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Nonlinearly charged three-dimensional black holes in the Einstein-dilaton gravity theory

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Abstract. In this article, the new black hole solutions to the Einstein-power-Maxwell-dilaton gravity theory have been investigated in a three-dimensional space time. The coupled scalar, electromagnetic and gravitational field equations have been solved in a spherically symmetric geometry and it has been shown that the dilatonic potential, as the solution to the scalar field equation, can be written in the form of a generalized Liouville potential. Also, two new classes of charged dilatonic BTZ black hole solutions, in the presence of power-law nonlinear electrodynamics, have been constructed out which are asymptotically non-flat and non-AdS. The conserved and thermodynamic quantities have been calculated from geometrical and thermodynamical approaches, separately. The consistency of the results of these two alternative approaches confirms the validity of the first law of black hole thermodynamics for both of the new black hole solutions. The black holes stability or phase transitions have been studied, making use of the canonical ensemble method. The points of type one and type two phase transitions as well as the ranges at which the black holes are stable have been indicated by considering the heat capacity of the new black hole solutions.

1 Introduction

There are several motivations for studying the three-dimensional exact black hole solutions as one of the interesting subjects for gravitational studies. The first comes from the fact that lower-dimensional space times are easier to study and three-dimensional solutions can help us to find a profound insight in the fundamental theories such as black hole physics and the quantum theory of gravity. The other arises from the AdS/CFT correspondence which relates the properties of a realized four-dimensional black hole with those of a quantum field theory in three dimensions [1–3]. Due to these facts and some other issues, studies of the three-dimensional manifolds and their attractive properties are still interesting objects.

The first studies on the three-dimensional black holes were done by Banados, Teitelboim, and Zanelli (BTZ). They showed that Einstein's field equations admit black hole solutions in $(2+1)$-dimensional space times with the negative cosmological constant [4,5]. Also, Chan and Mann are the first authors who investigated the charged three-dimensional black holes in the presence of a logarithmic dilaton field [6,7]. Although, the existence of the dilatonic black holes violates the no-hair conjecture, which originally stated that a black hole should be characterized only by its mass, angular momentum and electric charge [8,9], it has been shown by many authors that Einstein's gravity theory with a coupled scalar field admits exact hairy black hole solutions in three, four- and higher-dimensional space times [10–18].

The Maxwell theory of classical electrodynamics, as one of the successful theories of the twentieth century, is in agreement with a large amount of experimental tests but it confronted with the problem of infinite electric field and self-energy for the pointlike charged particles. A modification of Maxwell's theory to nonlinear theories of electrodynamics has been proposed originally to overcome these failures. Among the various proposed modifications are: the Born-Infeld [19–21], the logarithmic [22,23], the exponential [15,23,24], etc. As it is shown in [24], the usual theory of electrodynamics can be considered as the special case of the various proposed nonlinear theories. Also, they are less singular in comparison to Maxwell's theory of classical electrodynamics. Indeed, the nonlinear theories of electrodynamics are functions of Maxwell's invariant $F^{\mu\nu}F_{\mu\nu}$ and they are useful when the electromagnetic fields are highly strong such that the photon-photon interactions cannot be ignored. Even if the nonlinear theories of electrodynamics

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are proposed originally to remove the singularities of the classical theory of electrodynamics, nowadays they have provided useful instruments for studying the properties of the charged black holes. Investigation of the charged black hole solutions in the presence of Born-Infeled, logarithmic exponential, quadratic and power-law nonlinear electrodynamics has provided some new and interesting results in the context of geometrical physics and specially in the classical theory of black holes [15,24–31].

From the classical point of view, black holes are perfect absorbers; they do not emit anything and their physical temperature is absolute zero. It is well known that black holes can be considered as the thermodynamical systems with a temperature proportional to the surface gravity and entropy equal to one-fourth of the horizon area. Black holes can emit particles from the event horizon and the radiant spectrum is a pure thermal one. Since the radiation with a pure thermal spectrum cannot be recovered after black holes have evaporated and disappeared completely, the so-called information loss paradox, Hawking argued that the information could be preserved if the radiation spectrum were not a pure thermal one [32–38]. Now, thermodynamics of black holes is one of the most interesting research topics. There are several approaches for studying the black hole remnant or phase transition. Among them are thermodynamical geometry, canonical ensemble, grand canonical ensemble, etc. In the thermodynamical geometry, the points of phase transition can be determined by studying the divergence points of the thermodynamical Ricci scalar (see [39] and references therein). The thermal stability or phase transition of the black holes can be investigated by considering the behavior of the black hole heat capacity with the black hole charge as a constant [11,12,23,30,31]. In the grand canonical ensemble approach the determinant of the Hessian metrics enables one to study the thermal stability of the black holes [40–42].

The main goal of this work is to obtain the novel exact black hole solutions to the Einstein-power-Maxwell-dilaton gravity theory and to investigate the physical and thermodynamic properties of the solutions. Also, to check the validity of the thermodynamical first law as well as to perform a thermal stability or phase transition analysis for the new black hole solutions.

The paper is structured based on the following order. In sect. 2, by starting from a suitable three-dimensional Einstein dilatonic action coupled to a power-law nonlinear electrodynamics, we obtained the related field equations. We have solved the equations of the scalar, electromagnetic and tensor fields in a static spherically symmetric geometry and showed that the dilatonic potential can be written as the linear combination of two Liouville-type potentials. Also, two new classes of the black hole solutions, as the exact solutions to the Einstein-power-Maxwell-dilaton gravity theory have been constructed out, which are asymptotically non-flat and non-AdS. Section 3 is devoted to the study of the thermodynamic properties of the new charged black hole solutions. The black hole total charge and mass, as the conserved quantities, as well as the entropy and temperature associated with the black hole horizon have been obtained. Also, the electric potential of the black holes, relative to a reference point located at infinity relative to the horizon, has been obtained. In addition, through a Smarr-type mass formula, we have obtained the black hole mass as a function of the extensive parameters, charge and entropy. The intensive parameters, temperature and electric potential, conjugated to the extensive parameters, have been calculated from thermodynamical methods. The compatibility of the results of geometrical and thermodynamical approaches confirms the validity of the first law of black hole thermodynamics, for both classes of the new black hole solutions. Section 4 is dedicated to the investigation of the stability or phase transition of the black holes. Making use of the canonical ensemble method and regarding the black hole heat capacity, with the black hole charge as a constant, a black hole stability analysis has been performed and the points of type one and type two phase transitions as well as the ranges at which the black holes are locally stable have been determined, precisely. Some concluding results are summarized and discussed in sect. 5.

