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Dynamics of axially symmetric anisotropic modified holographic Ricci dark energy model in Brans-Dicke theory of gravitation

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Abstract. In this paper, we have derived field equations of Brans-Dicke (Phys. Rev. **124**, 925 (1961)) theory of gravitation with the help of an axially symmetric anisotropic Bianchi-type space-time in the presence of dark matter and anisotropic modified holographic Ricci dark energy. We have presented a cosmological model solving the field equations. We have used i) the hybrid expansion law, ii) a relation between metric potentials and iii) the modified holographic Ricci dark energy defined by Chen and Jing (Phys. Lett. B **679**, 144 (2009)) to solve the field equations. We have determined the cosmological parameters, namely, EoS parameter, matter energy density, anisotropic dark energy density, Skewness parameter, deceleration and jerk parameters. A detailed physical discussion of these dynamical parameters is presented through a graphical representation. We observe that we have a quintessence model which exhibits a smooth transition from decelerated phase to an accelerated phase of the universe. This situation is quite in agreement with the scenario of modern cosmology.

1 Introduction

The accelerated expansion of our universe is established by the recent cosmological observations [1,2]. It is said that an exotic type of unknown force with positive energy density and huge negative pressure known as "dark energy" (DE) is responsible for this cosmic acceleration. Although around 70% of the mass and energy content of our universe is occupied with this type of DE, its nature, even today, is an open problem in cosmology. Usually the behaviour of DE phenomena is characterized with the equation of state (EoS) parameter $\omega(= p/\rho)$, where p is pressure and ρ is energy density. The EoS parameter ω lying in the range $(-1, -1/3)$ represents the quintessence DE model, $\omega = -1$ describes the vacuum DE, commonly known as cosmological constant or ΛCDM model and $\omega < -1$ describes the DE model known as the phantom model. This phantom DE model can lead to a future unavoidable singularity of the space-time.

Cosmologists believe that the simplest DE model is the cosmological constant but it faces some theoretical problems [3,4] like "fine tuning problem" (some precisely small value) and "coincidence problem" (why dark matter and DE are of almost the same order at the present epoch even though the universe is expanding?). Hence, physicists from all over the globe have proposed various dynamical DE models, considered as an alternative way to solve these problems. Mainly these models are classified into two categories: scalar field models which include the quintessence, phantom, k-essence, tachyon, quintom [5–10] and the interacting DE models including the family of Chaplygin gas, braneworld, holographic DE (HDE), agegraphic DE models [11–16], etc. In another scenario, modification of standard Einstein-Hilbert action results in various modified theories of gravity. Some of the modified theories of gravity are Brans-Dicke scalar-tensor theory [17], $f(R)$ and $f(R,T)$ theories [18–20], (where R is the curvature scalar and T is the trace of the energy momentum tensor), etc. Even after all these attempts, the origin, evolution and true nature of DE have not been convincingly explained yet.

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Among the above-mentioned candidates for DE, a very interesting model which is attracting more and more attention is the so-called "holographic dark energy (HDE)". This is based on the holographic principle which states that the number of degrees freedom of a physical system should scale with its bounding area L^2 rather than with its volume [21] and it should be constrained by an infrared (IR) cut off. Cohen *et al.* [22], motivated by this principle, suggested that the vacuum energy density is proportional to the Hubble scale $L \approx H^{-1}$. Li [23] has defined the energy density of holographic dark energy as

$$\rho_\Lambda = 3c^2 M_{pl}^2 L^{-2}, \tag{1}$$

where L is the IR cut off radius, c is a constant and

$$M_{pl}^2 = \frac{1}{8\pi G} \tag{2}$$

is the Planck mass. The IR cut off has been considered as the Hubble radius or future event horizon. Later Gao *et al.* [24] have assumed that the future event horizon is replaced by the inverse of the Ricci scalar curvature, *i.e.*, $L \approx |R|^{-\frac{1}{2}}$. In this case the model is called Ricci dark energy model. Also Granda and Oliveros [25,26] proposed a new holographic Ricci dark energy model with energy density given by

$$\rho_\Lambda = \frac{3}{8\pi G}\left(\xi H^2 + \eta \dot{H}\right), \tag{3}$$

and, subsequently, Chen and Jing [27] have modified this model and named it as modified holographic Ricci dark energy (MHRDE) with energy density given by

$$\rho_\Lambda = \frac{3}{8\pi G}\left(\xi H^2 + \eta \dot{H} + \zeta \ddot{H} H^{-1}\right), \tag{4}$$

where an overhead dot indicates differentiation with respect to t and G is the gravitational constant which is taken as a function of time.

