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A mathematical analysis of a circular pipe in rate type fluid via Hankel transform

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Abstract. In this paper, helices of a generalized Oldroyd-B fluid have been analyzed through a horizontal circular pipe. The circular pipe is taken in the form of a circular cylinder. The analytical solutions are determined for velocities and shear stresses due to the unsteady helical flow of a generalized Oldroyd-B fluid. The general solutions are derived by using finite Hankel and discrete Laplace transforms to satisfy the imposed conditions and the governing equations. The special cases of our general solutions are also perused performing the same motion for fractional and ordinary Maxwell fluid, fractional and ordinary second-grade fluid and fractional and ordinary viscous fluid as well. The graphical illustration is depicted in order to explore how the two velocities and shear stresses profiles are impacted by different rheological parameters, for instance, fractional parameter, relaxation time, retardation time, material non-zero constant, dynamic viscosity and few others. Finally, ordinary and fractional operators have various similarities and differences on a circular pipe for helicoidal behavior of fluid flow.

1 Introduction

The flow of non-Newtonian fluids has been an active area of research for decades. Many non-Newtonian fluids comprise liquids such as suspension and colloidal fluids, shampoos, cosmetic products, ice cream, polymers, certain oils, paints and various others. Due to the complex characteristics and rheological properties of non-Newtonian fluids, they are classified in three main categories which include: i) rate type fluids (Burgers model, Oldroyd model, Maxwell model and Jeffery model); ii) differential type fluids (second as well as third-grade model); iii) integral type fluids (Sisko model). The Oldroyd-B fluid model is suggested for the prediction and forecasting of relaxation and retardation impacts simultaneously. Due to cylindrical geometry, the mathematical modeling of the Oldroyd-B fluid model is not an easy task for researchers. The Oldroyd-B fluid model in cylindrical geometry is rarely reported in the literature. For instance, Taylor [1] investigated the effects of a Newtonian fluid with two rotating circular cylinders for checking stability. Waters and King [2] studied the straight pipe of a circular cross-section for the unsteady flow of an elasto-viscous fluid. Rahaman and Ramkissoon [3] determined the closed form solution using the Fourier Bessel series for the unsteady flow of upper convected viscoelastic fluid in circular pipe. Hayat *et al.* [4] studied some basic flows for the Oldroyd-B fluid namely i) the non-periodic flows between two boundaries, ii) the symmetric flow with an arbitrary initial velocity, iii) the time-periodic Poiseuille flow due to an oscillating pressure gradient, iv) the Stokes problem and v) the modified Stokes problem and found exact solutions. Hayat *et al.* [5] analyzed the flows for cylindrical geometries for a second-grade fluid and obtained analytical solutions by employing integral transforms. Qi *et al.* [6] examined the generalized Oldroyd-B fluid model for Stokes' first problem. Fetecau *et al.* [7] explored the time-dependent rotation for the Taylor-Couette flow of an Oldroyd-B in cylindrical configuration. Fetecau *et al.* [8] studied the time-dependent shear stress in an Oldroyd-B fluid for axially Couette flow. Siddique *et al.* [9] found the exact solution for a generalized second-grade fluid for the rotational flow in cylindrical configuration. Fetecau *et al.* [10], examined the oscillating motions for an Oldroyd-B fluid in a cylindrical domain and established starting solutions using integral transforms

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techniques. Jamil *et al.* [11] investigated the flow of coaxial cylinders helicoidally for a viscoelastic fluid subject to given shear stresses on the boundary. Mahmood *et al.* [12] obtained exact solutions for a fractionalized non-Newtonian fluid flow in a cylindrical domain. Haitao *et al.* [13] analyzed the flows of a generalized fractional Oldroyd-B fluid for helecity using integral transforms and investigated analytical solutions. Imran *et al.* [14] worked on moving coaxial circular cylinders for the unsteady flow of a fractionalized viscoelastic fluid. Kamran *et al.* [15] investigated a fractionalized Oldroyd-B fluid for an unsteady rotational flow in a cylinder. Chunrui *et al.* [16] studied the effects of a heated generalized Oldroyd-B fluid in a porous medium with time-dependent shear stress. They calculated exact solutions from a fractionalized governing equation using Hankel and Laplace transforms. Of course, a number of research papers have been published in a similar domain, few of them we include here from [17–23].

Motivated from the above research work, our aim is to analyze the helices of a generalized (fractional) Oldroyd-B fluid through cylindrical configurations. The analytical solutions are carried out for velocities and shear stresses due to unsteady helical motion of a fractionalized Oldroyd-B fluid. The general solutions are investigated by implementing the finite Hankel and discrete Laplace transform fulfilling the imposed conditions on the rotational and accelerating behavior of a circular cylinder. Similar solutions have also been established by converting governing partial differential equations to ordinary differential equations. The special cases of our general solutions are also perused performing the same motion for fractional and ordinary Maxwell fluid, fractional and ordinary second-grade fluid and fractional and ordinary viscous fluid. Finally, graphical illustrations are depicted in order to explore how the two velocities and shear stresses' profiles are impacted by different rheological parameters, for instance, fractional parameter, relaxation time, retardation time, material non-zero constant, dynamic viscosity and few others.