2 The field equations and the black hole solutions

We start with the action of the three-dimensional charged black holes in the Einstein gravity theory coupled to a dilatonic potential. It can be written in the following general form [11,12,43,44]

$$I = -\frac{1}{16\pi}\int \sqrt{-g}\mathrm{d}^3x\left[\mathcal{R} - V(\phi) - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \mathcal{L}(\mathcal{F},\phi)\right]. \tag{1}$$

Here, $\mathcal{R}$ is the Ricci scalar. ϕ is the scalar field coupled to itself via the functional form $V(\phi)$. The last term is the coupled scalar-electrodynamic Lagrangian. Making use of the power-law nonlinear electrodynamics and in terms of the scalar-electromagnetic coupling constant α, it can be written in the following form [43,45–47]

$$\mathcal{L}(\mathcal{F},\phi) = \left(-\mathcal{F}e^{-2\alpha\phi}\right)^p, \tag{2}$$

where, $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ being the Maxwell invariant. In terms of the electromagnetic potential, A_μ, $F_{\mu\nu}$ is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and power p is known as the nonlinearity parameter. It is expected that in the case $p = 1$ the results of this theory reduce to the Einstein-Maxwell-dilaton gravity theory. Now, by varying the action (1), we get the following field equations:

$$\mathcal{R}_{\mu\nu} = V(\phi)g_{\mu\nu} + 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{L}(\mathcal{F},\phi) + 2\mathcal{L}_\mathcal{F}(\mathcal{F},\phi)\left(\mathcal{F}g_{\mu\nu} - F_{\mu\alpha}F_\nu{}^\alpha\right), \tag{3}$$

$$\nabla_\mu\left[\mathcal{L}_\mathcal{F}(\mathcal{F},\phi)F^{\mu\nu}\right] = 0, \qquad \mathcal{L}_\mathcal{F}(\mathcal{F},\phi) \equiv \frac{\partial}{\partial\mathcal{F}}\mathcal{L}(\mathcal{F},\phi), \tag{4}$$

$$4\Box\phi = \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} + 2\alpha p\mathcal{L}(\mathcal{F},\phi), \quad \phi = \phi(r), \tag{5}$$

for the gravitational, electromagnetic and scalar field equations, respectively. Assuming as a function of r, the only non-vanishing component of the electromagnetic field is $F_{tr} = -E(r) = h'(r)$, and we have $\mathcal{F} = -2E^2(r) = -2(h'(r))^2$. Throughout this paper, prime means derivative with respect to the argument.

We consider the following ansatz as the three-dimensional spherically symmetric solution to the gravitational field equations:

$$\mathrm{d}s^2 = -\Psi(r)\mathrm{d}t^2 + \frac{1}{\Psi(r)}\mathrm{d}r^2 + r^2R^2(r)\mathrm{d}\theta^2. \tag{6}$$

Making use of (6) in (3), we arrived at the following explicit form of the gravitational equations:

$$e_{tt} \equiv \Psi''(r) + \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)\Psi'(r) + 2V(\phi) + 2(p-1)\mathcal{L}(\mathcal{F},\phi) = 0, \tag{7}$$

$$e_{rr} \equiv e_{tt} + 2\Psi(r)\left(\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} + 2\phi'^2(r)\right) = 0, \tag{8}$$

$$e_{\theta\theta} \equiv \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)\Psi'(r) + \left(\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)}\right)\Psi(r) + V(\phi) + (2p-1)\mathcal{L}(\mathcal{F},\phi) = 0, \tag{9}$$

for tt, rr and $\theta\theta$ components, respectively. Noting eqs. (7) and (8) we obtain

$$\frac{R''(r)}{R(r)} + \frac{2}{r}\frac{R'(r)}{R(r)} + 2\phi'^2(r) = 0. \tag{10}$$

The differential equation (10) can be written in the following form:

$$\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\ln R(r) + \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln R(r) + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln R(r)\right)^2 + 2\phi'^2(r) = 0. \tag{11}$$

From eq. (11), one can argue that $R(r)$ must be an exponential function of $\phi(r)$. Therefore, we can write $R(r) = e^{2\beta\phi(r)}$ in eq. (11), and show that $\phi = \phi(r)$ satisfies the following differential equation:

$$\beta\phi'' + (1+2\beta^2)\phi'^2 + \frac{2\beta}{r}\phi' = 0. \tag{12}$$

It is easy to write the solution of eq. (12) in terms of a positive constant b as

$$\phi(r) = \gamma\ln\left(\frac{b}{r}\right), \quad \text{with } \gamma = \beta(1+2\beta^2)^{-1}. \tag{13}$$

Here, we are interested in studying the effects of the exponential solution (*i.e.*, $R(r) = e^{2\beta\phi(r)}$) with both $\beta = \alpha$ and $\beta \neq \alpha$ on the thermodynamics behavior of the three-dimensional nonlinearly charged dilatonic black hole solutions. The cases of $\beta = \alpha$ and $\beta \neq \alpha$, with Maxwell's electromagnetic theory, have been considered in [11] and [12], respectively. Here, we are interested in extending this idea to the charged black hole solutions in the presence of power-law nonlinear electrodynamics. To do so, we proceed to solve the field equations, making use of the scalar fields given by eq. (13).

Regarding these solutions together with eqs. (4) and (6), we have

$$[1+2\gamma(\alpha p-\beta)]r^{2\gamma(\alpha p-\beta)}[h'(r)]^{2p-1} + (2p-1)r^{1+2\gamma(\alpha p-\beta)}h''(r)[h'(r)]^{2p-2} = 0, \quad p \neq \frac{1}{2}.$$

The solution to the above differential equation can be written in the following form:

$$\begin{cases} h(r) = -\dfrac{q(2p-1)}{2p-2-A}\,r^{1-\frac{A+1}{2p-1}}, & p \neq \dfrac{1}{2}, \\ F_{tr}(r) = q\,r^{-\frac{1+A}{2p-1}}, \end{cases} \tag{14}$$

where $A = 2\gamma(\alpha p - \beta)$ and q is an integration constant related to the total electric charge of the black hole. It will be calculated in the following section. It must be noted that in order for the potential function $h(r)$ to be physically reasonable (*i.e.* zero at infinity), the statement $1 - \frac{A+1}{2p-1}$ must be negative.