The time variation of the gravitational constant G is the natural consequence of Dirac's "Large Number Hypothesis" [28]. Inspired by this hypothesis, many attempts have been made to obtain physical and cosmological results about the universe. The idea of Brans-Dicke (BD) scalar-tensor theory [17] and its generalization to other forms of scalar-tensor theories like general scalar-tensor theories with the special conditions given by Wagoner [29] arose from variable-G theories. In these theories the inverse of gravitational constant G is replaced by a scalar field ϕ coupling to gravity through a new parameter. Further, it was proven through various observations that G could be a function of time [30–32]. Setare [33,34] and Jamil *et al.* [35] have investigated interacting HDE models with or without varying G in order to explain the current status of the universe. Sharif and Jawad [36] have explored interacting modified HDE in the Kaluza-Klein universe.

BD theory introduces a scalar field ϕ, in addition to the metric tensor fields g_{ij}, which plays the role of inverse of the gravitational constant G. The gravitational field equations of BD theory for the combined scalar and tensor fields are given by

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - w\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) - \phi^{-1}\left(\phi_{i;j} - g_{ij}\phi_{;k}{}^{,k}\right) \tag{5}$$

and

$$\phi_{;k}{}^{,k} = 8\pi(3+2w)^{-1}T, \tag{6}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor, R is the scalar curvature, w is the BD coupling parameter T_{ij} is the stress energy tensor of the matter and the comma and semicolon denote partial and covariant differentiations, respectively.

Also, we have the energy conservation equation

$$T^{ij}{}_{,j} = 0. \tag{7}$$

Since the family of HDE density belongs to a dynamical cosmological model, a dynamical frame instead of Einstein general theory is necessary to describe it. Hence it is justified to study the family of HDE models in the framework of BD theory.

Recent experimental data and theoretical arguments support the existence of an amount of anisotropy in the early stage of evolution of the universe which evolves into an isotropy at late times. Hence it becomes necessary to investigate the evolution of the universe with the anisotropic background. Since Bianchi-type space-times are spatially homogeneous and anisotropic, several Bianchi-type cosmological models both in general relativity and in modified theories of gravitation have been studied in the presence of different physical matter distributions. In particular, Kiran *et al.* [37] have obtained minimally interacting Bianchi type-V dark energy models in BD scalar-tensor theory of gravitation. Adhav *et al.* [38] have studied interacting holographic dark energy models in Bianchi space-times. Recently, many authors in the literature have investigated HDE and MHRDE cosmological models within the framework of Bianchi space-times using the constant deceleration parameter and the time varying deceleration parameter [39–42]. Jawad *et al.* [43] have discussed MHRDE in Chameleon BD cosmology with non-minimally matter coupling of the scalar field and its thermodynamic consequence. Rao and Prasanthi [44] have explored some Bianchi-type MHRDE models in Saez-Ballester scalar-tensor theory of gravity with a variable deceleration parameter. Santhi *et al.* [45,46] have studied the Bianchi-type-III and -VI$_0$ MHRDE model within the framework of general relativity and BD theory, and discussed various dynamical properties of the models. Reddy [47] has discussed the Bianchi-type-V MHRDE model of the universe in Saez-Ballester scalar-tensor theory of gravitation for different dynamical average scale factors.

Motivated by the above discussion, we investigate, in this paper, the axially symmetric Bianchi-type modified holographic Ricci dark energy model in BD theory of gravitation. Our model has not been, so far, considered in the literature. Also, this model gives a clear and simple cosmological evolution of the dark energy universe. The outline of this paper is as follows: in sect. 2 we derive the BD field equations. Section 3 deals with the solution and presentation of the model. Section 4 is concerned with physical and kinematical parameters of the model, and their physical discussion using graphical representations is presented in sect. 5. The last section contains our concluding remarks.

