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Excitation on the para-Bose states: Nonclassical properties

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Abstract. In this paper, we use a para-Bose operator to construct new kinds of excited para-Bose states. These states may be considered as appropriate and linear combinations of the para-Bose Fock states. We prove that these states satisfy a closure relation that is expressed uniquely in terms of the Meijer G-function. We examine the nonclassical properties of these states, by evaluating para-Bose Fock state distribution, Mandel's parameter, second-order coherence function and quadrature squeezing. It results that the introduced excited para-Bose states show both the nonclassical and semiclassical statistics on their Fock state distribution. We also show that these states lack second-order coherence, *i.e.* they are not full coherent. Interestingly, the non-classicality of these states is controlled by the deformation parameter and number of excitation, where both parameters might be feasible for fine tuning in the trapped-ion quantum simulation.

1 Introduction

Studying non-classical states of light is a very important topic of quantum optics. In the last two decades, a significant attention has been given to the preparation and manipulation of various nonclassical states [1–6], because of their important resource for the modern quantum technology. It has been observed that for a more effective realization of quantum information processing with the continuous variables such as quantum teleportation, dense coding, quantum key distribution and quantum cloning, nonclassical states are potentially required [7–15]. An interesting class of nonclassical states consists of the photon-added coherent states or excited coherent states [16], which was constructed by the repeated application of the photon creation operator on the coherent states (CS), that is,

$$|z, m\rangle = a^{\dagger\, m}|z\rangle, \tag{1}$$

where m is the number of added photons and $|z\rangle$ is the CS which can be obtained through the operation of the unitary displacement operator $D(z) = \exp\left(za^{\dagger} - \bar{z}a\right)$ on the vacuum state of the single-mode radiation field, $|0\rangle$, that is,

$$|z\rangle = e^{(za^{\dagger} - \bar{z}a)}|0\rangle, \tag{2}$$

where, the complex number z is the coherent parameter. The excited CS intermediate between the classical-quantum regime, because they reduce the $|z\rangle$ and the quantum Fock state $|n\rangle$ in limits $m \to 0$ and $z \to 0$, respectively. From a practical point of view, Zavatta *et al.* prepared the state $|z, 1\rangle$ using parametric down-conversion in a nonlinear crystal [17] and detected non-classical characterization through quantum homodyne tomography [18]. Also, it can be obtained using an excited two level atom passing through a high quality cavity [19]. Moreover, Kalamidas *et al.* extended this method in order to generate the state $|z, 2\rangle$ [20], and also a way of generating excited CS with an arbitrary number of photons was proposed through conditional measurement on a beam splitter [21]. In ref. [22], it has been shown that the excited CS belong to the non-linear CS with special non-linearity functions. They exhibit quadrature squeezing and sub-Poissonian statistics of the field. Additionally, negativity in the Wigner function provides another

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signature for their non-classical features [16], and makes them more suitable for quantum information processing applications. In particular, it has been recognized that in the long distance quantum communication, these states show better performance than other nonclassical states [23–27]. In recent years addition of photons to some other kinds of quantum states such as the squeezed vacuum state [28], even (or odd) CS [29,30], displaced quantum states [31,32], thermal as well as the hypergeometric states [33–38], deformed CS [39–41] and some exactly solvable potentials were constructed [42–44].

On the other hand, in recent years a lot of interest has been paid to the extension and deformation of the boson oscillator algebra [45–49]. One of the most interesting deformed oscillator algebras is the para-Bose algebra or Wigner-Heisenberg algebra (WHA) [50]. This algebra is an extension of the Heisenberg algebra by the parity operator $\hat{R}$ associated with the following (anti-)commutation relations

$$\left[\mathbf{a}, \mathbf{a}^{\dagger}\right] = 1 + 2\lambda\hat{R}, \tag{3a}$$

$$\left\{\hat{R}, \mathbf{a}\right\} = \left\{\hat{R}, \mathbf{a}^{\dagger}\right\} = 0, \tag{3b}$$

where λ is a real positive integer called the Wigner parameter. Clearly, the Heisenberg algebra is recovered in the limit, $\lambda = 0$. The corresponding number operator is given by

$$\hat{\mathbf{N}} = \mathbf{a}^{\dagger}\mathbf{a} + \lambda\,(\hat{R} - 1), \tag{4}$$

which commutes with the parity operator and satisfies the following commutation relations

$$\left[\hat{\mathbf{N}}, \mathbf{a}\right] = -\mathbf{a},\ \left[\hat{\mathbf{N}}, \mathbf{a}^{\dagger}\right] = \mathbf{a}^{\dagger}. \tag{5}$$

In terms of the even and odd para-Bose Fock states [51], defined by the associated Laguerre polynomials [52] $L_n^{\alpha}(x) = \frac{1}{n!}x^{-\alpha}e^{x}(\frac{\mathrm{d}}{\mathrm{d}x})^n(x^{n+\alpha}e^{-x})$ with $(Re(\alpha) > -1)$,

$$\psi_{2n}(x) = \langle x|2n,\lambda\rangle = \sqrt{\frac{n!}{\Gamma(n+\lambda+\frac{1}{2})}}\,(-1)^n|x|^{\lambda}e^{-\frac{x^2}{2}}\,L_n^{(\lambda-\frac{1}{2})}(x^2), \tag{6a}$$

$$\psi_{2n+1}(x) = \langle x|2n+1,\lambda\rangle = \sqrt{\frac{n!}{\Gamma(n+\lambda+\frac{3}{2})}}\,(-1)^n x|x|^{\lambda}e^{-\frac{x^2}{2}}\,L_n^{(\lambda+\frac{1}{2})}(x^2), \tag{6b}$$

the WHA in (3) is realized as below

$$\mathbf{a}\,|2n,\lambda\rangle = \sqrt{2n}\,|2n-1,\lambda\rangle\,, \qquad \mathbf{a}\,|2n+1,\lambda\rangle = \sqrt{2n+2\lambda+1}\,|2n,\lambda\rangle\,, \tag{7a}$$

$$\mathbf{a}^{\dagger}\,|2n,\lambda\rangle = \sqrt{2n+2\lambda+1}\,|2n+1,\lambda\rangle\,, \qquad \mathbf{a}^{\dagger}\,|2n+1,\lambda\rangle = \sqrt{2n+2}\,|2n+2,\lambda\rangle\,, \tag{7b}$$

$$\hat{R}\,|n,\lambda\rangle = (-1)^n\,|n,\lambda\rangle\,, \tag{7c}$$

where the differential explicit forms of the operators $\mathbf{a}$ and $\mathbf{a}^{\dagger}$ can be expressed in terms of the bosonic annihilation (creation) operators a $(a^{\dagger})$, as

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{\lambda}{x}\,\hat{R}\right) = \hat{a} - \frac{\lambda}{\sqrt{2}\,x}\hat{R}, \tag{8a}$$

$$\hat{\mathbf{a}}^{\dagger} = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x + \frac{\lambda}{x}\,\hat{R}\right) = \hat{a}^{\dagger} + \frac{\lambda}{\sqrt{2}\,x}\hat{R}. \tag{8b}$$

In the $\lambda \to 0$ limit, the annihilation (creation) operators $\hat{\mathbf{a}}(\hat{\mathbf{a}}^{\dagger})$ reduce to the annihilation (creation) operators of the simple harmonic oscillator and the para-Bose Fock states (6) go over into the even and odd bosonic Fock states

$$\psi_{2n}(x) = \langle x|2n\rangle = \left(\sqrt{\pi}\,2^{2n}(2n)!\right)^{-\frac{1}{2}}\,e^{-\frac{x^2}{2}}\,H_{2n}(x), \tag{9a}$$

$$\psi_{2n+1}(x) = \langle x|2n+1\rangle = \left(\sqrt{\pi}\,2^{2n+1}(2n+1)!\right)^{-\frac{1}{2}}\,e^{-\frac{x^2}{2}}\,H_{2n+1}(x), \tag{9b}$$

where $H_n(x)$ is the Hermite polynomial [52].

It is worth mentioning that the WHA is clearly connected to the algebra of f-oscillators defined in ref. [53]. Indeed, considering $\hat{R}$ as a function of the bosonic number operator, one may show the point. Similar conclusions are obtained in the context of super-symmetric quantum mechanics, where CS generated by super-symmetry [54], also the latter gives rise to new oscillator-like structures in which coupled super-symmetry and ladder structures beyond the harmonic oscillator was given [55]. It must be noted that the realization of the WHA results in the new superposition of only even or odd para-Bose Fock basis, such as the para-Bose cat states [56,57] and generalized compass states [58] as eigenstates of the annihilation operators $\mathbf{a}^2$ and $\mathbf{a}^4$, respectively. These states include rather different nonclassical properties than the well known Schrödinger ones. For instance, contrary to the Schrödinger cat state, both even and odd para-Bose cats possess squeezing property and obey the sub-Poissonian statistics. Furthermore, the even compass states illustrate fully super-Poissonian statistics for $\lambda > 0$, while fully sub-Poissonian statistics appear for the odd compass states with $\lambda < 0$. It was demonstrated that these states can be generated in a parity deformed Jaynes-Cummings model describing a two-level atom interaction with a quantized electromagnetic field in the presence of a centrifugally external classical field [59]. It has been shown that in the parity-deformed Jaynes-Cumming model with dissipation if the field is initially prepared in the deformed cat state, the generated maximally entangled state becomes robust. Also, the WHA have been utilized to study the binomial and negative binomial states for the para-Bose oscillator [60,61].