Now, eq. (9) can be rewritten as

$$\Psi'(r) - \frac{2\beta\gamma}{r}\Psi(r) + r(1+2\beta^2)[V(\phi) + (2p-1)\mathcal{L}(\mathcal{F},\phi)] = 0. \tag{15}$$

For solving this equation for the metric function $\Psi(r)$, we need to calculate the functional form of $V(\phi(r))$ as the function of the radial coordinate. For this purpose we return to the scalar field equation (5). It can be written as

$$\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} + \frac{4\gamma}{r}\left(\Psi'(r) - \frac{2\beta\gamma}{r}\Psi(r)\right) + 2\alpha p\mathcal{L}(\mathcal{F},\phi) = 0. \tag{16}$$

The combination of the coupled differential equations (15) and (16) leads to the following first-order differential equation for the scalar potential

$$\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} - 4\beta V(\phi) + [2\beta + p(\alpha - 4\beta)]2^{p+1}F_{tr}^{2p}e^{-2\alpha p\phi} = 0. \tag{17}$$

The solution to the differential (17) can be written as

$$V(\phi) = 2\Lambda e^{4\beta\phi} + 2\Lambda_0 e^{4\beta_0\phi}, \tag{18}$$

where

$$\Lambda_0 = \frac{2^{p-1}q^{2p}(2p-1)\Upsilon_1}{b^{\frac{2p(A+1)}{2p-1}}}, \quad \text{with } \Upsilon_1 = \frac{\beta(4\beta p - \alpha p - 2\beta)}{p(1+\alpha\beta-4\beta^2)+2\beta^2} \qquad \text{and} \qquad \beta_0 = \frac{p(1+\alpha\beta)}{2(2p-1)\beta}. \tag{19}$$

It is notable that the solution given by eq. (18) can be considered as the generalized form of the Liouville scalar potential. Also, it must be noted that in the absence of dilatonic field ϕ, we have $V(\phi = 0) = 2\Lambda = -2\ell^{-2}$ and the action (1) reduces to that of Einstein-Λ-Maxwell theory [48,49].

Now, making use of eqs. (14), (15) and (18) the metric function $\Psi(r)$ can be obtained as

$$\Psi(r) = \begin{cases} -m\,r^{2/3} + 3\left[\dfrac{2(b^2r)^{\frac{2}{3}}}{\ell^2}\ln\left(\dfrac{r}{L}\right) - \dfrac{3q^{2p}(2p-1)^2}{b^{2(\mathcal{B}-1)}}\Upsilon(\beta=1)\left(\dfrac{b}{r}\right)^{\frac{2(\alpha p+3-5p)}{3(2p-1)}}\right], & \text{for } \beta = 1,\\ -m\,r^{2\beta\gamma} - (1+2\beta^2)^2\left[\dfrac{\Lambda b^2}{1-\beta^2}\left(\dfrac{b}{r}\right)^{4\beta\gamma-2} + \dfrac{q^{2p}(2p-1)^2}{b^{2(B-1)}}\Upsilon(\beta)\left(\dfrac{b}{r}\right)^{2\beta_0\gamma-2}\right], & \text{for } \beta \neq 1, \end{cases} \tag{20}$$

where L is a dimensional constant and

$$B = \frac{p(1+A)}{2p-1}, \qquad \mathcal{B} = \frac{p(1+2\alpha p)}{3(2p-1)},$$
$$\Upsilon(\beta) = 2^{p-1}(1+\Upsilon_1)\Upsilon_2, \quad \text{with } \Upsilon_2 = \left[(2p-1)(1+\beta^2) - p(1+\alpha\beta)\right]^{-1}. \tag{21}$$

Note that in the case $p = 1$ the metric function (20) is compatible with that of ref. [12]. The plots of metric functions $\Psi(r)$, presented in eq. (20), have been shown in figs. 1 and 3 and figs. 2 and 4 for $\beta = \alpha$ and $\beta \neq \alpha$ cases, respectively. From the curves of figs. 1–4, it is understood that, for the suitably fixed parameters, the metric functions $\Psi(r)$ can produce two horizon, extreme and naked singularity black holes for both of $\beta = 1$ and $\beta \neq 1$ cases.

Now, we investigate the space time singularities regarding the Ricci and Kretschmann scalars. They can be written in the following forms:

$$R = 4\gamma^2(1+4\beta^2)\frac{\Psi(r)}{r^2} - \frac{2}{1+2\beta^2}\frac{\Psi'(r)}{r} - \Psi''(r), \tag{22}$$

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = 16\gamma^3\left(\frac{\Psi(r)}{r^2}\right)^2 - \frac{8\beta^2}{(1+2\beta^2)^3}\frac{\Psi(r)\Psi'(r)}{r^3} + \frac{2}{(1+2\beta^2)^2}\left(\frac{\Psi'(r)}{r}\right)^2 + (\Psi'')^2. \tag{23}$$

Making use of the metric function (20) in eqs. (22) and (23), one can show that the Ricci and Kretschmann scalars are finite for finite values of r. Also, it is easy to show that

$$\lim_{r\longrightarrow\infty}\mathcal{R} = 0, \qquad \text{and} \qquad \lim_{r\longrightarrow 0}\mathcal{R} = \infty,$$
$$\lim_{r\longrightarrow\infty}\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda} = 0, \qquad \text{and} \qquad \lim_{r\longrightarrow 0}\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda} = \infty,$$

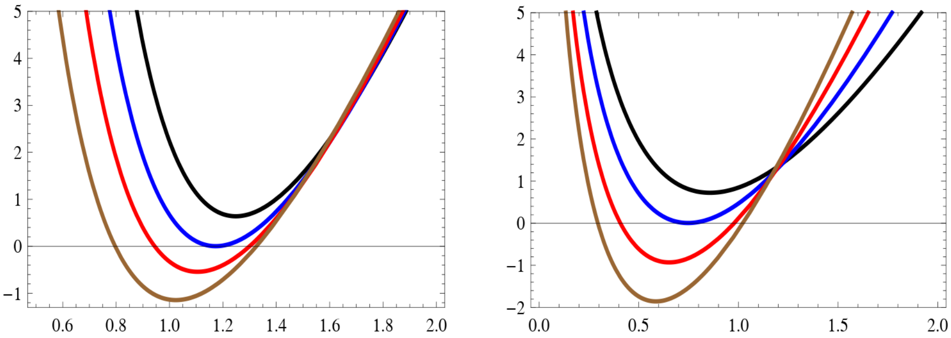

Fig. 1. $\Psi(r)$ *versus* r for $M = 0.5$, $Q = 0.5$, $\ell = 1$, $L = 1$ and $\beta = \alpha = 1$, eq. (20). Left: $b = 2$ and $p = 0.585$, 0.5925, 0.6, 0.61 for black, blue, red and brown curves, respectively. Right: $p = 0.7$ and $b = 1.56$, 1.68, 1.82, 1.95 for black, blue, red and brown curves, respectively.