2 Brans-Dicke field equation

In this section, we re-write BD field equations in the presence of anisotropic matter distribution and modified holographic Ricci dark energy and derive them explicitly with the help of anisotropic axially symmetric metric given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - X^2\mathrm{d}x^2 - Y^2\left(\mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{8}$$

where X and Y are functions of cosmic time t only.

In this particular case, BD field equations (5)–(7) take the form

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi\phi^{-1}\left(T'_{ij} + \overline{T}_{ij}\right) - w\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) - \varphi^{-1}\left(\phi_{i;j} - g_{ij}\phi_{,k}{}^{,k}\right), \tag{9}$$

where T'_{ij} and $\overline{T}_{ij}$ are energy-momentum tensors for matter and modified holographic Ricci dark energy respectively which are defined as

$$T'_{ij} = \rho_M u_i u_j, \quad i,j = 1,2,3,4, \tag{10}$$

$$\overline{T}_{ij} = (p_\Lambda + \rho_\Lambda)u_i u_j - p_\Lambda g_{ij}, \tag{11}$$

where ρ_M is the matter energy density, p_Λ is the energy density of the modified holographic Ricci dark energy and the other symbols have their usual meaning.

Now, parameterizing, we have, from eq. (11)

$$\begin{aligned}\overline{T}_i^j &= \mathrm{diag}[-1, \omega_x, \omega_y, \omega_z]\rho_\Lambda\\ &= \mathrm{diag}[-1, \omega, (\omega+\delta), (\omega+\gamma)]\rho_\Lambda,\end{aligned} \tag{12}$$

where ω we have used the EoS parameter given by

$$\omega\rho_\Lambda = P_\Lambda, \tag{13}$$

and ω_x, ω_y, ω_z are the directional EoS parameters along the x-, y- and z-axis, respectively. For the sake of simplicity we choose $\omega_x = \omega$ and the skewness parameters δ and γ are the deviations from ω on the y and z axes, respectively. Also the assumed axial symmetry yields

$$\delta = \gamma. \tag{14}$$

The scalar field satisfies the wave equation

$$\phi^{;k}_{;k} = \frac{8\pi}{3+2w}(T' + \overline{T}) \tag{15}$$

and the energy conservation equation takes the form

$$(T'_{ij} + \overline{T}_{ij})_{;j} = 0. \tag{16}$$

Now using a co-moving coordinate system and eqs. (9)–(16) we obtain the field equations of BD theory for the metric (1) in the following explicit form:

$$\frac{\ddot{X}}{X}+\frac{\ddot{Y}}{Y}+\frac{\dot{X}\dot{Y}}{XY}-\frac{w}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\ddot{\phi}}{\phi}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}\right)=-8\pi\phi^{-1}(\omega+\delta)\rho_\Lambda, \tag{17}$$

$$2\frac{\ddot{Y}}{Y}+\frac{\dot{Y}^2}{Y^2}-\frac{w}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\ddot{\phi}}{\phi}-2\frac{\dot{\phi}}{\phi}\frac{\dot{Y}}{Y}=-8\pi\phi^{-1}\omega\rho_\Lambda, \tag{18}$$

$$2\frac{\dot{X}\dot{Y}}{XY}+\frac{\dot{Y}^2}{Y^2}+\frac{w}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)=8\pi\phi^{-1}(\rho_M+\rho_\Lambda), \tag{19}$$

$$\ddot{\phi}+\dot{\phi}\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)=8\pi(3+2w)^{-1}(\rho_M+\rho_\Lambda-(3\omega+2\delta)) \tag{20}$$

and the conservation equation takes the form

$$\dot{\rho}_M+\dot{\rho}_\Lambda+\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)\rho_M+\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)(1+\omega)\rho_\Lambda+2\rho_\Lambda\delta\frac{\dot{Y}}{Y}=0. \tag{21}$$

Here an overhead dot denotes differentiation with respect to cosmic time t.