Beside the many previous extensions and generalizations, an appropriate and normalized superposition of the para-Bose Fock states $|n,\lambda\rangle$ as a Gilmore-Perelomov like para-Bose states were given in ref. [62]:

$$
\begin{aligned}
|z,\lambda\rangle &= e^{z\mathbf{a}^\dagger - \bar{z}\mathbf{a}}|0,\lambda\rangle \\
&= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\,z)^{2n}}{(2n)!} \sqrt{n!\left(\lambda+\frac{1}{2}\right)_n}\; {}_1F_1\left(-\lambda, n+\frac{1}{2}; \frac{|z|^2}{2}\right) |2n,\lambda\rangle \\
&\quad + e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\,z)^{2n+1}}{(2n+1)!} \sqrt{n!\left(\lambda+\frac{1}{2}\right)_{n+1}}\; {}_1F_1\left(-\lambda, n+\frac{3}{2}; \frac{|z|^2}{2}\right) |2n+1,\lambda\rangle .
\end{aligned} \tag{10}
$$

where the Gilmore-Perelomov CS are recovered in the limit $\lambda = 0$. The states $|z,\lambda\rangle$ satisfy the over-completeness showing both non-classical and semi-classical behavior on their photon number distribution. Its quantum mechanical features are analytically controlled by the parameters λ and z. A particular form of $|z,\lambda\rangle$ can be experimentally realized in a scheme involving a single trapped ion driven by two pairs of orthogonal fields and tuned to the first red- and blue-sideband transitions [63].

In this paper, we introduce an excitation on the para-Bose states $|z,\lambda\rangle$. We will provide an analytic expression for these states and show that they can be considered as finite superpositions of generalized excited even and odd states that admit the resolution of the identities through positive definite measures. We will perform a comparison of non-classicality between the introduced states and those obtained by photon adding on the coherent and cat states of a simple harmonic oscillator. In addition, we will show that excited para-Bose states can be generated based on the interaction of a two-level atom with a single-mode quantized field in the presence of a centrifugal classical field. Finally, a conclusion is presented.

2 Excitation on the para-Bose states $|z,\lambda\rangle$

The excited para-Bose states are obtained by repeated application of the para-Bose creation operator $\mathbf{a}^\dagger$ on the para-Bose states, $|z,\lambda\rangle$ as follows:

$$
\begin{aligned}
|z,\lambda,m\rangle &= \mathbf{a}^{\dagger\, m}\,|z,\lambda\rangle \\
&= e^{-\frac{|z|^2}{2}} \Bigg\{ \sum_{n=0}^{\infty} \frac{(\sqrt{2}\,z)^{2n}\left[n!(\lambda+\frac{1}{2})_n \prod_{j=1}^{[\frac{m+1}{2}]}(2n+2j+2\lambda-1)\prod_{j=1}^{[\frac{m}{2}]}(2n+2j)\right]^{\frac{1}{2}}}{(2n)!} \\
&\quad \times\, {}_1F_1\left(-\lambda, n+\frac{1}{2}; \frac{|z|^2}{2}\right) |2n+m\rangle \\
&\quad + \sum_{n=0}^{\infty} \frac{(\sqrt{2}\,z)^{2n+1}\left[n!(\lambda+\frac{1}{2})_{n+1} \prod_{j=1}^{[\frac{m}{2}]}(2n+2j+2\lambda+1)\prod_{j=1}^{[\frac{m+1}{2}]}(2n+2j)\right]^{\frac{1}{2}}}{(2n+1)!} \\
&\quad \times\, {}_1F_1\left(-\lambda, n+\frac{3}{2}; \frac{|z|^2}{2}\right) |2n+m+1\rangle \Bigg\},
\end{aligned} \tag{11}
$$

where the notations $(x)_j$ and ${}_1F_1(a,b;x)$ denote the Pochhammer symbol and confluent hypergeometric function of the first kind, respectively. Clearly, the states $|z,\lambda,m\rangle$ reduce to the $|z,\lambda\rangle$ and standard excited CS as $m=0$ and $\lambda=0$, respectively. Using the summation formula associated with the confluent hypergeometric function ${}_1F_1(a,b;x)$ [52] and depending on whether the excitation number m is even or odd, $|z,\lambda,m\rangle$ can be given as follows:

$$|z,\lambda,m\rangle^{(e)} = N^{(e)}_{\lambda,m}\, e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \left(\frac{|z|^2}{2}\right)^k \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!} \left[|z,\lambda,m\rangle^{(+)}_k + |z,\lambda,m\rangle^{(-)}_k\right], \tag{12a}$$

$$|z,\lambda,m\rangle^{(o)} = N^{(o)}_{\lambda,m}\, e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \left(\frac{|z|^2}{2}\right)^k \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!} \left[\left|z,\lambda+\frac{1}{2},m\right\rangle^{(+)}_k + \left|z,\lambda+\frac{1}{2},m\right\rangle^{(-)}_k\right], \tag{12b}$$

where, $N^{e(o)}_{\lambda,m}$ are normalization coefficients to be determined below and the normalized state $|z,\lambda,m\rangle^{(\pm)}_k$ are

$$|z,\lambda,m\rangle^{(+)}_k = M^{(+)}_k(\lambda,m) \sum_{n=0}^{\infty} \left(\frac{z}{\sqrt{2}}\right)^{2n} \frac{\sqrt{(n+\lambda+\frac{m-1}{2})!(n+\frac{m}{2})!}}{n!\,(n+k-\frac{1}{2})!}\, |2n+m,\lambda\rangle, \tag{13a}$$

$$|z,\lambda,m\rangle^{(-)}_k = M^{(-)}_k(\lambda,m) \sum_{n=0}^{\infty} \left(\frac{z}{\sqrt{2}}\right)^{2n+1} \frac{\sqrt{(n+\lambda+\frac{m+1}{2})!(n+\frac{m}{2})!}}{n!\,(n+k+\frac{1}{2})!}\, |2n+m+1,\lambda\rangle, \tag{13b}$$

where the normalization coefficients $M^{(\pm)}_k(\lambda,m)$ are

$$M^{(+)}_k(\lambda,m) = \left[\frac{(\frac{m}{2})!\,(\lambda+\frac{m}{2}-\frac{1}{2})!\,{}_2F_3\left(\left[\frac{m}{2}+1,\lambda+\frac{m+1}{2}\right],\left[1,k+\frac{1}{2},k+\frac{1}{2}\right];\frac{|z|^4}{4}\right)}{\Gamma\left(k+\frac{1}{2}\right)^2}\right]^{-\frac{1}{2}}, \tag{14a}$$

$$M^{(-)}_k(\lambda,m) = \left[\frac{(\frac{m}{2})!\,(\lambda+\frac{m}{2}+\frac{1}{2})!|z|^2\,{}_2F_3\left(\left[\frac{m}{2}+1,\lambda+\frac{m+3}{2}\right],\left[1,k+\frac{3}{2},k+\frac{3}{2}\right];\frac{|z|^4}{4}\right)}{2\Gamma\left(k+\frac{3}{2}\right)^2}\right]^{-\frac{1}{2}}. \tag{14b}$$

The states $|z,\lambda,m\rangle^{(\pm)}_k$ include a wide class of excited even and odd CS, already discussed in refs. [29,30,43,56], then we will call them as *generalized excited para-Bose cats*. Due to the fact that the coefficient $\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)}$ in eqs. (12) is zero for a given integer λ less than k, then excited para-Bose states, $|z,\lambda,m\rangle$ reduce to a finite superposition of $|z,\lambda,m\rangle^{(\pm)}_k$, *i.e.*

$$|z,\lambda,m\rangle^{(e)} = N^{(e)}_{\lambda,m}\, e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\lambda} \left(\frac{|z|^2}{2}\right)^k \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!} \left(|z,\lambda,m\rangle^{(+)}_k + |z,\lambda,m\rangle^{(-)}_k\right), \tag{15a}$$

$$|z,\lambda,m\rangle^{(o)} = N^{(o)}_{\lambda,m}\, e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\lambda} \left(\frac{|z|^2}{2}\right)^k \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!} \left(\left|z,\lambda+\frac{1}{2},m\right\rangle^{(+)}_k + \left|z,\lambda+\frac{1}{2},m\right\rangle^{(-)}_k\right). \tag{15b}$$