It should be noted that factional calculus is the subject of the science of differentiation and it was firstly originated in 1695 by L'Hopital. Literature reveals the existence of various fractional operators in which the definitions of fractional operator along with significant properties have been studied by several mathematicians and scientists, for instance, the Riemann-Liouville fractional derivative, theLiouville-Caputo fractional derivative, the Caputo-Erdelyi-Kober fractional derivative, the Caputo-Hadamard fractional derivative, the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives [24–26]. The Riemann-Liouville and Liouville-Caputo fractional derivatives involve the convolution of the local derivative of a given function with the power-law function. Caputo and Fabrizio presented new definitions of fractional operators without a singular kernel based in the decay exponential law, and recently, Atangana and Baleanu presented another interesting operator based on the non-singular and non-local generalized stretched Mittag-Leffler function. Interesting applications of these fractional operators can be found in [27–35].

Moreover, the literature includes discussions of many generalized fractional differential equations, fractional integro-differential and fractional difference equations related to models in the fields of fluid mechanics, viscoelasticity, electromagnetic, acoustics, chemistry, biology, physics, neuron modeling and material sciences; for more details, see [36–42] and the references therein.

2 Governing equations with the helices of a cylinder

Assume the velocity field in the form of U and the extra stress in the form S are given by [43]

$$U = u_1(r,t)e_\theta + u_2(r,t)e_z, \qquad S = S(r,t), \tag{1}$$

where (r,θ,z) are the cylindrical coordinates and e_θ, e_z are unit vectors in the direction of θ and z. For this type of flows, the constraint of incompressibility is inevitably fulfilled. At start, the fluid is in rest position (up to the moment $t=0$)

$$U(r,0) = 0, \qquad S(r,0) = 0. \tag{2}$$

The meaningful equations, for such motions of the Oldroyd-B fluid due to the rotational symmetry, are given by [43–45]

$$\frac{\partial u_1(r,t)}{\partial t}\left[\alpha\frac{\partial}{\partial t}+1\right] - v\left[\frac{\partial^2}{\partial t^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\left[\alpha_r\frac{\partial}{\partial t}+1\right]u_1(r,t) = 0, \tag{3}$$

$$\frac{\partial u_2(r,t)}{\partial t}\left[\alpha\frac{\partial}{\partial t}+1\right] - v\left[\frac{\partial^2}{\partial t^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\alpha_r\frac{\partial}{\partial t}+1\right]u_2(r,t) = 0, \tag{4}$$

$$\left[\alpha\frac{\partial}{\partial t}+1\right]\tau_1(r,t) - \mu\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\left[\alpha_r\frac{\partial}{\partial t}+1\right]u_1(r,t) = 0, \tag{5}$$

$$\left[\alpha\frac{\partial}{\partial t}+1\right]\tau_2(r,t) - \mu\left[\alpha_r\frac{\partial}{\partial t}+1\right]\frac{\partial}{\partial r}u_2(r,t) = 0. \tag{6}$$

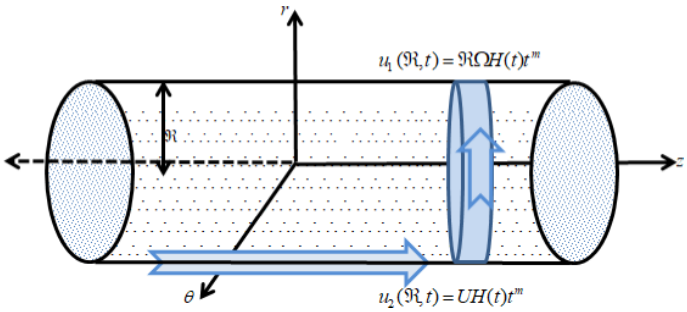

Fig. 1. Geometry of the problem.

The partial differential equations corresponding to an incompressible Liouville-Caputo fractionalized Oldroyd-B fluid performing the same motion are

$$\frac{\partial u_1(r,t)}{\partial t}\left[\alpha\frac{\partial^\eta}{\partial t^\eta}+1\right]-v\left[\frac{\partial^2}{\partial t^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\left[\alpha_r\frac{\partial^\eta}{\partial t^\eta}+1\right]u_1(r,t)=0, \tag{7}$$

$$\frac{\partial u_2(r,t)}{\partial t}\left[\alpha\frac{\partial^\eta}{\partial t^\eta}+1\right]-v\left[\frac{\partial^2}{\partial t^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\alpha_r\frac{\partial^\eta}{\partial t^\eta}+1\right]u_2(r,t)=0, \tag{8}$$

$$\left[\alpha\frac{\partial^\eta}{\partial t^\eta}+1\right]\tau_1(r,t)-\mu\left[\frac{\partial}{\partial r}-\frac{1}{r}\right]\left[\alpha_r\frac{\partial^\eta}{\partial t^\eta}+1\right]u_1(r,t)=0, \tag{9}$$