3 Modified holographic Ricci dark energy model

In this section, we present a modified holographic Ricci dark energy model by solving the field equations using some physically valid conditions.

The field equations (17)–(21) are a set of four independent equations (the conservation eq. (21) being the consequence of eqs. (17)–(20)) in seven unknowns X, Y, ϕ, ρ_M, ρ_Λ, ω and δ. Hence to get a determinate solution we need three more conditions. Therefore we use the following physically viable conditions which have been extensively used in the literature.

i) The shear scalar σ^2 is proportional to the scalar expansion θ, which gives us [48]

$$X=cY^k, \tag{22}$$

where c is a constant which can be set equal to unity without loss of generality so that we have

$$X=Y^k. \tag{23}$$

ii) We use the hybrid law expansion for the average scale factor given by Akarsu *et al.* [49] as

$$a(t)=(XY^2)^{\frac{1}{3}}=a_0t^{\alpha_1}e^{\alpha_2 t}, \tag{24}$$

where α_1 and α_2 are non-negative constants. Here, when $\alpha_1=0$ we get the exponential law and when $\alpha_2=0$ we obtain power law. Thus, eq. (24) gives the combination of exponential and power law which is usually known as hybrid expansion law. This choice of average scale factor leads to a time-dependent deceleration parameter. The solution gives the inflation and radiation dominance era with subsequent transition from the decelerating to the accelerating phase of the universe. This type of average scale factor has already been considered by many authors [50,51].

iii) We use modified holographic Ricci dark energy defined by eq. (4) (using the fact that ϕ plays the role of G^{-1} in BD theory) in the form

$$\rho_\Lambda=\frac{3\phi}{8\pi}\left(\xi H^2+\eta\dot{H}+\zeta\ddot{H}H^{-1}\right). \tag{25}$$

Also, since the field equations are highly non-linear we use a power law between BD scalar field and the average scale factor of the universe in the form

$$\phi\,\alpha\,a^m,$$

so that

$$\phi=\phi_0a^m, \tag{26}$$

where ϕ_0 and m are positive constants. This power law has been used by Johri and Sudharsan [52] and Johri and Desikan [53] to study the evolution of the universe in BD theory.

Now from eqs. (23) and (24) we obtain the metric potentials as

$$X = \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^{\frac{3k}{k+2}}, \qquad Y = \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^{\frac{3}{k+2}}, \tag{27}$$

so that the metric (8) can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^{\frac{6k}{k+2}} \mathrm{d}x^2 - \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^{\frac{6}{k+2}} \left(\mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{28}$$

Also, from eqs. (24) and (26) the BD scalar field in the model is given by

$$\phi = \phi_0 \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^m. \tag{29}$$

Hence the axially symmetric modified holographic Ricci dark energy model in BD theory is given by the space-time (28) with the BD scalar field given by eq. (29).

4 Cosmological parameters of the model

We shall now define and determine the cosmological parameters of the universe given by eqs. (28) and (29), which play an important role in cosmology. We shall also discuss the physical behaviour of these parameters using graphical representation.

The spatial volume of the universe given by eq. (28) is

$$V = a^3(t) = XY^2 = \left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^3. \tag{30}$$

The average Hubble parameter is

$$H = \frac{1}{3}\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right) = \left(\frac{\alpha_1}{t} + \alpha_2\right). \tag{31}$$

The scalar expansion in the universe is

$$\theta = 3H = \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y} = \left(\frac{\alpha_1}{t} + \alpha_2\right). \tag{32}$$

The shear scalar is

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{3}\left[\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right]^2 = 3\left(\frac{k-1}{k+2}\right)^2 \left(\frac{\alpha_1}{t} + \alpha_2\right)^2. \tag{33}$$

The anisotropy parameter is

$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = 2\left(\frac{k-1}{k+2}\right)^2. \tag{34}$$

Using eq. (32), the deceleration parameter is obtained as

$$q = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right) - 1 = -1 + \frac{\alpha_1}{(\alpha_1 + \alpha_2 t)^2}. \tag{35}$$