It is straightforward to prove that the state $|z,\lambda,m\rangle^{(+)}_k$ is orthogonal to $|z,\lambda,m\rangle^{(-)}_k$, *i.e.*, ${}^{(-)}{}_k\langle z,\lambda,m|z,\lambda,m\rangle^{(+)}_k = 0$. It follows that by using ${}^{e(o)}\langle z,\lambda,m|z,\lambda,m\rangle^{e(o)} = 1$ the normalization coefficients $N^{e(o)}_{\lambda,m}$ are determined as follows

$$N^{(e)}_{\lambda,m} = \mathbf{e}^{\frac{|z|^2}{2}} \left[\sum_{k=0}^{\lambda} \left(\frac{|z|^2}{2}\right)^{2k} \left(\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!}\right)^2 \frac{\left[M^{(+)}_k(\lambda,m)\right]^2 + \left[M^{(-)}_k(\lambda,m)\right]^2}{\left[M^{(+)}_k(\lambda,m)M^{(-)}_k(\lambda,m)\right]^2}\right]^{-\frac{1}{2}}, \tag{16a}$$

$$N^{(o)}_{\lambda,m} = \mathbf{e}^{\frac{|z|^2}{2}} \left[\sum_{k=0}^{\lambda} \left(\frac{|z|^2}{2}\right)^{2k} \left(\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!}\right)^2 \frac{\left[M^{(+)}_k\left(\lambda+\frac{1}{2},m\right)\right]^2 + \left[M^{(-)}_k\left(\lambda+\frac{1}{2},m\right)\right]^2}{\left[M^{(+)}_k\left(\lambda+\frac{1}{2},m\right)M^{(-)}_k\left(\lambda+\frac{1}{2},m\right)\right]^2}\right]^{-\frac{1}{2}}. \tag{16b}$$

Now, we should introduce appropriate measures, $d^{(\pm)}_{m,k}(z,\bar{z},\lambda)$, so that the states $|z,\lambda,m\rangle^{(\pm)}_k$ resolve the following closure relation, which is usually inherited to the generalized CS [64–66]:

$$\int_{\mathbb{C}} |z,\lambda,m\rangle^{(+)}_{k}\,{}^{(+)}_{k}\langle z,\lambda,m|\, d^{(+)}_{m,k}(z,\bar{z},\lambda) = \sum_{n=0}^{\infty} |2n+m,\lambda\rangle\langle 2n+m,\lambda| = I^{+}_{\lambda,m}, \tag{17a}$$

$$\int_{\mathbb{C}} |z,\lambda,m\rangle^{(-)}_{k}\,{}^{(-)}_{k}\langle z,\lambda,m|\, d^{(-)}_{m,k}(z,\bar{z},\lambda) = \sum_{n=0}^{\infty} |2n+m+1,\lambda\rangle\langle 2n+m+1,\lambda| = I^{-}_{\lambda,m}. \tag{17b}$$

Having the closure relation for any set of generalized CS is relevant because this automatically proposes the delta-distribution as the P-representation of such states [66]. Then, following [64,65], the fields represented by such states would have a classical analog. However, it has been shown that they have nonclassical properties [66]. In the polar coordinate representations of $z = r\,e^{i\phi}$ with $0 \le r < \infty$, $0 \le \phi \le 2\pi$, by considering the measure $d^{(\pm)}_{m,k}(z,\bar{z},\lambda)$ as

$$d^{(\pm)}_{m,k}(z,\bar{z},\lambda) = \mathcal{W}^{(\pm)}_{m,k}(r,\lambda) r\,\mathrm{d}r\,\mathrm{d}\phi, \tag{18}$$

and substituting eqs. (13) into (17), the problem is reduced to finding a positive weight function $\mathcal{W}^{(\pm)}_{m,k}(r,\lambda)$ satisfying the following integral relations:

$$\int_0^{\infty} \mathrm{d}r \left(\frac{r}{\sqrt{2}}\right)^{4n+1} \left[M^{(+)}_k(\lambda,m)\right]^2 \mathcal{W}^{(+)}_{k,m}(r,\lambda) = \frac{\left[n!\,\Gamma\left(n+k+\frac{1}{2}\right)\right]^2}{2\sqrt{2}\,\pi\,\Gamma\left(n+\lambda+\frac{m+1}{2}\right)\Gamma\left(n+\frac{m}{2}+1\right)} \tag{19a}$$

$$\int_0^{\infty} \mathrm{d}r \left(\frac{r}{\sqrt{2}}\right)^{4n+3} \left[M^{(-)}_k(\lambda,m)\right]^2 \mathcal{W}^{(-)}_{k,m}(r,\lambda) = \frac{\left[n!\,\Gamma\left(n+k+\frac{3}{2}\right)\right]^2}{2\sqrt{2}\,\pi\,\Gamma\left(n+\lambda+\frac{m+3}{2}\right)\Gamma\left(n+\frac{m}{2}+1\right)}. \tag{19b}$$

Using the integral relations for the Meijers G-functions (see $\frac{7-811}{4}$ in [52]), the measures permitting the resolution of the identities (16) are given in terms of Meijers G-functions, *i.e.*

$$\mathcal{W}^{(+)}_{k,m}(r,\lambda) = \frac{r^6}{8\pi\left[M^{(+)}_k(\lambda,m)\right]^2}\, G^{4,0}_{2,4}\left(\frac{r^4}{4}\,\middle|\, \begin{matrix} \frac{m}{2}-1,\, \lambda+\frac{m}{2}-\frac{3}{2} \\ -1,\, -1,\, k-\frac{3}{2},\, k-\frac{3}{2} \end{matrix}\right), \tag{20a}$$

$$\mathcal{W}^{(-)}_{k,m}(r,\lambda) = \frac{r^4}{4\pi\left[M^{(-)}_k(\lambda,m)\right]^2}\, G^{4,0}_{2,4}\left(\frac{r^4}{4}\,\middle|\, \begin{matrix} \frac{m}{2}-1,\, \lambda+\frac{m}{2}-\frac{1}{2} \\ -1,\, -1,\, k-\frac{1}{2},\, k-\frac{1}{2} \end{matrix}\right). \tag{20b}$$

The above equations confirm the fact that deformed oscillator algebras give rise to generalized CS leading to closure relations expressed uniquely in terms of the Meijer G-function [66]. We have plotted the changes of functions $\mathcal{W}^{(\pm)}_{m,k}(r,\lambda)$ in terms of $|z|$ for different values of λ, m, and k in fig. 1. These figures confirm the positive definiteness of the $\mathcal{W}^{(\pm)}_{m,k}(r,\lambda)$ in all regions of $r = |z|$.

3 Nonclassical properties of the excited para-Bose states and generalized excited para-Bose cats

In this section, we examine some non-classical features of the generalized excited para-Bose cats and excited para-Bose states which have been introduced in eqs. (13) and (15), respectively. We have found that nonclassical properties of $|z,\lambda,m\rangle^{(o)}$ are similar to $|z,\lambda,m\rangle^{(e)}$. Therefore, we have restricted ourselves in this section to studding non-classical properties of excited para-Bose states with even excitation number. The non-classicality of a state may be established through signs such as quadrature squeezing, higher order squeezing, negativity of the Wigner function, sub-Poissonian statistics of Fock-number distribution and second-order coherence. Although, each of the above criteria is indeed sufficient for a state to belong to non-classical states [67], we will pay attention to quadrature squeezing and Fock state statistics and second-order coherence of the introduced states.

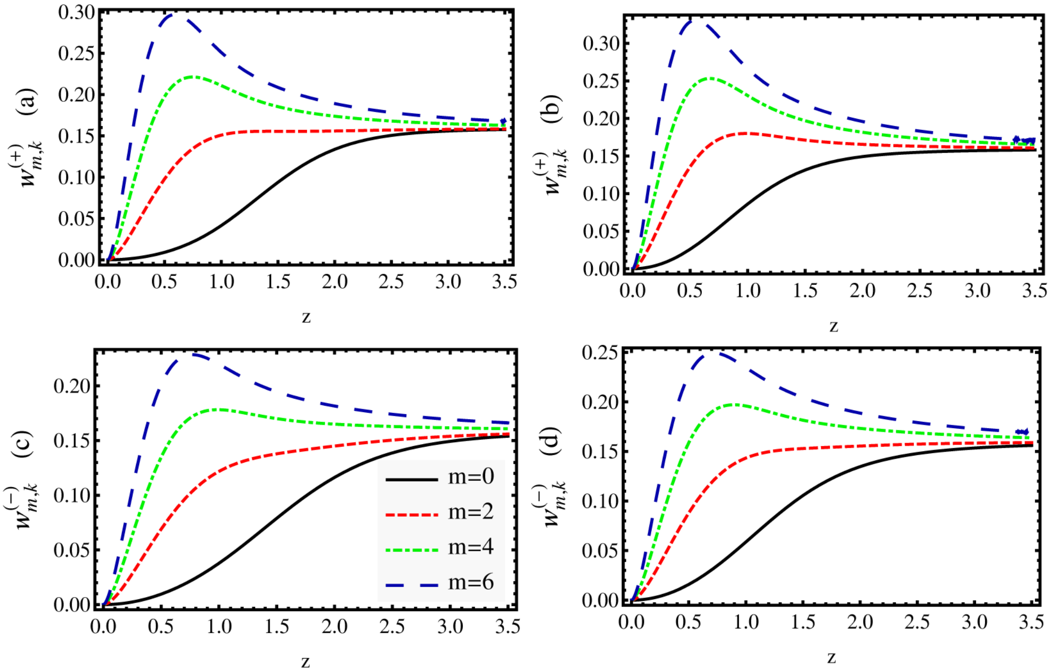

Fig. 1. Variation of weight functions $\mathcal{W}^{(+)}_{m,k}(r,\lambda)$ (in (a) and (b)) and $\mathcal{W}^{(-)}_{m,k}(r,\lambda)$ (in (c) and (d)) *versus* the real coherent parameter $z = r$ for $k = 2$ and different values of m. In (a) and (c), $\lambda = 1$ and in (b) and (d), $\lambda = 2$.