$$\left[\alpha\frac{\partial^\eta}{\partial t^\eta}+1\right]\tau_2(r,t)-\mu\left[\alpha_r\frac{\partial^\eta}{\partial t^\eta}+1\right]\frac{\partial}{\partial r}u_2(r,t)=0, \tag{10}$$

where η is the fractional parameter and the fractional differential operator, the so-called Liouville-Caputo fractional operator, $\frac{\partial^\eta}{\partial t^\eta}$ is defined by [46,47]

$$\frac{\partial^\eta}{\partial t^\eta}g(t)=\begin{cases}\dfrac{1}{\Gamma(1-\eta)}\displaystyle\int_0^t\frac{g'(\epsilon)}{(t-\epsilon)^\eta}\mathrm{d}\epsilon, & 0<\eta<1,\\ \dfrac{\partial}{\partial t}g(t) & \eta=1.\end{cases} \tag{11}$$

We consider an incompressible fractional Oldroyd-B fluid at rest in a helical cylinder of radius R. At time $t=0^+$, the helical cylinder starts to rotate about its axis $r=0$ having Ωt^m (angular velocity) and sliding due to linear velocity Ut^m along with same axis. The fluid is gradually moved due to shear. Its velocity being of eq. (1), the eqs. (7)–(10) are governing partial differential equations. Such a fluid is typically called a helical fluid, in light of this fact; such streamlines are called helices which can be observed from the geometry of flow. The suitable conditions (initial and boundary conditions) are

$$\frac{\partial}{\partial t}u_1(r,0)=\frac{\partial}{\partial t}u_2(r,0)=u_1(r,0)=u_2(r,0)=\tau_1(r,0)=\tau_2(r,0)=0, \tag{12}$$

$$u_1(R,t)=R\Omega H(t)t^m,$$
$$u_2(R,t)=UH(t)t^m. \tag{13}$$

Figure 1 shows the geometry of the problem.

3 Calculation of problem

3.1 Analytical solutions of velocity field

In order to investigate the solution of the fractional partial differential equations (7)-(8), we apply the discrete Laplace transform and using initial and boundary conditions (12)-(13), we arrive at

$$\frac{1+\alpha\xi^\eta}{v(1+\alpha_r\xi^\eta)}\bar{u}_1(r,\xi)=\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\bar{u}_1(r,\xi), \tag{14}$$

$$\frac{1+\alpha\xi^\eta}{v(1+\alpha_r\xi^\eta)}\bar{u}_2(r,\xi)=\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\bar{u}_2(r,\xi), \tag{15}$$

where $\bar{u}_1(r,\xi)$ and $\bar{u}_2(r,\xi)$ are the Laplace transforms of $u_1(r,t)$ and $u_2(r,t)$; ξ is the Laplace transform parameter having to fulfill the following conditions:

$$u_1(R,\xi) = \frac{R\Omega\Gamma(m+1)}{\xi^{m+1}},$$
$$u_2(R,\xi) = \frac{U\Gamma(m+1)}{\xi^{m+1}}. \tag{16}$$

Using the Hankel transform on eqs. (14)-(15) with eq. (16) and keeping in mind eqs. (A.1)–(A.4) given in the appendix, we arrive at

$$\left[vr_m^2 + \frac{1+\alpha\xi^\eta}{1+\alpha_r\xi^\eta}\right]\bar{u}_{1H}(r_m,\xi) = \frac{vr_m J_2(Rr_m)R^2\Omega\Gamma(m+1)}{\xi^{m+1}}, \tag{17}$$
$$\left[vr_n^2 + \frac{1+\alpha\xi^\eta}{1+\alpha_r\xi^\eta}\right]\bar{u}_{2H}(r_n,\xi) = \frac{vr_n J_2(Rr_n)RU\Gamma(m+1)}{\xi^{m+1}}. \tag{18}$$

The suitable presentation equations (17)-(18) contain equivalent forms as

$$\bar{u}_{1H}(r_m,\xi) = \frac{J_2(Rr_m)\Gamma(m+1)R^2\Omega}{r_m\xi^{m+1}} + \frac{J_2(Rr_m)\Gamma(m+1)R^2\Omega[\xi^n(\alpha_r vr_m-\xi_1)+(vr_m^2-\xi_1\xi_2)]}{vr_m\xi_1\xi^{m+1}(\xi^n+\xi_2)}, \tag{19}$$
$$\bar{u}_{2H}(r_n,\xi) = \frac{J_1(Rr_m)\Gamma(m+1)RU}{r_n\xi^{m+1}} + \frac{J_1(Rr_n)\Gamma(m+1)RU[\xi^n(\alpha_r vr_n-\xi_3)+(vr_n^2-\xi_3\xi_4)]}{vr_n\xi_3\xi^{m+1}(\xi^n+\xi_4)}, \tag{20}$$

where $\xi_1=\alpha+\alpha_r vr_m^2$, $\xi_2=\frac{1+vr_m^2}{\xi_1}$, $\xi_3=\alpha+\alpha_r vr_n^2$, $\xi_4=\frac{1+vr_n^2}{\xi_3}$.