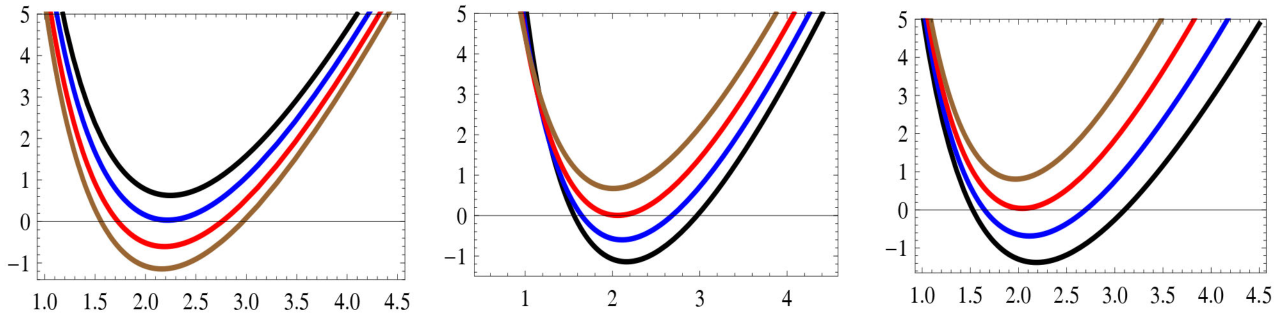

Fig. 2. $\Psi(r)$ *versus* r for $M = 0.5$, $Q = 1$, $\ell = 1$, $L = 1$, $\beta = 1$ and $\alpha \neq \beta$, eq. (20). Left: $p = 0.8$, $b = 1.2$ and $\alpha = 2.35$, 2.396, 2.45, 2.5 for black, blue, red and green curves, respectively. Middle: $b = 1.2$, $\alpha = 2.5$ and $p = 0.8$, 0.825, 0.85, 0.875 for black, blue, red and green curves, respectively. Right: $p = 0.8$, $\alpha = 2.5$ and $b = 1.19$, 1.22, 1.253, 1.29 for black, blue, red and green curves, respectively.

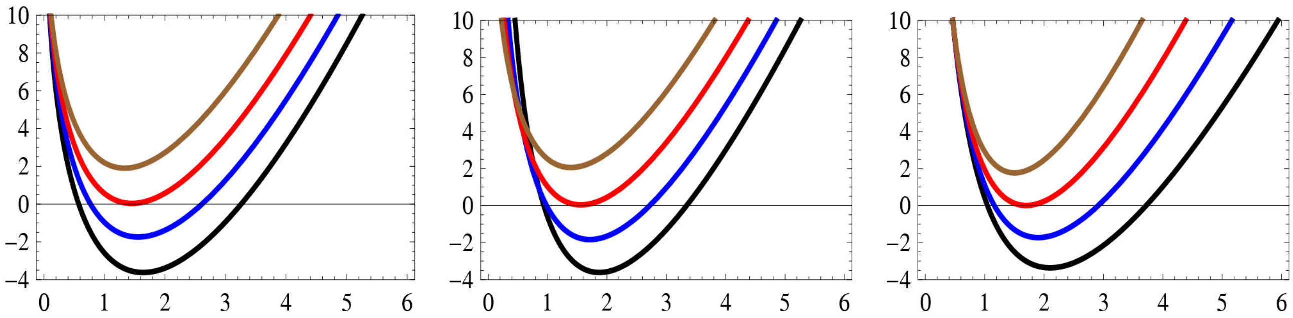

Fig. 3. $\Psi(r)$ *versus* r for $M = 2$, $Q = 0.6$, $\ell = 1$, $\beta \neq 1$ and $\beta = \alpha$, eq. (20). Left: $b = 1.5$, $p = 0.8$ and $\alpha = 0.6$, 0.635, 0.6615, 0.685 for black, blue, red and brown curves, respectively. Middle: $b = 1.5$, $\alpha = 0.73$ and $p = 0.69$, 0.72, 0.744, 0.765 for black, blue, red and brown curves, respectively. Right: $p = 0.7$, $\alpha = 0.76$ and $b = 1.37$, 1.42, 1.48, 1.55 for black, blue, red and brown curves, respectively.

from which, one can conclude that there is an essential singularity located at $r = 0$ and the asymptotic behavior of the solutions is neither flat nor AdS. Therefore, the inclusion of the scalar field modifies the asymptotic behavior of the solutions.

Up to now, we have obtained two new classes of static and spherically symmetric exact solutions to the coupled field equations of the Einstein-dilaton gravity theory presented in eq. (20). It must be noted that the solutions can be interpreted as black holes provided that the two following conditions are satisfied, simultaneously: 1) The existence of the singularity which could be determined by divergencies of the curvature scalars as presented in eqs. (22) and (23). 2) The appearance of at least one event horizon which has been illustrated by figs. 1–4. Therefore, our solutions are certainly black holes. Also, it is worth noting that the dilatonic black hole solutions, we just obtained, are non-singular and they are interesting to study because they can provide a deeper insight into their QFT counterpart.

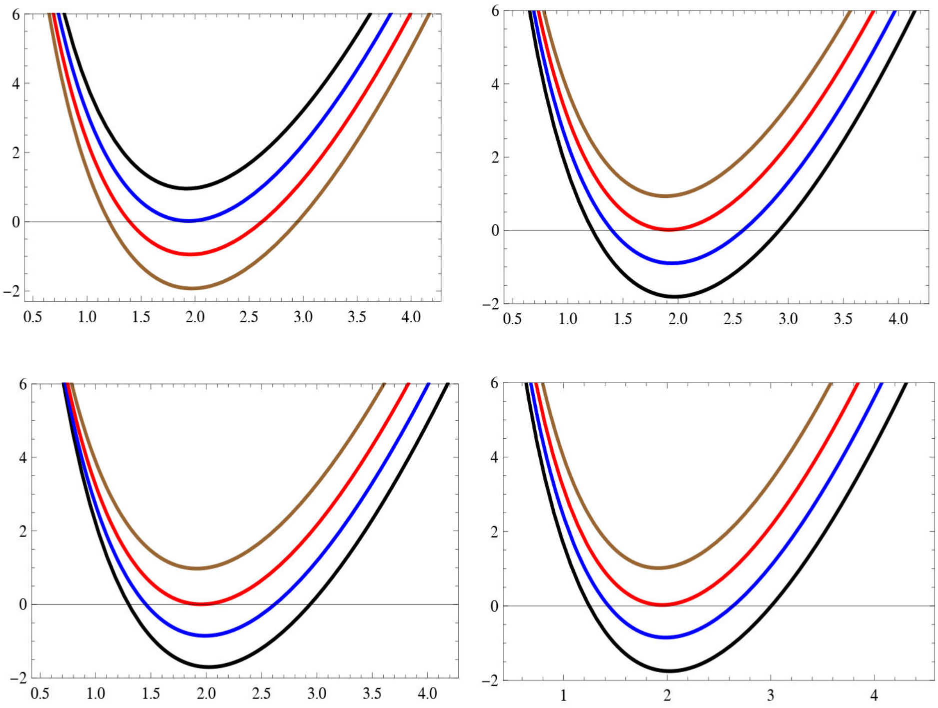

Fig. 4. $\Psi(r)$ *versus* r for $M = 2.5$, $Q = 1.25$, $\ell = 1$, $\beta \neq 1$ and $\beta \neq \alpha$, eq. (20). Top left: $b = 2$, $p = 1.2$, $\beta = 0.5$ and $\alpha = 1.985$, 2.025, 2.07, 2.12 for black, blue, red and brown curves, respectively. Top right: $b = 2$, $p = 1.2$, $\alpha = 2.1$ and $\beta = 0.49$, 0.52, 0.542, 0.56 for black, blue, red and brown curves, respectively. Bottom left: $b = 2$, $\alpha = 2$, $\beta = 0.5$ and $p = 1.135$, 1.16, 1.184, 1.21 for black, blue, red and brown curves, respectively. Bottom right: $\beta = 0.5$, $p = 1.2$, $\alpha = 2$ and $b = 1.84$, 1.9, 1.96, 2.03 for black, blue, red and brown curves, respectively.