3.1 Para-Bose Fock state distribution and Klyshko's criterion

The probability of detecting n para-Bose particles in a state $|\psi\rangle$ is described by the distribution $P_n = |\langle n|\psi\rangle|^2$. For generalized excited para-Bose cats, we find

$$P^{(+)}_{2n}(m,k) = \left|\langle 2n|z,\lambda,m\rangle^{(+)}_k\right|^2 = \frac{(n+\lambda-\frac{1}{2})!(n)!\left[M^{(+)}_k(\lambda,m)\right]^2}{\left[(n-\frac{m}{2})!\ (n+k-\frac{1+m}{2})!\right]^2}\left(\frac{|z|^2}{2}\right)^{2n-m}, \tag{21a}$$

$$P^{(-)}_{2n+1}(m,k) = \left|\langle 2n+1|z,\lambda,m\rangle^{(-)}_k\right|^2 = \frac{(n+\lambda+\frac{1}{2})!(n)!\left[M^{(-)}_k(\lambda,m)\right]^2\left(\frac{|z|^2}{2}\right)^{2n-m+1}}{\left[(n-\frac{m}{2})!\ (n+\frac{1-m}{2}+k)!\right]^2}, \tag{21b}$$

$$P^{(+)}_{2n+1}(m,k) = P^{(-)}_{2n}(m,k) = 0. \tag{21c}$$

It is obvious that for $2n < m$, $P^{(+)}_{2n}(m,k)$ and $P^{(-)}_{2n+1}(m,k)$ are zero, it can be viewed as holes in the Fock-number distribution of $|z,\lambda,m\rangle^{(\pm)}_k$. It has been shown that the Glauber-Sudarshan P-function in such a case cannot be interpreted as a classical probability distribution and thus a state including holes in its Fock-number distribution corresponds to a nonclassical state [68–71]. These states may be used for optical data storage such that each hole being associated with some signals (say yes, $|1\rangle$, or $|+\rangle$) and its absence being associated with the opposite signals (no, $|0\rangle$, or $|-\rangle$) [72]. Some theoretical schemes to generate controlled holes in the Fock-number space of quantized electromagnetic fields have been proposed in the contexts of cavity-QED [73,74] and traveling waves [75,76] while the photon addition is a feasible way to make a hole in the Fock-vector space.

Similarly, for the excited para-Bose state, we have the expressions

$$P^{(e)}_n(m) = \begin{cases} e^{-|z|^2}\left[N^{(e)}_{\lambda,m}\sum_{k=0}^{\lambda}\left(\frac{|z|^2}{2}\right)^k\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!}\left(\frac{P^{(+)}_{2n}(m,k)}{M^{(+)}_k(\lambda,m)}\right)\right]^2, & n \text{ even}, \quad (22a)\\ e^{-|z|^2}\left[N^{(o)}_{\lambda,m}\sum_{k=0}^{\lambda}\left(\frac{|z|^2}{2}\right)^k\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)k!}\left(\frac{P^{(-)}_{2n+1}(m,k)}{M^{(-)}_k(\lambda,m)}\right)\right]^2, & n \text{ odd}. \quad (22b)\end{cases}$$

With the help of the above distribution functions, non-classicality of the excited para-Bose states and generalized excited para-Bose cats may be examined using Klyshko's criterion. Generally, for any Fock-number distribution P_n,

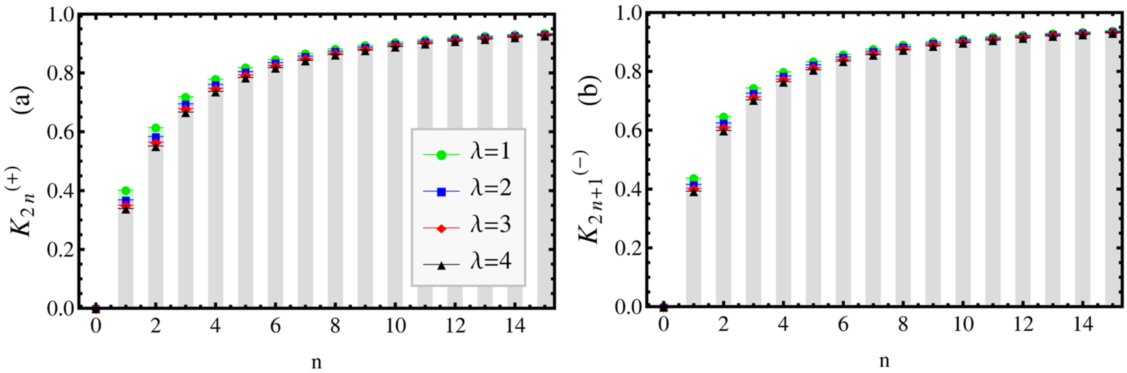

Fig. 2. Klyshko's criterion (a) $K_{2n}^{(+)}$ and (b) $K_{2n+1}^{(-)}$ associated with the generalized excited even para-Bose states and generalized excited odd para-Bose cats, respectively with different deformation parameter λ by setting $m = 2$ and $k = 2$.

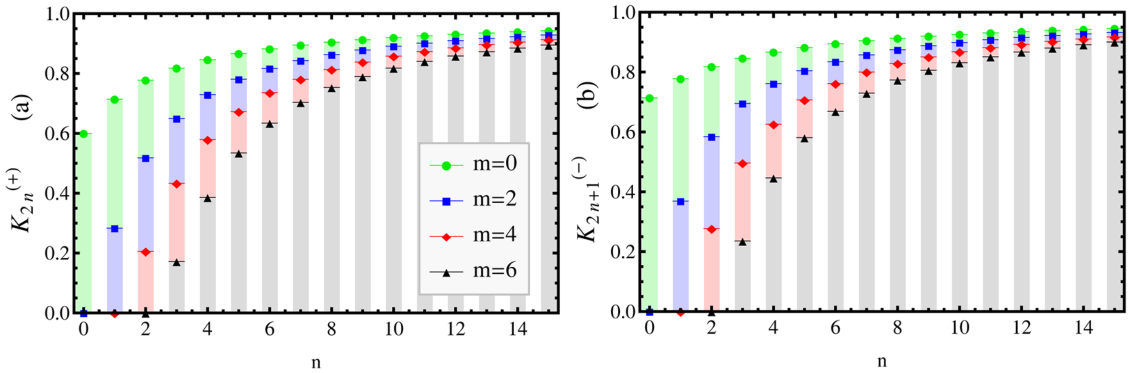

Fig. 3. Klyshko's criterion (a) $K_{2n}^{(+)}$ and (b) $K_{2n+1}^{(-)}$ associated with the generalized excited even para-Bose cats and generalized excited odd para-Bose states, respectively with different photon addition number m by setting $\lambda = 1$ and $k = 1$.

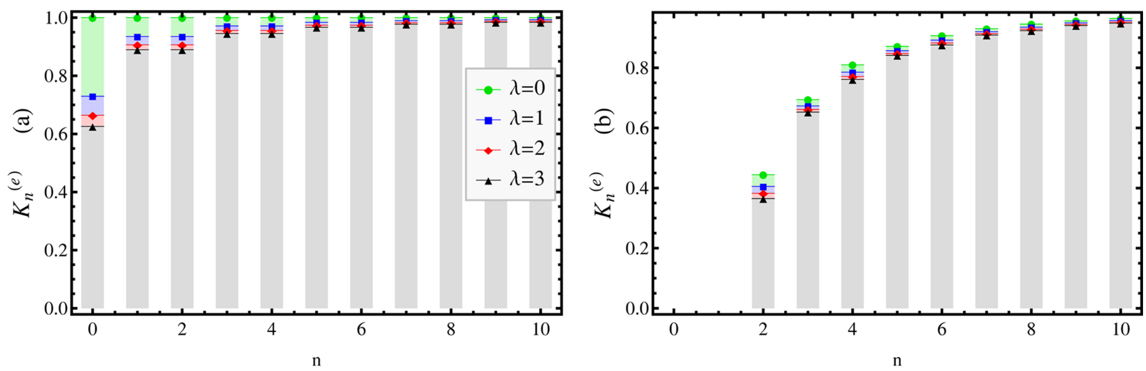

Fig. 4. Klyshko's criterion $K_n^{(e)}$ associated with the excited para-Bose states with $z = 0.2$ and different λ by setting (a) $m = 0$ and (b) $m = 2$.