The cosmic jerk parameter j in cosmology is defined as the third derivative of the scale factor with respect to the cosmic time and is given by [54]

$$j(t) = \frac{1}{H^3}\frac{\dddot{a}}{a} = q + 2q^2 - \frac{\dot{q}}{H}. \tag{36}$$

This is used to discuss the models close to ΛCDM in cosmology. It is believed that the transition of the universe from decelerated phase to accelerated phase occurs for models with negative value of the deceleration parameter and positive value of the jerk parameter. The jerk parameter for the ΛCDM model has a constant jerk, $j = 1$. In this model, we get

$$j(t) = 1 - \frac{3\alpha_1}{(\alpha_1 + \alpha_2 t)^2} + \frac{2\alpha_1}{(\alpha_1 + \alpha_2 t)^3}. \tag{37}$$

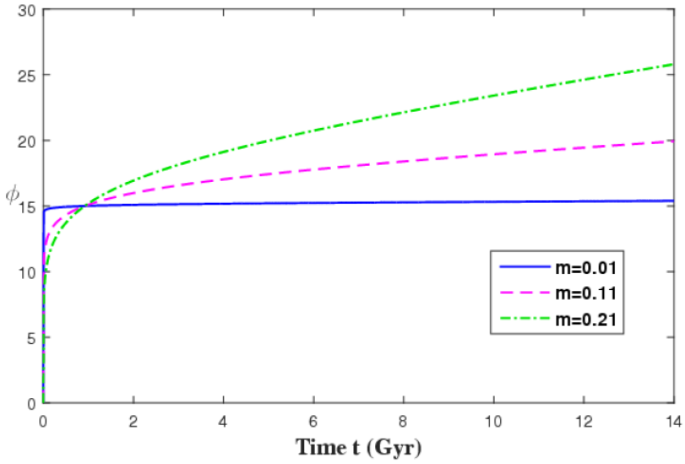

Fig. 1. Plot of BD scalar field *versus* cosmic time t for $a_0 = 1$, $\phi_0 = 15$, $\alpha_1 = 0.66$ and $\alpha_2 = 0.06$.

From eqs. (25) and (32) the energy density of the modified holographic dark energy in the universe is obtained as

$$\rho_\Lambda = \frac{3\phi_o\left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^m}{8\pi}\left[\left(\xi\left(\frac{\alpha_1}{t}+\alpha_2\right)^2 - \eta\frac{\alpha_1}{t^2} + 2\zeta\frac{\alpha_1}{t^3}\right)\right]. \tag{38}$$

The energy density of matter can be obtained from eqs. (19), (28), (29) and (38) as

$$\rho_M = \frac{3\phi_o\left(a_0 t^{\alpha_1} e^{\alpha_2 t}\right)^m}{8\pi}\left(\frac{\alpha_1}{t}+\alpha_2\right)^2\left[\frac{[18(2k+1)+6m(k+2)^2+wm^2(k+2)^2]}{2(k+2)^2}\right] - \left(\xi\left(\frac{\alpha_1}{t}+\alpha_2\right)^2 - \eta\frac{\alpha_1}{t^2} + 2\zeta\frac{\alpha_1}{t^3}\right). \tag{39}$$

From eqs. (17), (18), (28), (29) and (38) the skewness parameters are given by

$$\delta = \gamma = \frac{(k-1)[\alpha_1 t - t(\alpha_1 t + \alpha_2)^2(m+3)]}{(k+2)[\xi t(\alpha_1 + \alpha_2 t)^2 - \eta\alpha_1 t + 2\zeta\alpha_1]}. \tag{40}$$

From eqs. (18), (28), (29) and (38) we get the EoS parameter in the model as

$$\omega = -\left[\frac{t(\alpha_1+\alpha_2 t)^2[6(2m+9)+m^2(k+2)^2(2-w)] - 2\alpha_1 t(k+2)[m(k+2)+6]}{6(k+2)^2[\xi t(\alpha_1+\alpha_2 t)^2 - \eta\alpha_1 t + 2\zeta\alpha_1]}\right]. \tag{41}$$

5 Physical discussion of the results

The above results facilitate the discussion of the physical behavior of the model (29). It may be observed that the spatial volume increases with time. This confirms the spatial expansion of the universe. It can be seen that the above physical parameters like θ, σ^2 and H of the model diverse at initial epoch, *i.e.*, at $t = 0$ while all of them converges to α_2 as t approaches infinity. It may also be noted that when $k = 1$, the anisotropy parameter vanishes and the model becomes isotropic and shear free at late times.