3 Black hole thermodynamics

The aim of this section is to check the validity of the first law of black hole thermodynamics for both of the new charged dilatonic BTZ black holes introduced here. For this purpose, we calculate the conserved and thermodynamical quantities related to either of the black hole solutions. The black hole entropy as a pure geometrical quantity can be obtained from the well-known entropy-area law. It is equal to one quarter of the black hole surface area and for our new black hole solutions it can be written in the following form

$$S = \frac{A}{4} = \begin{cases} \dfrac{\pi r_+}{2}\left(\dfrac{b}{r_+}\right)^{2/3}, & \text{for } \beta = 1, \\[2ex] \dfrac{\pi r_+}{2}\left(\dfrac{b}{r_+}\right)^{2\beta\gamma}, & \text{for } \beta \neq 1. \end{cases} \tag{24}$$

which reduces to the entropy relation of the BTZ black holes in the absence of dilatonic parameters ($\beta = 0 = \gamma$). Indeed, the area law is a nearly universal law and dilatonic black holes are not exceptions. It is well-known that the entropy-area law is a direct result of the general relativity on one hand and, as proved by Jacobson, Einstein's field equations can be derived by assuming the universality of the area law, on the other hand [50].

The other thermodynamical quantity which can be calculated geometrically is the Hawking temperature associated with the black hole horizon $r = r_+$. In terms of the surface gravity κ it can be written as $T = \frac{\kappa}{2\pi}$, where

$$\kappa = \sqrt{-\frac{1}{2}\left(\nabla_\mu \chi_\nu\right)\left(\nabla^\mu \chi^\nu\right)}.$$

The four-vector χ^ν is known as the null killing vector of the horizon. Taking $\chi^\nu = (-1, 0, 0, 0)$, we have $\chi_\nu = (\Psi(r_+), 0, 0, 0)$ and hence $\nabla_\mu \chi_\nu \nabla^\mu \chi^\nu = -\frac{1}{2}\left(\frac{\mathrm{d}\Psi(r)}{\mathrm{d}r}\right)_{r=r_+}$. Therefore, we have $T = \frac{1}{4\pi}\left(\frac{\mathrm{d}\Psi(r)}{\mathrm{d}r}\right)_{r=r_+}$ and noting eq. (20)

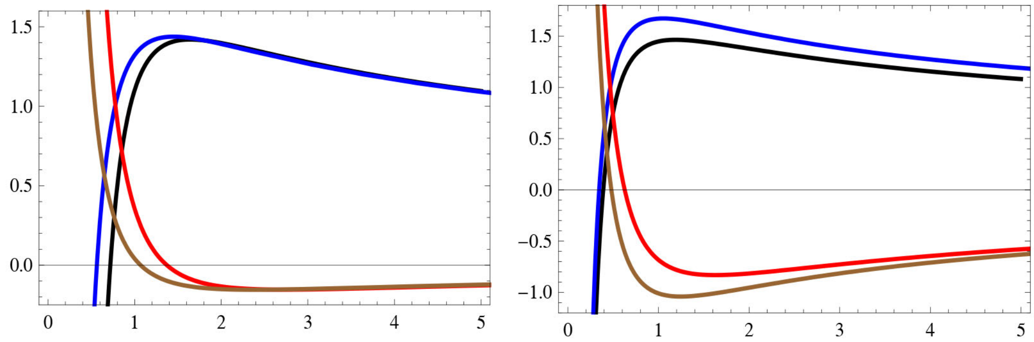

Fig. 5. T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+, for $\ell = 1$, $Q = 0.5$ and $\beta = \alpha = 1$, eqs. (26) and (35). Left: $b = 2.8$, (T, $p = 0.6$, 0.62 for black, blue curves, respectively) and $(2\,(\partial^2 M/\partial S^2)_Q$, $p = 0.6$, 0.62 for red and brown curves, respectively). Right: $p = 0.65$, (T, $b = 2.8$, 3 for black, blue curves, respectively) and $(10\,(\partial^2 M/\partial S^2)_Q$, $b = 2.8$, 3 for red and brown curves, respectively).

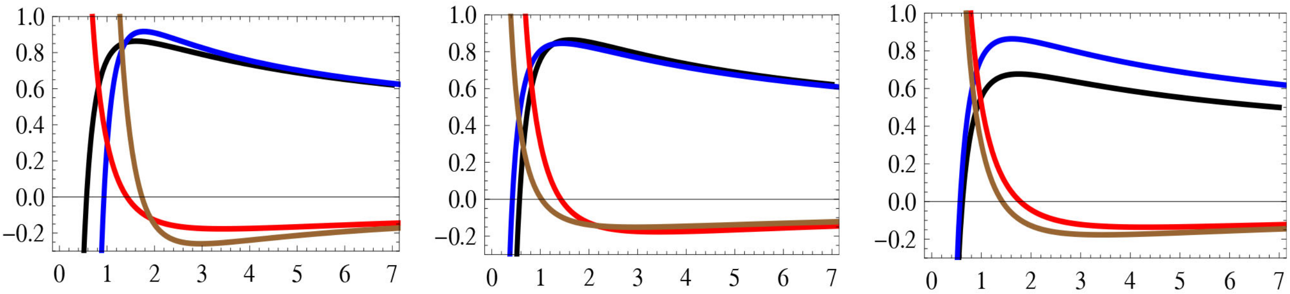

Fig. 6. T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+, for $\ell = 1$, $Q = 0.5$ and $\beta = 1$, $\alpha \neq \beta$, eqs. (26) and (35). Left: $p = 0.8$, $b = 2$, (T, $\alpha = 2.5$, 5 for black, blue curves, respectively) and $(4\,(\partial^2 M/\partial S^2)_Q$, $\alpha = 2.5$, 5 for red and brown curves, respectively). Middle: $b = 2$, $\alpha = 2.5$ (T, $p = 0.8$, 0.9 for black, blue curves, respectively) and $(4\,(\partial^2 M/\partial S^2)_Q$, $p = 0.8$, 0.9 for red and brown curves, respectively). Right: $p = 0.8$, $\alpha = 2.5$ (T, $b = 1.7$, 2 for black, blue curves, respectively) and $(4\,(\partial^2 M/\partial S^2)_Q$, $b = 1.7$, 2 for red and brown curves, respectively).