Klyshko's criterion [77] is defined as

$$K_n = \frac{(n+2)P_n P_{n+2}}{(n+1)P_{n+1}^2}, \tag{23}$$

in which, for any classical state such as CS or thermal state, the inequality $K_n \geq 1$ is fulfilled for all n. In other words, if this inequality is violated for any of K_n, the state is determined to be nonclassical. Using the Fock-number distribution from eqs. (21) and (22), one can easily evaluate Klyshko's criterion for the generalized excited para-Bose cats and excited para-Bose states, respectively. For specific parameter values, we have depicted the variation of K_n for generalized excited para-Bose cats and excited para-Bose states in figs. 2–5. These figures demonstrate the non-classicality of both generalized excited para-Bose cats and excited para-Bose states which decreases (increases) with λ (m). Meanwhile, we see that the amount of Klyshko's criterion approaches to one for large n which indicates that Fock state distribution of the introduced states is reduced to Poissonian.

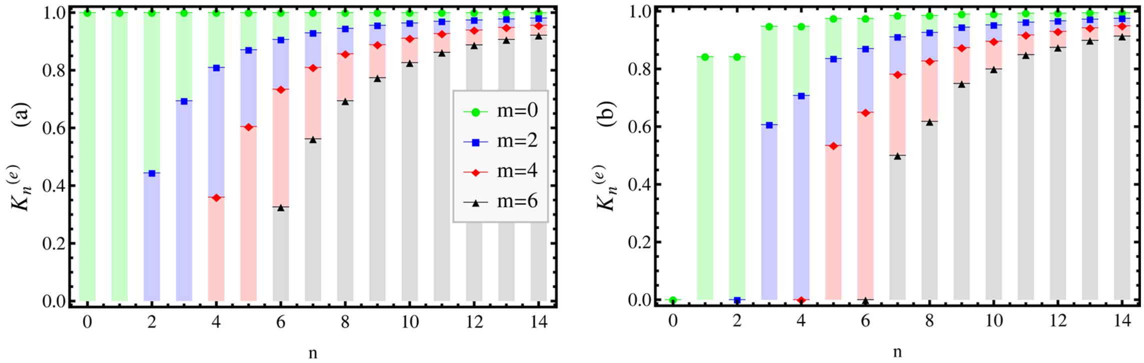

Fig. 5. Klyshko's criterion $K_n^{(e)}$ associated with the excited para-Bose states with $z = 1$ and different photon addition number m by setting (a) $\lambda = 0$ and (b) $\lambda = 1$.

3.2 Fock state statistics

The Fock state statistics of a quantum state is investigated by evaluating Mandel's parameter, which can be defined as

$$Q = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle} - 1. \tag{24}$$

In fact, a quantum state exhibits super-Poissonian, Poissonian and sub-Poissonian statistics, respectively, if $Q > 0$, $Q = 0$ and $Q < 0$. In order to calculate Mandel's parameter, we first obtain expressions for $\hat{N}$ and $\hat{N}^2$ corresponding to the introduced states given in eqs. (13), and (15), respectively:

$$\langle \hat{N} \rangle_k^{(+)} = \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\lambda+\frac{m+1}{2}\right) m \left[M_k^{(+)}(\lambda,m)\right]^2}{\left[\Gamma\left(k+\frac{1}{2}\right)\right]^2} \times {}_3F_4\left(\left[\frac{m}{2}+1, \frac{m}{2}+1, \lambda+\frac{m+1}{2}\right]\left[1, \frac{m}{2}, k+\frac{1}{2}, k+\frac{1}{2}\right]; \frac{|z|^2}{4}\right), \tag{25a}$$

$$\langle \hat{N} \rangle_k^{(-)} = \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\lambda+\frac{m+3}{2}\right)(m+1)\left[M_k^{(-)}(\lambda,m)\right]^2}{2\left[\Gamma\left(k+\frac{3}{2}\right)\right]^2} \times |z|^2\, {}_3F_4\left(\left[\frac{m}{2}+1, \frac{m+3}{2}, \lambda+\frac{m+3}{2}\right]\left[1, \frac{m+1}{2}, k+\frac{3}{2}, k+\frac{3}{2}\right]; \frac{|z|^2}{4}\right), \tag{25b}$$

$$\langle \hat{N}^2 \rangle_k^{(+)} = \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\lambda+\frac{m+1}{2}\right) m^2 \left[M_k^{(+)}(\lambda,m)\right]^2(\lambda,m)}{\left[\Gamma\left(k+\frac{1}{2}\right)\right]^2} \times {}_4F_5\left(\left[\frac{m}{2}+1, \frac{m}{2}+1, \frac{m}{2}+1, \lambda+\frac{m+1}{2}\right]\left[1, \frac{m}{2}, \frac{m}{2}, k+\frac{1}{2}, k+\frac{1}{2}\right]; \frac{|z|^2}{4}\right), \tag{25c}$$

$$\langle \hat{N}^2 \rangle_k^{(-)} = \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\lambda+\frac{m+3}{2}\right)(m+1)^2\left[M_k^{(-)}(\lambda,m)\right]^2}{2\left[\Gamma\left(k+\frac{3}{2}\right)\right]^2}|z|^2 \times {}_4F_5\left(\left[\frac{m}{2}+1, \frac{m+3}{2}, \frac{m+3}{2}, \lambda+\frac{m+3}{2}\right]\left[1, \frac{m+1}{2}, \frac{m+1}{2}, k+\frac{3}{2}, k+\frac{3}{2}\right]; \frac{|z|^2}{4}\right), \tag{25d}$$

$$\langle \hat{N} \rangle^{(e)} = \sum_{k=0}^{\lambda}\left(\frac{N_{\lambda,m}^{(e)}\Gamma(k-\lambda)\left(\frac{|z|^2}{2}\right)^k e^{-\frac{|z|^2}{2}}}{\Gamma(-\lambda)k!}\right)^2\left[\frac{\langle \hat{N} \rangle_k^{(+)}}{\left[M_k^{(+)}(\lambda,m)\right]^2} + \frac{\langle \hat{N} \rangle_k^{(-)}}{\left[M_k^{(-)}(\lambda,m)\right]^2}\right], \tag{25e}$$

$$\langle \hat{N}^2 \rangle^{(e)} = \sum_{k=0}^{\lambda}\left(\frac{N_{\lambda,m}^{(e)}\Gamma(k-\lambda)\left(\frac{|z|^2}{2}\right)^k e^{-\frac{|z|^2}{2}}}{\Gamma(-\lambda)k!}\right)^2\left[\frac{\langle \hat{N}^2 \rangle_k^{(+)}}{\left[M_k^{(+)}(\lambda,m)\right]^2} + \frac{\langle \hat{N}^2 \rangle_k^{(-)}}{\left[M_k^{(-)}(\lambda,m)\right]^2}\right]. \tag{25f}$$