The behaviour of the BD scalar field ϕ *versus* cosmic time is shown in fig. 1. It can be seen that this scalar field increases with time for all three values of m. Hence the corresponding kinetic energy of the model decreases and approaches to zero with the passage of time. In fig. 2, we have plotted the MHRDE and matter energy densities in terms of cosmic time t. It is observed that both energy densities are positive throughout the evolution, attain a constant value at the present epoch and vanish in the future. Also, we observe that the energy density of MHRDE ρ_Λ is almost not affected by the BD scalar field ϕ. But the energy density of matter increases with the BD scalar field and has no effect in the present epoch. Figure 3 represents the behaviour of the skewness parameter with cosmic time t for different values of m. It is observed the skewness parameter is always positive and vanishes for $k = 1$. We also observed that the skewness parameter increases as the BD scalar field increases.

The EoS parameter for $m = 0.01, 0.11, 0.21$ with respect to cosmic time t is shown in fig. 4. Also, the constants $a_0 = 1$, $\phi_0 = 15$, $k = 0.98$, $\xi = 1.5$, $\eta = 0.5$ and $\zeta = 0.01$ have been taken such that the energy densities remain positive throughout the evolution. It is observed that the EoS parameter of our MHRDE model is a decreasing function of time and later on it attains a constant value. As the BD scalar field ϕ increases the EoS parameter ω moves towards

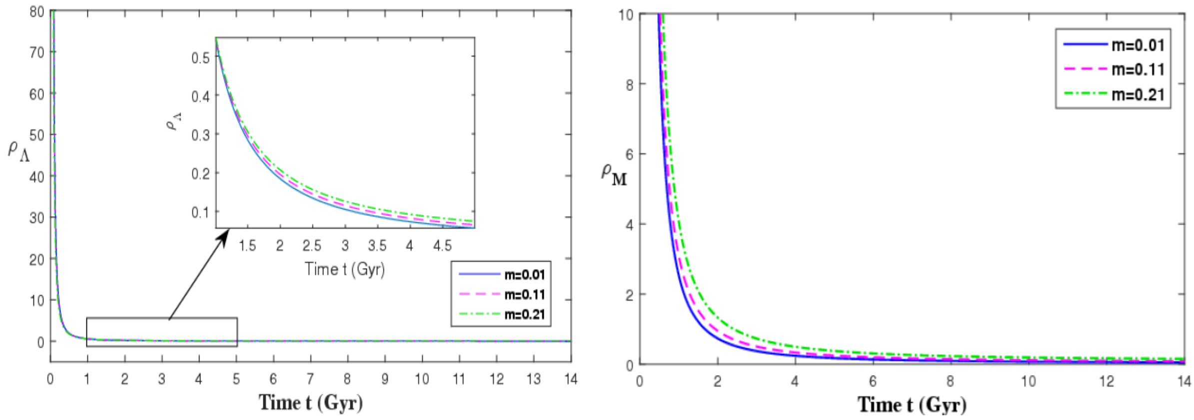

Fig. 2. Plot of energy densities *versus* time t for $k = 0.98$, $a_0 = 1$, $\phi_0 = 15$, $\alpha_1 = 0.66$, $\alpha_2 = 0.06$, $\xi = 1.5$, $\eta = 0.5$ and $\zeta = 0.01$.

