |
| | 0.05 | 0.011101 | 0.0099137 | 0.0102192 |
| | 0.075 | 0.00494941 | 0.0025081 | 0.0039915 |

(a) Our results. (b) Results obtained in [43]. (c) Results obtained in [44].

In order to test the accuracy of our results, we performed a comparison of our obtained energy spectrum in special cases with those of obtained previously in the literatures. These results are found to be in perfect agreement with the findings of other authors. Also, the other important result is that, in the nonrelativistic limit, the result that we obtained for energy equation of the relativistic Klein-Gordon equation for deformed generalized Deng-Fan potential plus deformed Eckart potential is coincide with that obtained in ref. [36] for nonrelativistic Schrödinger equation with this potential.



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