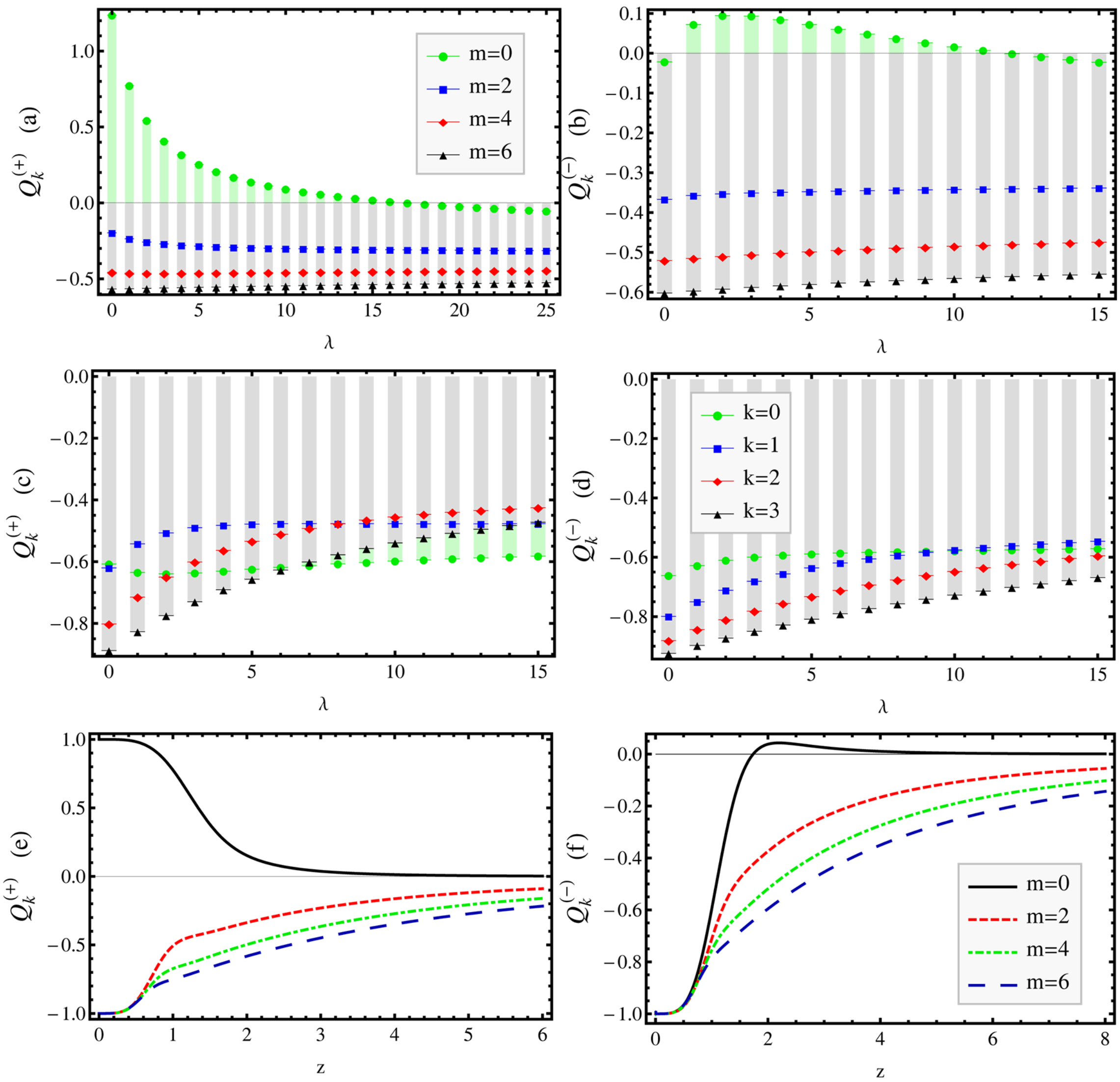

Fig. 6. Mandel's parameter with coherent and deformation parameters for the generalized excited even para-Bose cat states and generalized excited odd para-Bose cat states in (a), (c), (e) and (b), (d), (f), respectively. In (a) and (b), $z = 2$ and $k = 2$; while $z = 1$ and $m = 2$ in (c) and (d). Also, $k = 1$ and $\lambda = 2$ in (e) and (f).

In fig. 6, variations of Mandel's parameter for some members generalized excited para-Bose cats have been plotted for different values of m. It can be observed that these states always obey the sub-Poissonian statistics and their non-classicality increase with the excitation number (see figs. 6(a), 6(b), 6(e) and 6(f)). Additionally, for given coherent parameter z and excitation number m, the non-classicality of generalized excited para-Bose cats increases with k as shown in figs. 6(c) and 6(d). The changes of Mandel's parameter associated with the excited para-Bose states have been plotted for different values of m in fig. 7. We recover the photon number statistics of excited CS [16] for $\lambda = 0$. As shown, adding photons to the CS of simple harmonic oscillator makes its statistics sub-Poissonian, whereas the Fock-number statistics of para-Bose states obey from super-Poissonian for large values of λ. It is clearly observed that for a given excitation number m, there exists a family of parameters z and $\lambda \geq 1$ that changes the Fock-number statistics of excited para-Bose states into Poissonian; the states with lower (larger) λ include sub-(super-)Poissonian statistics.

3.3 Quadrature squeezing

Now, we investigate the squeezing of the excited para-Bose states and generalized excited para-Bose cats in the quadrature operators $\hat{x}$ and $\hat{p}$. With the help of commutation relation 3(a) the commutation relation between quadratures $\hat{x}$ and $\hat{p}$ becomes

$$[\hat{x}, \hat{p}] = i(1 + 2\lambda\hat{R}), \tag{26}$$

it follows that

$$\langle\sigma_{\hat{x}\hat{x}}\rangle\langle\sigma_{\hat{p}\hat{p}}\rangle \geq \frac{|\langle 1+2\lambda\hat{R}\rangle|^2}{4}\,, \tag{27}$$

where $\langle\sigma_{\hat{x}\hat{y}}\rangle = \frac{\langle\hat{x}\hat{y}+\hat{y}\hat{x}\rangle}{2} - \langle\hat{x}\rangle\langle\hat{y}\rangle$. A state is squeezed if the condition $\langle\sigma_{\hat{x}\hat{x}}\rangle < \frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}$ or $\langle\sigma_{\hat{p}\hat{p}}\rangle < \frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}$ is fulfilled [78, 79]. To characterize squeezing in the components $\hat{x}$ and $\hat{p}$, we introduce parameters

$$S_x = \frac{\langle\sigma_{xx}\rangle - \frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}}{\frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}}, \tag{28a}$$

$$S_p = \frac{\langle\sigma_{pp}\rangle - \frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}}{\frac{|\langle 1+2\lambda\hat{R}\rangle|}{2}}. \tag{28b}$$

We will say that the state has squeezing if it satisfies the inequalities $-1 \leq S_x < 0$ or $-1 \leq S_p < 0$, respectively. In order to calculate squeezing parameters for the excited para-Bose states and generalized excited para-Bose cats, we need the following expectation values:

$$\langle\mathbf{a}\rangle_k^{(\pm)} = \langle\mathbf{a}^\dagger\rangle_k^{(\pm)} = 0, \tag{29a}$$

$$\langle\mathbf{a}^2\rangle_k^{(+)} = \frac{\Gamma\left(\frac{m}{2}+2\right)\Gamma\left(\lambda+\frac{m+3}{2}\right)\left[M_k^{(+)}(\lambda,m)\right]^2 z^2}{(2k+1)\left[\Gamma\left(k+\frac{1}{2}\right)\right]^2}\,{}_2F_3\left(\left[\frac{m}{2}+2,\lambda+\frac{m+3}{2}\right]\left[2,k+\frac{3}{2},k+\frac{1}{2}\right];\frac{|z|^4}{4}\right), \tag{29b}$$

$$\langle\mathbf{a}^2\rangle_k^{(-)} = \frac{\Gamma\left(\frac{m}{2}+2\right)\Gamma\left(\lambda+\frac{m+5}{2}\right)\left[M_k^{(-)}(\lambda,m)\right]^2 |z|^2 z^2}{(2k+3)\left[\Gamma\left(k+\frac{3}{2}\right)\right]^2}\,{}_2F_3\left(\left[\frac{m}{2}+2,\lambda+\frac{m+5}{2}\right]\left[2,k+\frac{5}{2},k+\frac{3}{2}\right];\frac{|z|^4}{4}\right), \tag{29c}$$

$$\langle\mathbf{a}\rangle_k^{(e)} = \langle\mathbf{a}^\dagger\rangle_k^{(e)} = 0, \tag{29d}$$

$$\langle\mathbf{a}^2\rangle^{(e)} = \sum_{k=0}^{\lambda}\left(\frac{N_{\lambda,m}^{(e)}\Gamma(k-\lambda)\left(\frac{|z|^2}{2}\right)^k \mathrm{e}^{-\frac{|z|^2}{2}}}{\Gamma(-\lambda)k!}\right)^2\left[\frac{\langle\mathbf{a}^2\rangle_k^{(+)}}{\left[M_k^{(+)}(\lambda,m)\right]^2}+\frac{\langle\mathbf{a}^2\rangle_k^{(-)}}{\left[M_k^{(-)}(\lambda,m)\right]^2}\right]. \tag{29e}$$

The other required expectation values have been given in eqs. (24) in the paper.

In figs. 8 and 9, we have plotted the squeezing parameters S_p for generalized excited para-Bose cats *versus* λ for different values of z and k. We choose $m = 0, 2, 4$. In figs. 8(a) and 9(a), it is observed that the Schrödinger cat states [80–82] can be recovered for $k = 0, \lambda = 0$ and $m = 0$. These figures confirm that the generalized excited para-Bose cats do not have squeezing [29,30]. However, for a given photon addition number m, there exists a family of parameters k, λ and z that generalized excited para-Bose cats belonging to this parameters family will have squeezing (see figs. 8(a)–8(d) and 9(a)–9(d)). According to the diagrams placed in fig. 10, it appears that for some coherent parameter ($z < 0.8$), photon addition cannot make a squeezing in the CS, whereas the possibility of observing squeezing exists for para-Bose states with large deformation parameter λ.