one can obtain the black hole temperature as follows:

$$T = \begin{cases} \dfrac{3}{2\pi r_+}\left[\dfrac{b^2}{\ell^2}\left(\dfrac{r_+}{b}\right)^{2/3} + \dfrac{q^{2p}(2p-1)}{b^{2(\mathcal{B}-1)}}\,(\alpha p + 2 - 3p)\,\Upsilon(\beta=1)\left(\dfrac{b}{r_+}\right)^{\frac{2(\alpha p+3-5p)}{3(2p-1)}}\right], & \text{for } \beta = 1, \\[2ex] \dfrac{1+2\beta^2}{2\pi r_+}\left[\dfrac{b^2}{\ell^2}\left(\dfrac{b}{r_+}\right)^{4\beta\gamma-2} + \dfrac{q^{2p}(2p-1)^2}{b^{2(B-1)}}\,(2\beta_0\beta - 1 - \beta^2)\,\Upsilon(\beta)\left(\dfrac{b}{r_+}\right)^{4\beta_0\gamma-2}\right], & \text{for } \beta \neq 1. \end{cases} \tag{25}$$

The black hole temperature (25) reduces to that of ref. [12], if one lets $p = 1$. Note that we have used the relation $\Psi(r_+) = 0$ for eliminating the mass parameter m from the obtained equations. Also, it must be noted that extreme black holes occur if q and r_+ are chosen such that $T = 0$. Now, making use of eq. (25) we can obtain the horizon radius of the extreme black holes in the following forms:

$$r_{ext} = \begin{cases} b\left[\dfrac{q^{2p}\ell^2(2p-1)}{b^{2\mathcal{B}}}\Upsilon(\beta=1)\,(3p-\alpha p-2)\right]^{\frac{3(2p-1)}{2(\alpha p+2-3p)}}, & \text{for } \beta = 1, \\[2ex] b\left[\dfrac{q^{2p}\ell^2(2p-1)^2}{b^{2B}}\,(1+\beta^2-2\beta_0\beta)\,\Upsilon(\beta)\right]^{\frac{1}{4(\beta_0-\beta)\gamma}}, & \text{for } \beta \neq 1. \end{cases} \tag{26}$$

In order to investigate the effects of dilatonic and nonlinearity parameters (*i.e.* α, β, b and p) on the horizon temperature of the black holes, the plots of black hole temperature *versus* horizon radius are shown in figs. 5–8 for both of $\alpha = \beta$ and $\alpha \neq \beta$ cases, separately. They show that, for the properly fixed parameters, the physical black holes with positive temperature are those for which $r_+ > r_{ext}$ and un-physical black holes, having negative temperature, occur if $r_+ < r_{ext}$.

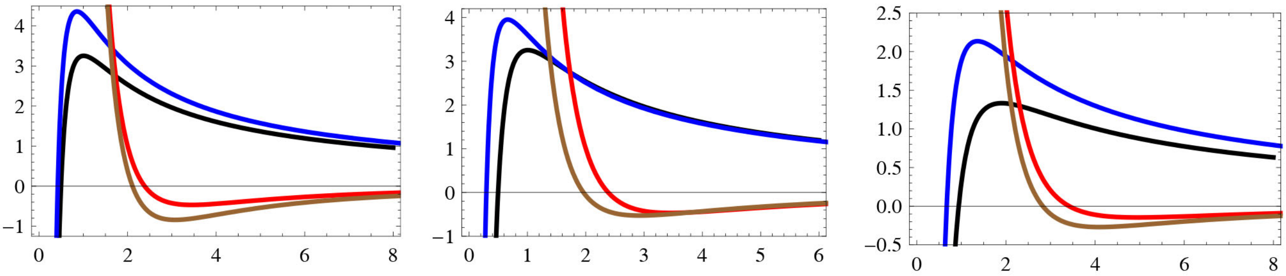

Fig. 7. T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+, for $\ell = 1$, $Q = 0.5$ and $\beta \neq 1$, $\alpha = \beta$, eqs. (26) and (35). Left: $p = 0.6$, $b = 2$, (T, $\alpha = 2$, 2.2 for black, blue curves, respectively) and ($(\partial^2 M/\partial S^2)_Q$, $\alpha = 2$, 2.2 for red and brown curves, respectively). Middle: $b = 2$, $\alpha = 2$ (T, $p = 0.6$, 0.62 for black, blue curves, respectively) and ($(\partial^2 M/\partial S^2)_Q$, $p = 0.6$, 0.62 for red and brown curves, respectively). Right: $p = 0.6$, $\alpha = 2$ (T, $b = 1.6$, 1.8 for black, blue curves, respectively) and ($(\partial^2 M/\partial S^2)_Q$, $b = 1.6$, 1.8 for red and brown curves, respectively).