Applying the inverse Hankel transform to eqs. (19)-(20) and using eqs. (A.5)-(A.6) given in the appendix, we obtain

$$\bar{u}_1(r,\xi) = \frac{r\Gamma(m+1)\Omega}{\xi^{m+1}} + \sum_{m=1}^{\infty}\frac{J_1(rr_m)}{r_mJ_2(Rr_m)}\frac{2\Gamma(m+1)\Omega}{\xi_1}\left[\frac{(\alpha_r vr_m-\xi_1)\xi^{\eta-m}}{\xi(\xi^\eta+\xi_2)} + \frac{(vr_m-\xi_1\xi_2)\xi^m}{\xi(\xi^\eta+\xi_2)}\right], \tag{21}$$
$$\bar{u}_2(r,\xi) = \frac{\Gamma(m+1)U}{\xi^{m+1}} + \sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\frac{2\Gamma(m+1)U}{R\xi_3}\left[\frac{(\alpha_r vr_m-\xi_3)\xi^{\eta-m}}{\xi(\xi^\eta+\xi_4)} + \frac{(vr_m-\xi_3\xi_4)\xi^m}{\xi(\xi^\eta+\xi_4)}\right], \tag{22}$$

inverting eqs. (21)-(22) by the Laplace transform and using eq. (A.7) given in the appendix, we obtained the solution for velocity field as

$$u_1(r,t) = r\Omega H(t)t^m + \sum_{m=1}^{\infty}\frac{J_1(rr_m)}{r_mJ_2(Rr_m)}\frac{2\Omega H(t)\Gamma(m+1)}{\xi_1}\times\left\{(\alpha_r vr_m-\xi_1)E_{\eta-m}\left[-\xi_2t^{\eta-m}\right]+(vr_m-\xi_1\xi_2)E_m\left[-\xi_2t^m\right]\right\}, \tag{23}$$

$$u_2(r,t) = UH(t)t^m + \sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(Rr_n)}\frac{2U\Gamma(m+1)}{R\xi_3}\times\left\{(\alpha_r vr_n-\xi_3)E_{\eta-m}\left[-\xi_4t^{\eta-m}\right]+(vr_m-\xi_3\xi_4)E_m\left[-\xi_2t^m\right]\right\}. \tag{24}$$

3.2 Analytical solutions of shear stress

By applying the discrete Laplace transform to eqs. (9)-(10) and using initial and boundary conditions (12)-(13), we arrive at

$$\bar{\tau}_1(r,\xi) = \mu\left[\frac{\partial\bar{u}_1(r,\xi)}{\partial r} - \frac{\bar{u}_1(r,\xi)}{r}\right]\frac{\alpha_r\xi^\eta+1}{\alpha\xi^\eta+1}, \tag{25}$$
$$\bar{\tau}_2(r,\xi) = \mu\left[\frac{\alpha_r\xi^\eta+1}{\alpha\xi^\eta+1}\frac{\partial\bar{u}_2(r,\xi)}{\partial r}\right]. \tag{26}$$

Now calculating the following quantities $\frac{\partial \bar{u}_1(r,\xi)}{\partial r} - \frac{\bar{u}_1(r,\xi)}{r}$ and $\frac{\partial \bar{u}_2(r,\xi)}{\partial r}$ and substituting them in eqs. (25)-(26), we get

$$\bar{\tau}_1(r,\xi) = \mu \left\{ \frac{2\Omega\Gamma(m+1)}{\xi_1(1-r)} \sum_{m=1}^{\infty} \frac{J_2(rr_m)}{J_2(Rr_m)} \left[\frac{(\alpha_r v r_m - \xi_1)\xi^{\eta-m}}{\xi(\xi^\eta + \xi_2)} + \frac{(vr_m - \xi_1\xi_2)\xi^m}{\xi(\xi^\eta + \xi_2)} \right] \right\} \frac{\alpha_r \xi^\eta + 1}{\alpha\xi^\eta + 1}, \tag{27}$$

$$\bar{\tau}_2(r,\xi) = \mu \left\{ \frac{2U\Gamma(m+1)}{\xi_3 R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(Rr_n)} \left[\frac{(\alpha_r v r_n - \xi_3)\xi^{\eta-m}}{\xi(\xi^\eta + \xi_4)} + \frac{(vr_m - \xi_3\xi_4)\xi^m}{\xi(\xi^\eta + \xi_4)} \right] \right\} \frac{\alpha_r \xi^\eta + 1}{\alpha\xi^\eta + 1}. \tag{28}$$