3.4 Second-order correlation function

Now we calculate the second-order coherence function of the generalized excited para-Bose cats and excited para-Bose states given in (13) and (15), respectively. For this purpose, we define the second-order coherence function for excited para-Bose states as

$$g^{(2)}(0) = \frac{\langle z,\lambda,m|\mathbf{a}^{\dagger^2}\mathbf{a}^2|z,\lambda,m\rangle}{\langle z,\lambda,m|\mathbf{a}^\dagger\mathbf{a}|z,\lambda,m\rangle^2}\,. \tag{30}$$

the necessary condition of coherence is that $g^{(2)}(0) = 1$. In other words, the states are nonclassical if $g^{(2)}(0) < 1$. By means of analytical calculation, the second-order coherence functions, $g^{(2)}(0)$ of the generalized excited para-Bose cats and excited para-Bose states are plotted in figs. 11 and 12, respectively. From fig. 11, we see that for a given coherent parameter $z = 1$ the states $|z,\lambda,m\rangle_k^{(+)}$ with $m \geq 1$ satisfy $g^{(2)}(0) < 1$ condition for any values of k and λ, while this condition is satisfied for the states $|z,\lambda,m\rangle_k^{(-)}$ for any values of m, k and λ. Moreover, in fig. 12 we recover the second-order coherence function of PACS, $|z,m\rangle$, [16] for $\lambda = 0$. As shown, the second-order coherence function of PACS is less than one for any values of z, whereas nonclassical behavior of para-Bose states $|z,\lambda,m\rangle^{(e)}$ with $z \geq 0.2$

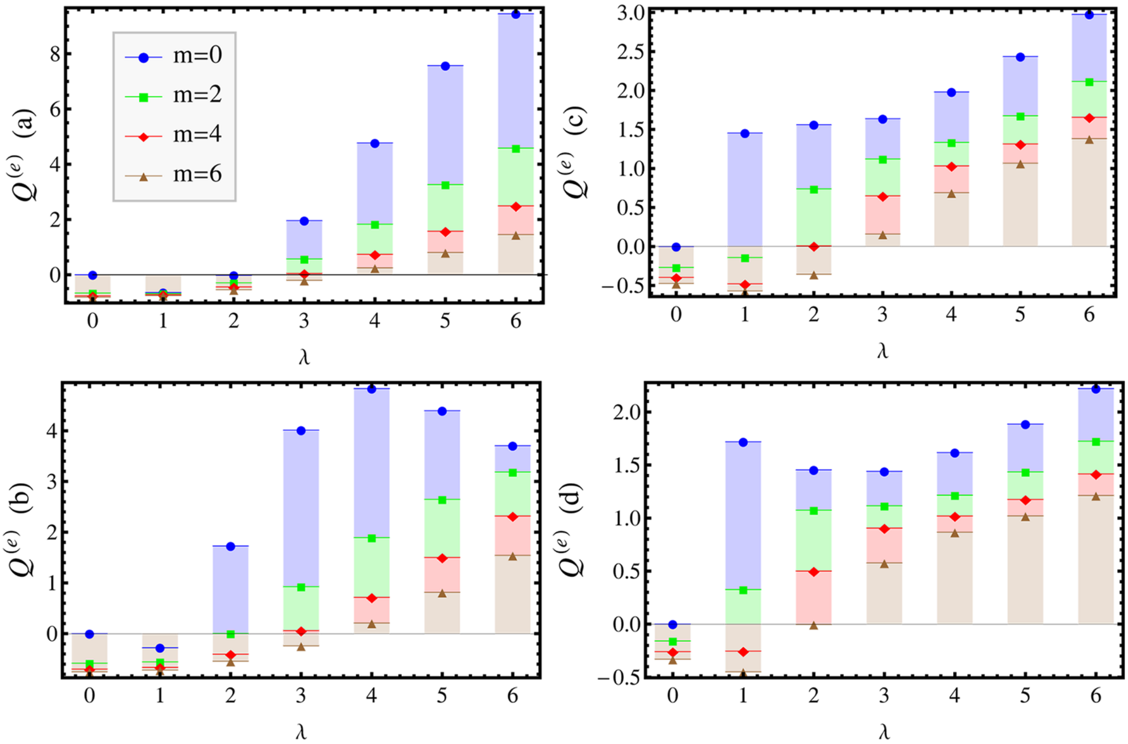

Fig. 7. Mandel's parameter as a function of λ for excited even para-Bose cat states with different deformation parameter m by setting (a) $z = 0.9$, (b) $z = 1.2$, (c) $z = 3$ and (d) $z = 4.4$.

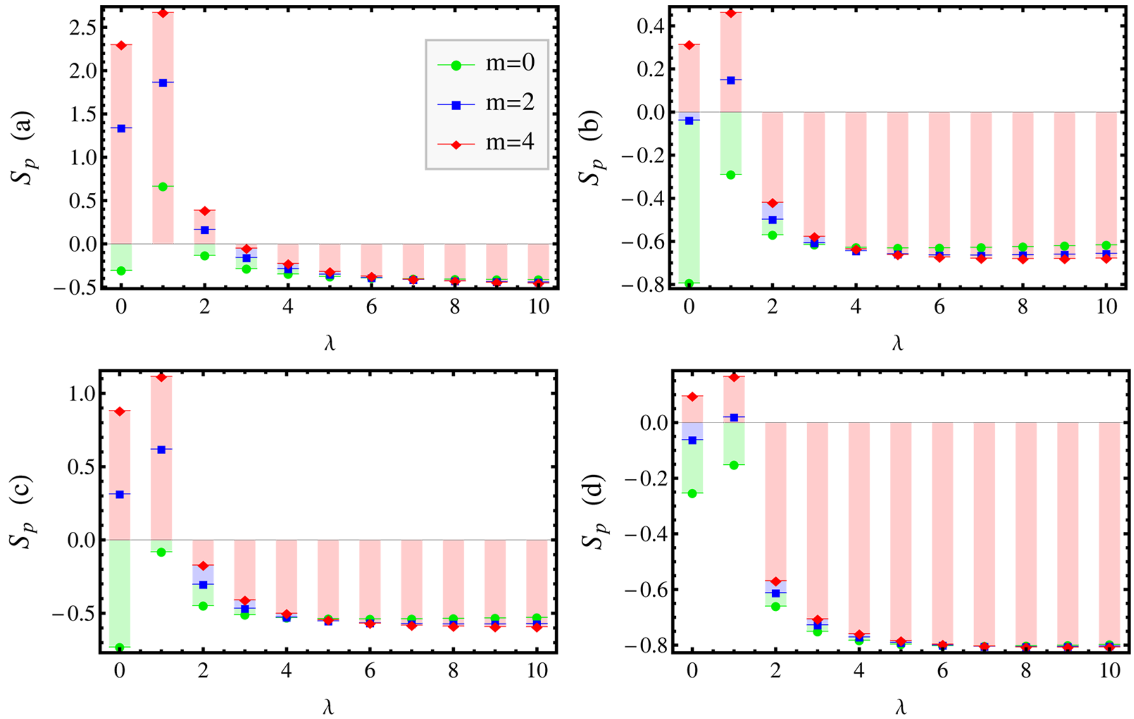

Fig. 8. S_p as a function of λ for generalized excited even para-Bose cats with different deformation parameter m by setting (a) $z = 1.2$, $k = 0$ (b) $z = 3$ and $k = 5$, (c) $z = 2$ and $k = 2$ and (d) $z = 5$ and $k = 3$.

and $m > 2$ is observed only for $\lambda < 3$. As a result, the states $|z, \lambda, m\rangle^{(e)}$ and $|z, \lambda, m\rangle_k^{(\pm)}$ are not fully coherent, so according to the discussion presented in ref. [66], they cannot be considered "classical" in the sense established by Glauber in his quantum theory of optical coherence [64,65].

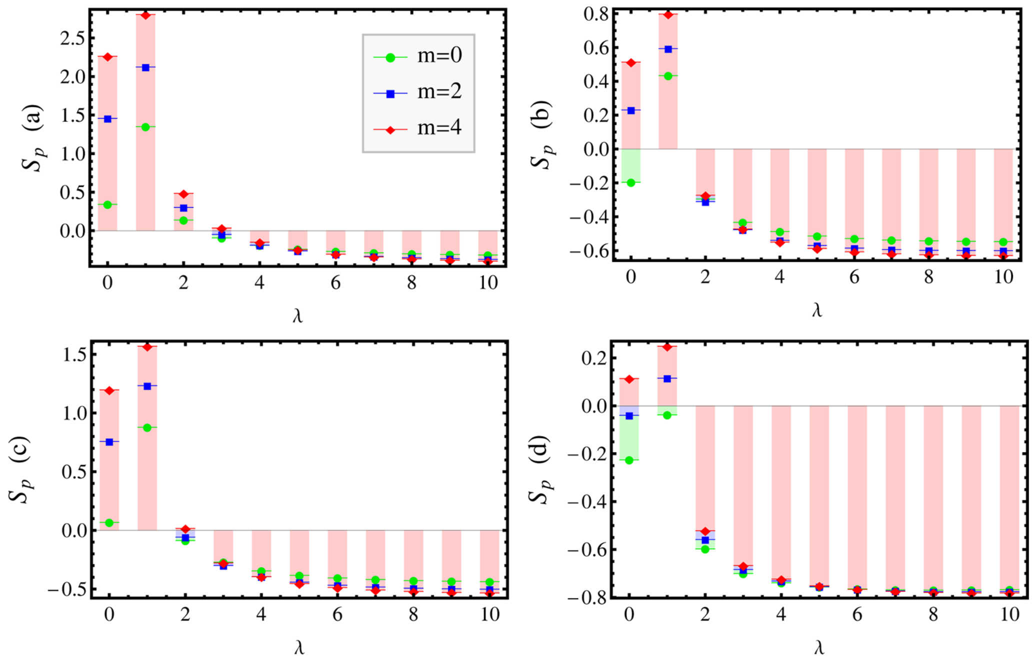

Fig. 9. S_p as a function of λ for generalized excited odd para-Bose cats with the parameters given in fig. 8.