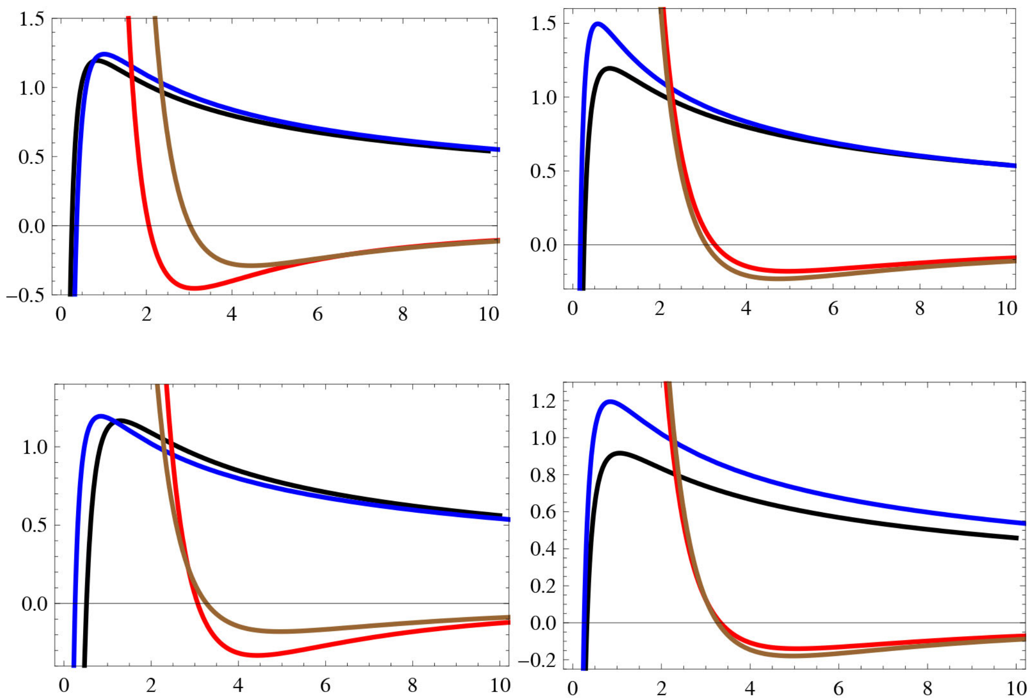

Fig. 8. T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+, for $\ell = 1$, $Q = 0.5$ and $\beta \neq 1$, $\alpha \neq \beta$, eqs. (26) and (35). Top left: $p = 0.8$, $b = 2$, $\beta = 1.2$ ($0.1T$, $\alpha = 2$, 2.5 for black, blue curves, respectively) and ($10\,(\partial^2 M/\partial S^2)_Q$, $\alpha = 2$, 2.5 for red and brown curves, respectively). Top right: $p = 0.8$, $b = 2$, $\alpha = 2$ (T, $\beta = 1.2$, 1.3 for black, blue curves, respectively) and ($10\,(\partial^2 M/\partial S^2)_Q$, $\beta = 1.2$, 1.3 for red and brown curves, respectively). Bottom left: $b = 2$, $\alpha = 2$, $\beta = 1.2$ (T, $p = 0.8$, 0.7 for black, blue curves, respectively) and ($10\,(\partial^2 M/\partial S^2)_Q$, $p = 0.8$, 0.7 for red and brown curves, respectively). Bottom right: $p = 0.8$, $\alpha = 2$, $\beta = 1.2$ (T, $b = 1.8$, 2 for black, blue curves, respectively) and ($10\,(\partial^2 M/\partial S^2)_Q$, $b = 1.8$, 2 for red and brown curves, respectively).

The electric potential Φ of black holes, measured by an observer located at infinity with respect to the horizon, can be calculated making use of the following standard relation [51–58]:

$$\Phi = A_\mu \chi^\mu|_{\text{reference}} - A_\mu \chi^\mu|_{r=r_+}, \tag{27}$$

where $\chi = C\partial_t$ is the null generator of the horizon and C is an arbitrary constant to be determined [45–47]. Noting eqs. (14) and (27) we obtained the black hole's electric potential on the horizon as

$$\Phi = \begin{cases} \dfrac{3Cq(2p-1)}{2(\alpha p - 3p + 2)} r_+^{1-\frac{2\alpha p+1}{3(2p-1)}}, & \text{for } \beta = 1, \\[2ex] \dfrac{Cq(2p-1)}{A+2-2p} r_+^{1-\frac{A+1}{2p-1}}, & \text{for } \beta \neq 1. \end{cases} \tag{28}$$

The conserved electric charge of the black holes can be obtained by calculating the total electric flux measured by an observer located at infinity with respect to the horizon (*i.e.*, $r \to \infty$) [52–58]. With this issue in mind and making use of eq. (14) together with the help of Gauss's law, after some simple calculations, we arrived at

$$Q = \begin{cases} p\, 2^{p-2} q^{2p-1}\, b^{\frac{2}{3}(1-\alpha p)}, & \text{for } \beta = 1, \\ \dfrac{p\, 2^{p-2}}{b^{A}} q^{2p-1}, & \text{for } \beta \neq 1, \end{cases} \tag{29}$$

which reduces to that of charged BTZ black holes in the absence of the dilatonic field. Also, it is compatible with the results of our previous work in the case $p = 1$ [12].

The other conserved quantity to be calculated is the black hole mass. As mentioned before, it can be obtained in terms of the mass parameter m. It is a matter of calculation to show that the total mass of the new nonlinearly charged dilatonic BTZ black holes is [11,12,59,60]

$$M = \begin{cases} \dfrac{m}{24} b^{2/3}, & \text{for } \beta = 1, \\ \dfrac{m\,\gamma}{8\beta} b^{2\beta\gamma}, & \text{for } \beta \neq 1, \end{cases} \tag{30}$$

which is compatible with the mass of charged BTZ black holes when the dilatonic potential disappears.

Now, we are in the position to investigate the consistency of these quantities with the thermodynamical first law. From eqs. (20), (24) and (29), we can obtain the black hole mass as the function of extensive parameters S and Q. To do so, we use the relation $\Psi(r_+) = 0$. The Smarr-type mass formula for both of the new black holes can be obtained as

$$M(S,Q) = \begin{cases} \dfrac{1}{4}\left[\dfrac{b^2}{\ell^2}\ln\left(\dfrac{r_+(S)}{L}\right) - \dfrac{3q^{2p}(2p-1)^2}{2b^{2(\mathcal{B}-1)}}\Upsilon(\beta=1)\left(\dfrac{b}{r_+(S)}\right)^{\frac{2(\alpha p+3-5p)}{3(2p-1)}}\right], & \text{for } \beta = 1, \\ -\dfrac{\beta}{8\gamma}\left[\dfrac{\Lambda b^2}{1-\beta^2}\left(\dfrac{b}{r_+(S)}\right)^{4\beta\gamma-2} + \dfrac{q^{2p}(2p-1)^2}{b^{2(B-1)}}\Upsilon(\beta)\left(\dfrac{b}{r_+(S)}\right)^{4\beta_0\gamma-2}\right], & \text{for } \beta \neq 1. \end{cases} \tag{31}$$

It is a matter of calculation to show that the intensive parameters T and Φ, conjugate to the black hole entropy and charge, satisfy the following relations

$$\left(\frac{\partial M}{\partial S}\right)_Q = T \qquad \text{and} \qquad \left(\frac{\partial M}{\partial Q}\right)_S = \Phi, \tag{32}$$

provided that C be chosen as [45–47]

$$C = -2^{-p}(1+2\beta^2)(A+2-2p)\Upsilon. \tag{33}$$

Therefore, we proved that the first law of black hole thermodynamics is valid, for both classes of the new nonlinearly charged dilatonic BTZ black holes, in the following form:

$$\mathrm{d}M(S,Q) = T\mathrm{d}S + \Phi\mathrm{d}Q. \tag{34}$$

Here, S and Q are known as the thermodynamical extensive parameters and T and Φ are intensive parameters conjugated to S and Q, respectively. Equation (34) shows that, although the conserved and thermodynamical quantities are affected by dilaton fields, the first law of black hole thermodynamics remains valid for the black hole solutions, obtained here.

4 Black hole solutions in the canonical ensemble

In this section, we investigate the thermal stability or phase transition of our new the black hole solutions, making use of the canonical ensemble method. To do so, we need to calculate the black hole heat capacity with the black hole charge as a constant. It is defined in the following form:

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = T\left(\frac{\partial^2 M}{\partial S^2}\right)_Q^{-1}, \tag{35}$$

where, the last step in eq. (35) comes from the fact that $T = (\partial M/\partial S)_Q$. From the thermodynamical point of view, the positivity of the black hole heat capacity C_Q or equivalently the positivity of $(\partial S/\partial T)_Q$ or $(\partial^2 M/\partial S^2)_Q$ is sufficient to ensure the local stability of the physical black holes. The unstable black holes undergo phase transitions to be stabilized. Type one phase transition takes place at the points where the black hole heat capacity vanishes. On the other hand, an unstable black hole undergoes type two phase transition at the divergent points of the black hole heat capacity, where the denominator of the heat capacity vanishes [51–58]. Regarding the above-mentioned points, we proceed to perform a thermal stability or phase transition analysis for both of the new black hole solutions we just obtained.