Simplifying eqs. (27) and (28), we get

$$\begin{aligned}\bar{\tau}_1(r,\xi) = {} & \frac{2\mu\Omega\Gamma(m+1)}{\xi_1(1-r)} \sum_{k=0}^{\infty} \alpha^k \sum_{m=1}^{\infty} \frac{J_2(rr_m)}{J_2(Rr_m)} \left\{ \frac{(\alpha_r v r_m - \xi_1)\xi^{\eta-m}}{\xi(\xi^\eta + \xi_2)} + \frac{(vr_m - \xi_1\xi_2)\xi^m}{\xi(\xi^\eta + \xi_2)} \right\} \\ & + \frac{2\mu\Omega\Gamma(m+1)\alpha_r}{\xi_1(1-r)} \sum_{k=0}^{\infty} \alpha^k \sum_{m=1}^{\infty} \frac{J_2(rr_m)}{J_2(Rr_m)} \left\{ \frac{(\alpha_r v r_m - \xi_1)\xi^{2\eta-m+\eta k}}{\xi(\xi^\eta + \xi_2)} + \frac{(vr_m - \xi_1\xi_2)\xi^{m+\eta+\eta k}}{\xi(\xi^\eta + \xi_2)} \right\},\end{aligned} \tag{29}$$

$$\begin{aligned}\bar{\tau}_2(r,\xi) = {} & \frac{2U\mu\Gamma(m+1)}{\xi_3 R} \sum_{k=0}^{\infty} \alpha^k \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(Rr_n)} \left\{ \frac{(\alpha_r v r_n - \xi_3)\xi^{\eta-m}}{\xi(\xi^\eta + \xi_4)} + \frac{(vr_m - \xi_3\xi_4)\xi^m}{\xi(\xi^\eta + \xi_4)} \right\} \\ & + \frac{2U\mu\Gamma(m+1)\alpha_r}{\xi_3 R} \sum_{k=0}^{\infty} \alpha^k \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(Rr_n)} \left\{ \frac{(\alpha_r v r_n - \xi_3)\xi^{2\eta-m+\eta k}}{\xi(\xi^\eta + \xi_4)} + \frac{(vr_m - \xi_3\xi_4)\xi^{m+\eta+\eta k}}{\xi(\xi^\eta + \xi_4)} \right\}.\end{aligned} \tag{30}$$

Inverting eqs. (29) and (30) by the Laplace transform and using eq. (A.7) given in the appendix, we get final solutions of shear stress as

$$\begin{aligned}\tau_1(r,t) = {} & \frac{2\mu\Omega\Gamma(m+1)}{\xi_1(1-r)} \sum_{k=0}^{\infty} \alpha^k \sum_{m=1}^{\infty} \frac{J_2(rr_m)}{J_2(Rr_m)} \left\{ (\alpha_r v r_m - \xi_1) E_{\eta-m}\left[-\xi_2 t^{\eta-m}\right] + (vr_m - \xi_1\xi_2) E_m\left[-\xi_2 t^m\right] \right\} \\ & + \frac{2\mu\Omega\Gamma(m+1)\alpha_r}{\xi_1(1-r)} \sum_{k=0}^{\infty} \alpha^k \sum_{m=1}^{\infty} \frac{J_2(rr_m)}{J_2(Rr_m)} \left\{ (\alpha_r v r_m - \xi_1) E_{2\eta-m+\eta k}\left[-\xi_2 t^{2\eta-m+\eta k}\right] \right. \\ & \left. + (vr_m - \xi_1\xi_2) E_{2\eta-m+\eta k}\left[-\xi_2 t^{2\eta-m+\eta k}\right] \right\}.\end{aligned} \tag{31}$$

$$\begin{aligned}\tau_2(r,t) = {} & \frac{2U\mu\Gamma(m+1)}{\xi_3 R} \sum_{k=0}^{\infty} \alpha^k \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(Rr_n)} \left\{ (\alpha_r v r_n - \xi_3) E_{\eta-m}\left[-\xi_4 t^{\eta-m}\right] + (vr_m - \xi_3\xi_4) E_m\left[-\xi_4 t^m\right] \right\} \\ & + \frac{2U\mu\Gamma(m+1)\alpha_r}{\xi_3 R} \sum_{k=0}^{\infty} \alpha^k \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{J_1(Rr_m)} \left\{ (\alpha_r v r_n - \xi_3) E_{2\eta-m+\eta k}\left[-\xi_4 t^{2\eta-m+\eta k}\right] \right. \\ & \left. + (vr_m - \xi_3\xi_4) E_{2\eta-m+\eta k}\left[-\xi_4 t^{2\eta-m+\eta k}\right] \right\}.\end{aligned} \tag{32}$$

The general solutions obtained can be reduced for few limiting solutions in ordinary and fractional forms. Letting $\eta = 1$ into eqs. (23)-(24) and (31)-(32), the solutions can be recovered for the ordinary Oldroyd-B fluid. Making $\alpha_r = 0$ into eqs. (23)-(24) and (31)-(32), the solutions obtained by Jamil *et al.* [43] can be recovered. In continuation, the rotational solutions can be recovered by substituting $\alpha = \alpha_r = 0$ and $\eta = 1$ investigated by Athar *et al.* [48]. The analytical solutions can also be traced out for the first Stokes problem by letting $m = 0$ into eqs. (23)-(24) and (31)-(32).

4 Results and conclusions

Figures 2(a)-(b) represent the behavior of a fluid at different values of time for angular velocity $u_1(y,t)$ and linear velocity $u_2(y,t)$. It is observed that both angular velocity and linear velocity are an increasing function with respect to time.