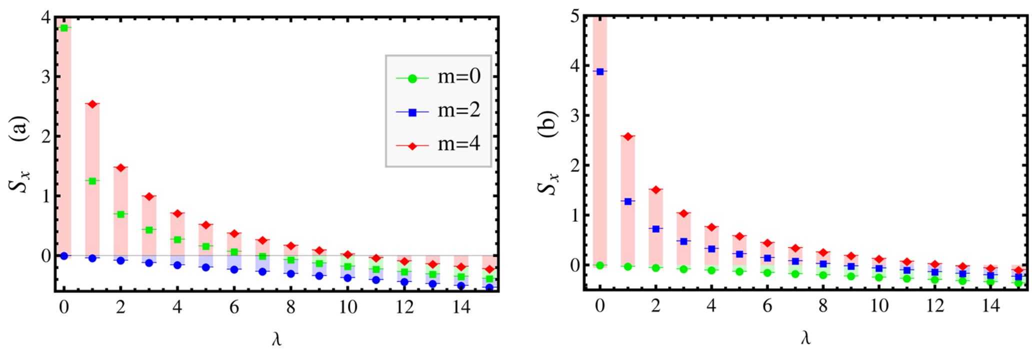

Fig. 10. S_x as a function of λ for the excited para-Bose states with different deformation parameter m by setting (a) $z = 0.08$ and (b) $z = 0.1$.

4 Generation of the excited para-Bose states $|\mathbf{z}, \boldsymbol{\lambda}, \mathbf{m}\rangle$

We have already mentioned that in a trapped-ion setup whose effective dynamics is equivalent to a pseudo-harmonic oscillator, $|z, \lambda\rangle$ can be generated through an effective dynamics of the vacuum state [63]. Now, we propose a theoretical framework to generate $|z, \lambda, m\rangle$ in practice. We consider a two-level atom interacting simultaneously with a quantized electromagnetic field and a centrifugally external classical field which is described by the following interaction Hamiltonian [59]

$$H_{int} = \hbar g \left(\mathbf{a}^\dagger \sigma_- + \mathbf{a}\, \sigma_+ \right). \tag{31}$$

Here, g is referred to the coupling constant between the atom and quantized field. The operators σ_+ and σ_- denote the atomic lowering and raising operators, respectively. We assume that the state of system is initially prepared as

$$|\Psi(0)\rangle = |z, \lambda\rangle\, |e\rangle, \tag{32}$$

where $|e\rangle$ is the excited state of the atom. After a small enough time, the state of system is evolved to

$$|\Psi(t)\rangle = |z, \lambda\rangle\, |e\rangle - igt\, \mathbf{a}^\dagger\, |z, \lambda\rangle\, |g\rangle, \tag{33}$$

where $|g\rangle$ is the ground state of atom. Thus, if the atom is detected in a ground state, the state of the field will be collapsed to $\mathbf{a}^\dagger |z, \lambda\rangle$. By iterating the above mentioned process, finally, the field state becomes $\mathbf{a}^{\dagger\, m} |z, \lambda\rangle$, as discussed above.

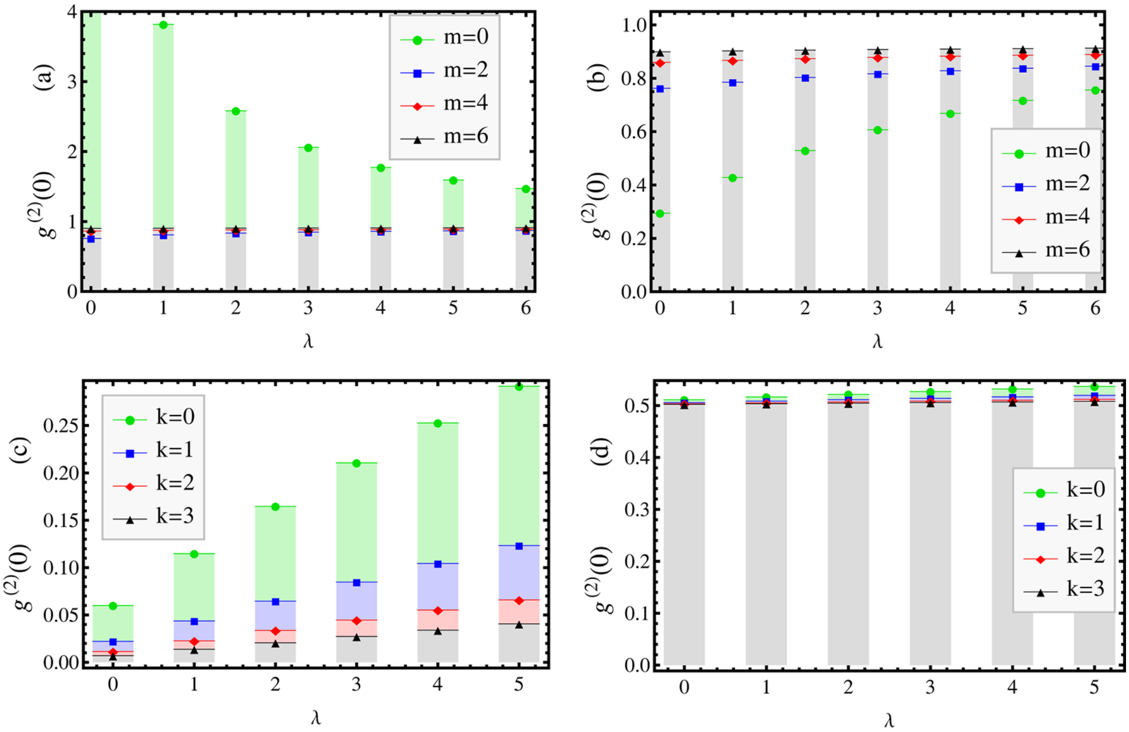

Fig. 11. $g^{(2)}(0)$ as a function of λ is shown for $|z, \lambda, m\rangle_k^{(+)}$ in (a) and (c). $g^{(2)}(0)$ as a function of λ is shown for $|z, \lambda, m\rangle_k^{(-)}$ in (b) and (d). In (a) and (b), $k = 1$ and $z = 1$; in (c) and (d) $m = 1$ and $z = 1$.

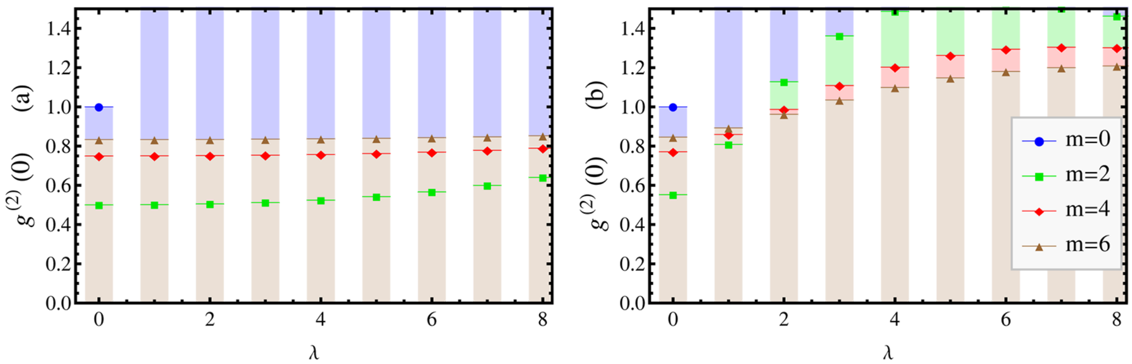

Fig. 12. $g^{(2)}(0)$ as a function of λ is shown for $|z, \lambda, m\rangle^{(e)}$ by setting (a) $z = 0.1$ and (b) $z = 0.2$.

5 Conclusions

In summary, we introduced excited states associated with a para-Bose oscillator. We showed that these states are finite-number superposition of a wide class of generalized excited para-Bose cats in which, the excited para-Bose cats can be considered as the last state of the superposition. The over-completeness of generalized excited para-Bose cat states have been established through positive definite measures. We continued with the study of nonclassical features of the excited para-Bose states and generalized excited para-Bose cats through investigation of normal order squeezing, sub-Poissonian statistics and second order correlation function. Furthermore, we studied the Klyshko's criterion of non-classicality. We found that the non-classicality of the introduced states can be controlled by the coherent parameter, z and excitation number, m. Meanwhile, a comparative study of excited CS (excited cats) and excited para-Bose states (generalized excited para-Bose cats) led to the following conclusions:

i) In contrast to the excited CS which always show nonclassical statistics on their Fock state distribution, excited para-Bose states show both nonclassical and semiclassical statistics.
ii) Although, the possibility of observing squeezing phenomena does not exist for excited cats, the generalized excited para-Bose cats exhibit squeezing for specific values of deformation and coherent parameters.

Finally, we proposed an experimentally feasible scheme to generate the excited para-Bose states.

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