Making use of eq. (31), the denominator of the black hole heat capacity can be calculated as

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q = \begin{cases} \dfrac{-2}{\pi^2\ell^2}\left(\dfrac{b}{r_+}\right)^{\frac{2}{3}}\left[1-\dfrac{4p-2\alpha p-3}{(\alpha p+2-3p)^{-1}}\dfrac{\ell^2 q^{2p}}{b^{2\mathcal{B}}}\Upsilon(\beta=1)\left(\dfrac{b}{r_+}\right)^{\frac{2(\alpha p+2-3p)}{3(2p-1)}}\right], & \text{for } \beta=1, \\ \dfrac{1-4\beta^4}{\pi^2\ell^2}\left(\dfrac{b}{r_+}\right)^{2\beta\gamma}\left[1-\dfrac{\ell^2 q^{2p}(2p-1)^2}{(1-2\beta^2)b^{2B}}\dfrac{4\beta_0\beta-1-2\beta^2}{(2\beta_0\beta-1-\beta^2)^{-1}}\Upsilon(\beta)\left(\dfrac{b}{r_+}\right)^{4(\beta_0-\beta)\gamma}\right], & \text{for } \beta\neq 1. \end{cases} \tag{36}$$

It is understood from eq. (36) that the denominator of the black hole heat capacity vanishes at the point

$$r_+ \equiv r_0 = b\left[\frac{4p-2\alpha p-3}{(\alpha p+2-3p)^{-1}}\frac{\ell^2 q^{2p}}{b^{2\mathcal{B}}}\Upsilon(\beta=1)\right]^{\frac{3(2p-1)}{2(\alpha p+2-3p)}}, \quad \text{for } \beta=1. \tag{37}$$

The plots of T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+, in terms of different values of dilatonic and nonlinearity parameters, are shown in figs. 5 and 6 for $\alpha=\beta=1$ and $\alpha\neq\beta=1$ cases, respectively. They show that the heat capacity of the black holes with the size satisfying the condition given by eq. (37), diverges and they undergo type two phase transition. Also, the black hole heat capacity vanishes at the $r_+ = r_{ext}$ (eq. (26)) and the type one phase transition takes place. In addition, for the black holes with the horizon radius in the range $r_{ext} < r_+ < r_0$ both the temperature and the denominator of the heat capacity are positive. As a result, black holes with the horizon radius in this range are locally stable.

On the other hand, the nonlinearly charged dilatonic BTZ black holes are unstable and undergo type two phase transition at the real root of eq. (36), which is located at

$$r_+ \equiv r_1 = b\left[\frac{\ell^2 q^{2p}(2p-1)^2}{(1-2\beta^2)b^{2B}}\frac{4\beta_0\beta-1-2\beta^2}{(2\beta_0\beta-1-\beta^2)^{-1}}\Upsilon(\beta)\right]^{\frac{1}{4(\beta_0-\beta)\gamma}}, \quad \text{for } \beta\neq 1. \tag{38}$$

Also, type one phase transition takes place at the point $r_+ = r_{ext}$, given by eq. (26), where the black hole heat capacity vanishes. The plots of T and $(\partial^2 M/\partial S^2)_Q$ *versus* r_+ are shown in figs. 7 and 8 for $\alpha=\beta\neq 1$ and $\alpha\neq\beta\neq 1$ cases, respectively. They show that the black holes with the horizon radius in the range $r_{ext} < r_+ < r_1$ have positive heat capacity and are thermally stable.

All of the plots shown in figs. 5–8 correspond to the case in which both r_{ext} and $r_0\,(r_1)$ exist, simultaneously. A notable point, which is not shown in figs. 5–8, is that there is an interesting case corresponding to the especial choice of the dilatonic and nonlinearity parameters for which r_{ext} exist but the denominator of the black hole heat capacity is positive and $r_0\,(r_1)$ does not. In this case no type two phase transition takes place. The $r_+ = r_{ext}$ is the only point of type one phase transition and both of the new black hole solutions are stable for $r_+ > r_{ext}$.

5 Conclusion

In this work we have investigated the new nonlinearly charged dilatonic BTZ black hole solutions, as the exact solutions to the to the field equations of the Einstein-power-Maxwell-dilaton gravity theory. By varying the proper action of the theory, we obtained the explicit form of the coupled scalar, electromagnetic and gravitational field equations. By introducing a static and spherically symmetric geometry, we found that the solution of the scalar field equation can be written in the form of a generalized Liouville dilatonic potential. Also, two new classes of charged dilatonic BTZ black hole solutions, as the exact solutions to the gravitational field equations, have been obtained in the presence of the power-Maxwell invariant as the nonlinear theory of electrodynamics. Regarding the Ricci and Kretschmann scalars, we found that there is a point of essential singularity located at the origin. Also, the asymptotic behavior of the solutions is neither flat nor AdS. The existence of the real roots of the metric functions together with the singular Ricci scalars is in favor of the black hole interpretation of the solutions. As it is shown in figs. 1–4, for both of the new black hole solutions the two horizon, extreme and naked singularity black holes can occur, if the parameters of the theory are fixed suitably.

Next, we studied the thermodynamics of the new black hole solutions. We have obtained the conserved charge and mass of the black holes. Also, by using the geometrical methods, we have calculated the temperature, entropy and electric potential for both of the new black hole solutions. We showed that the extreme, physical and un-physical black holes can occur if $r_+ = r_{ext}$, $r_+ > r_{ext}$ and $r_+ < r_{ext}$, respectively (eq. (26)). Through a Smarr-type mass formula, we have obtained the black hole mass as the function of the thermodynamical extensive parameters S and Q, from which we have obtained the intensive parameters T and Φ. Compatibility of the results obtained from thermodynamical and geometrical approaches proves the validity of the thermodynamical first law for both of the new black hole solutions.

At the final stage, from the canonical ensemble point of view, we have analyzed the thermal stability or phase transition for both of the new black hole solutions. Regarding the black hole heat capacity, with the black hole charge as a constant, we found that two following possibilities are considerable, separately. i) If the dilatonic and nonlinearity parameters are fixed such that both r_{ext} and $r_0(r_1)$ exist simultaneously, the black holes undergo type one phase transition at $r_+ = r_{ext}$ where the black hole heat capacity vanishes. There is a point of type two phase transition located at $r_+ = r_0(r_1)$ at which the black hole heat capacity diverges. The physical black holes with the horizon radius in the range $r_{ext} < r_+ < r_0(r_1)$ are locally stable (figs. 5–8). ii) If the parameters are chosen such that r_{ext} exist but the denominator of the heat capacity is positive everywhere and $r_0(r_1)$ does not exist, there is no type two phase transition. The unstable black holes undergo type one phase transition at $r_+ = r_{ext}$ and the physical black holes with the horizon radius in the range $r_+ > r_{ext}$ are locally stable. Note that this case is not shown in the figures.

Studies of the dynamical stability and finding the quasi-normal modes of these novel dilatonic charged BTZ black holes are interesting subjects for future works.

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