Figures 3(a)-(b) represents the non-linear velocities over the boundary. It is noted that angular velocity $u_1(y,t)$ has reciprocal behavior to linear velocity $u_2(y,t)$ over the complete domain.

The effects of viscosity for angular velocity $u_1(y,t)$ and linear velocity $u_2(y,t)$ are demonstrated in figs. 4(a)-(b). It is pointed out that both types of velocities have oscillating behavior of fluid flow. This may be due to the fact that the streamlines are helices in the cylinder.

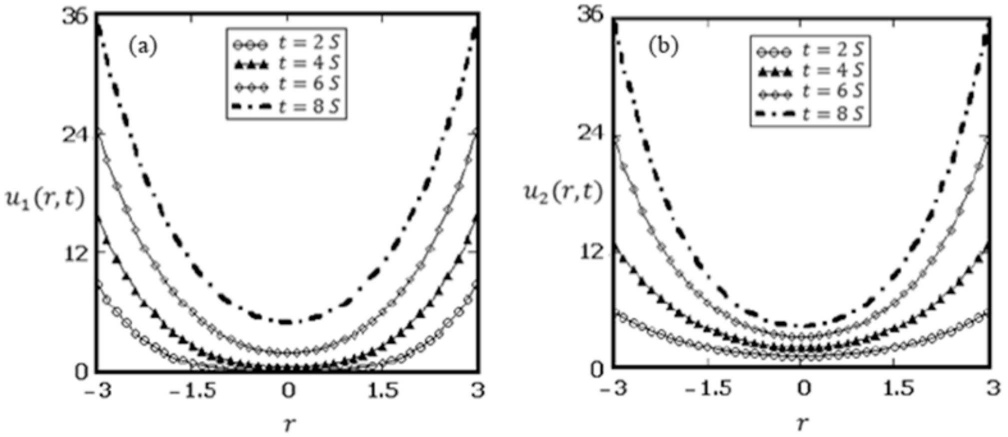

Fig. 2. Profile of velocity field for different values of t.

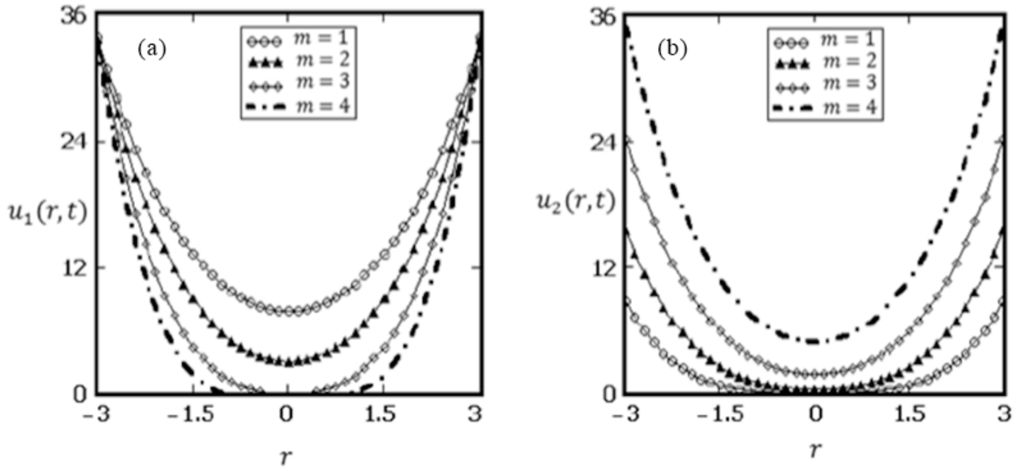

Fig. 3. Profile of velocity field for different values of m.

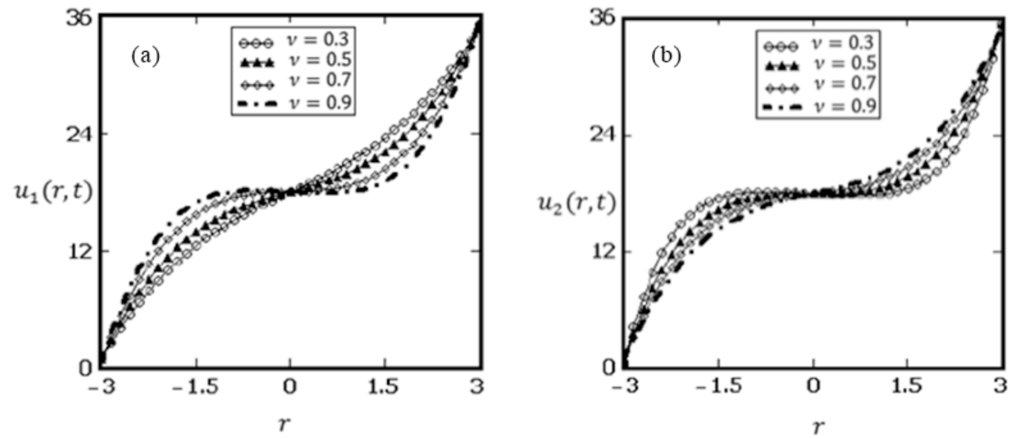

Fig. 4. Profile of velocity field for different values of v.

The relaxation and retardation time are depicted in figs. 5(a)-(b) and 6(a)-(b). It is observed that both parameters have an oscillating fluid motion. It is also noted that the relaxation and retardation time parameters have scattering and sequestrating effects on the whole boundary.

Figures 7(a)-(b) are drawn for angular velocity $u_1(y,t)$ for four types of ordinary and fractional models: i) ordinary and fractional Oldroyd-B fluid, ii) ordinary and fractional Maxwell fluid, iii) ordinary and fractional second-grade fluid and iv) ordinary and fractional Newtonian fluid. It is observed that fractional models have smattering behavior of fluid

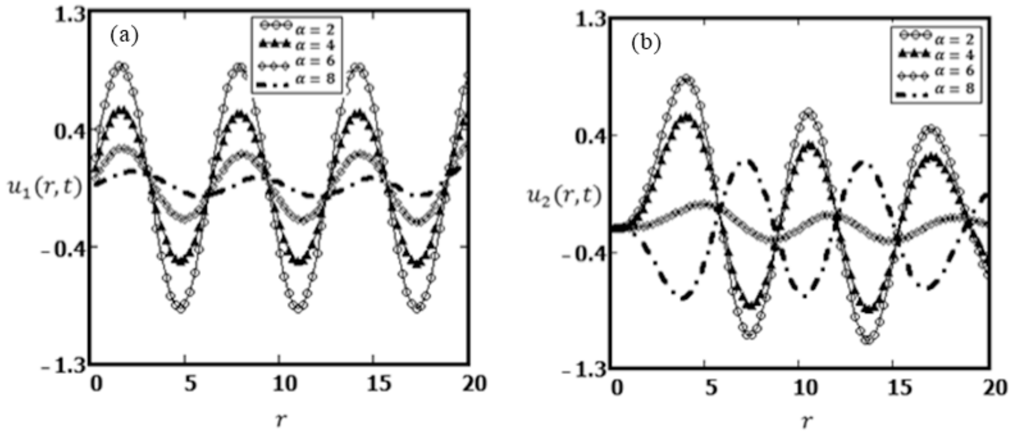

Fig. 5. Profile of velocity field for different values of α.

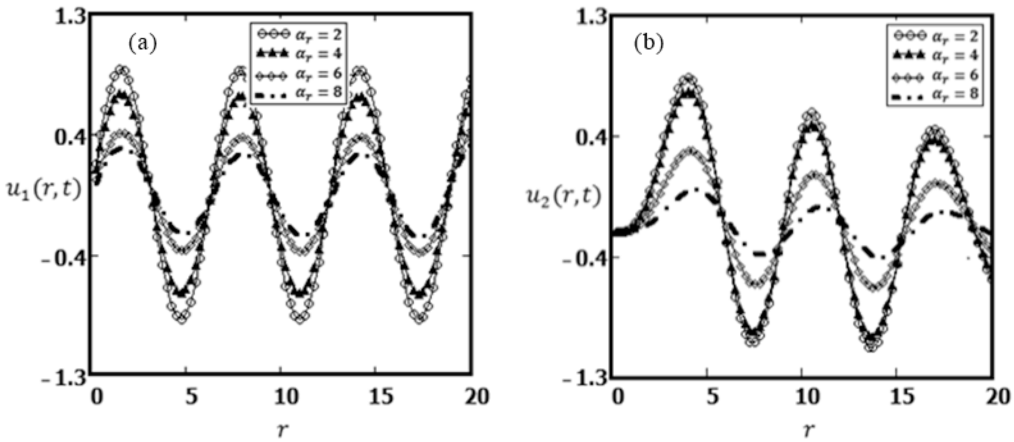

Fig. 6. Profile of velocity field for different values of α_r.

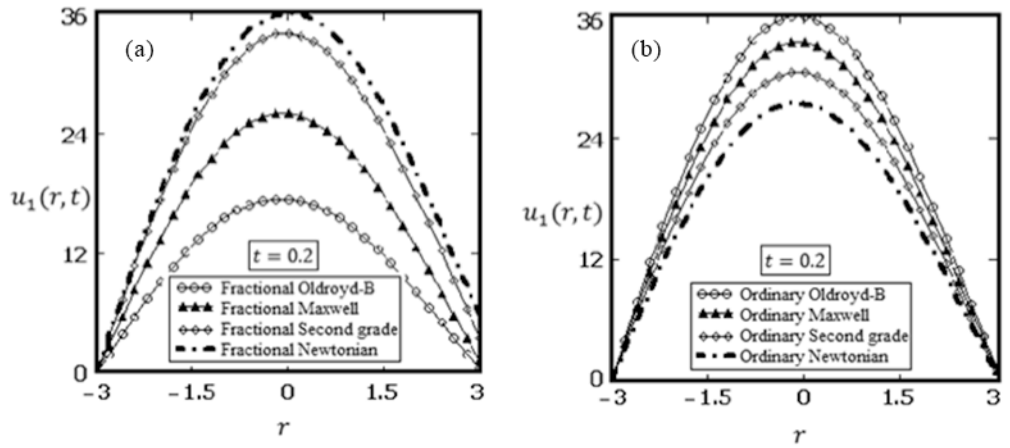

Fig. 7. Comparison of velocity field for fractional and ordinary models of fluids for smaller time $t = 0.2\,\mathrm{s}$.

and ordinary models have sequestrating behavior of fluid. On the contrary, figs. 8(a)-(b) is drawn for linear velocity $u_2(y,t)$ in which similar effects are noted. It is also worth pointing out that both angular velocity $u_1(y,t)$ and linear velocity $u_2(y,t)$ have reciprocal behavior either in fractional or ordinary models.

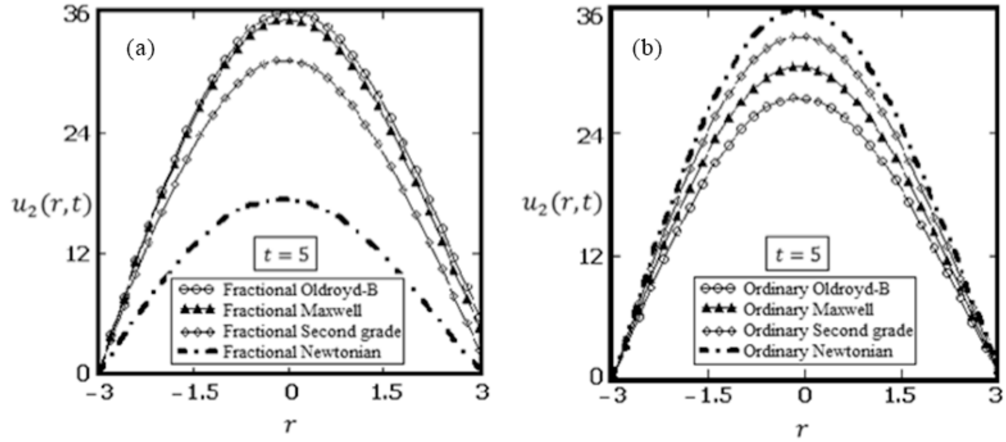

Fig. 8. Comparison of velocity field for fractional and ordinary models of fluids for smaller time $t = 5\,\mathrm{s}$.

5 Concluding remarks

In this work, the effects of helices on a generalized Oldroyd-B fluid have been analyzed through a horizontal circular pipe. The analytical solutions are determined for velocities and shear stresses due to unsteady helical flow of a fractionalized Oldroyd-B fluid. The general solutions are investigated by implementing the finite Hankel and discrete Laplace transform satisfying the imposed conditions on rotational and accelerating behavior of a circular cylinder. The similar solutions have also been established by converting governing partial differential equations to ordinary differential equations. The special cases of our general solutions are also perused performing the same motion for a fractional and ordinary Maxwell fluid, a fractional and ordinary second-grade fluid and a fractional and ordinary viscous fluid as well. In order to bring physical insights of helices of the circular cylinder, we have included graphical illustrations. Through depiction of graphs, helical flows showed how the velocities and shear stresses' profiles are impacted by different rheological parameters. In addition, some very important remarks are enumerated.

Appendix A.

$$J_2(Rr_f)Rr_f\bar{w}_1(R,t) - r_f^2\bar{w}_H(r_f,t) = \int_0^R \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{1}{r}\right) J_1(rr_f)\bar{w}_1 r\,\mathrm{d}r, \tag{A.1}$$

$$J_1(Rr_g)Rr_g\bar{v}_1(R,t) - r_g^2\bar{v}_H(r_g,t) = \int_0^R \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right) J_0(rr_g)\bar{v}_1 r\,\mathrm{d}r, \tag{A.2}$$

$$\bar{w}_{1H}(r_f,s) = \int_0^R J_1(rr_f)r\bar{w}_{1H}(r_f,s)\mathrm{d}r, \tag{A.3}$$

$$\bar{w}_{2H}(r_g,s) = \int_0^R J_0(rr_g)r\bar{w}_{2H}(r_g,s)\mathrm{d}r, \tag{A.4}$$

$$\bar{w}_1(r,s) = \frac{2}{R^2}\sum_{f=1}^{\infty}\bar{w}_{1H}(r_f,s)\frac{J_1(Rr_f)}{J_2^2(Rr_f)}, \qquad \bar{w}_2(r,s) = \frac{2}{R^2}\sum_{g=1}^{\infty}\bar{w}_{2H}(r_g,s)\frac{J_0(Rr_g)}{J_1^2(Rr_g)}, \tag{A.5}$$

$$\int_0^R J_1(rr_f)r^2\mathrm{d}r = \frac{R^2J_2(Rr_f)}{r_f}, \qquad \int_0^R J_0(rr_g)r\,\mathrm{d}r = \frac{RJ_1(Rr_g)}{r_g}, \tag{A.6}$$

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha+b}}{s(s^\alpha + a)}\right] = E_{\alpha+b}(-at^{\alpha+b}). \tag{A.7}$$

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Author contribution statement

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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