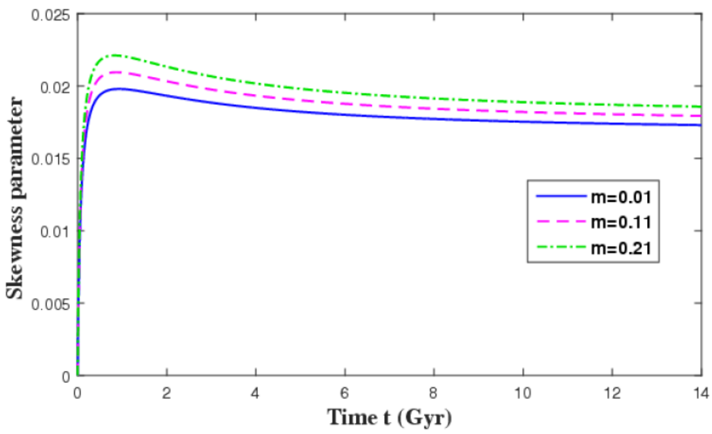

Fig. 3. Plot of skewness parameters *versus* time t for $k = 0.98$, $a_0 = 1$, $\phi_0 = 15$, $\alpha_1 = 0.66$, $\alpha_2 = 0.06$, $\xi = 1.5$, $\eta = 0.5$ and $\zeta = 0.01$.

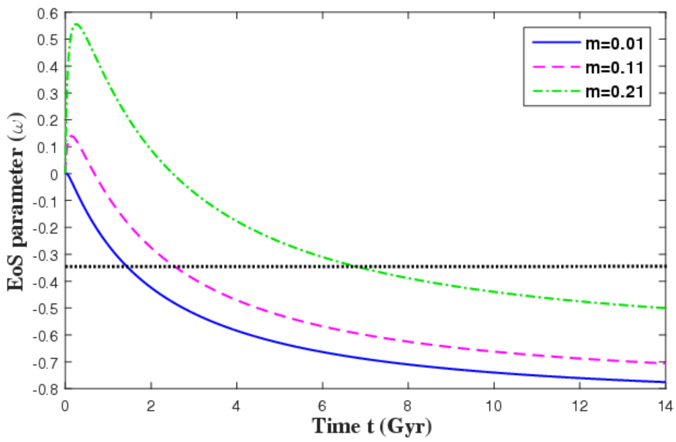

Fig. 4. Plot of the EoS parameter *versus* time t for $k = 0.98$, $a_0 = 1$, $\phi_0 = 15$, $\alpha_1 = 0.66$, $\alpha_2 = 0.06$, $\xi = 1.5$, $\eta = 0.5$ and $\zeta = 0.01$.

a lower quintessence region. It can be seen that at the very early stages of the universe the EoS parameter $\omega > 0$. In this case the EoS parameter of our MHRDE model may be playing an important role to represent the earlier standard matter-dominated era (*i.e.*, the deceleration phase) of the universe. Again, it is found that at some particular point of time, the EoS parameter becomes zero (dust-dominated universe). After that, the EoS of MHRDE enters into the negative region and attains a constant value in the quintessence region ($-1 < \omega < -1/3$) for all values of m, leading to an accelerated expansion phase. It is worthwhile to mention here that the EoS parameter of our MHRDE model in BD

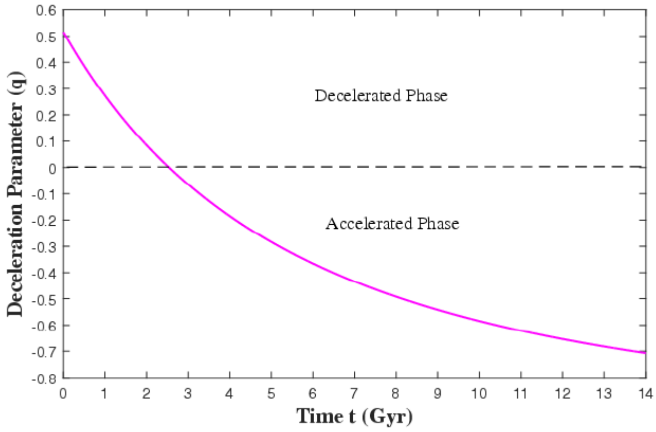

Fig. 5. Plot of the deceleration parameter *versus* time t for $\alpha_1 = 0.66$ and $\alpha_2 = 0.06$.

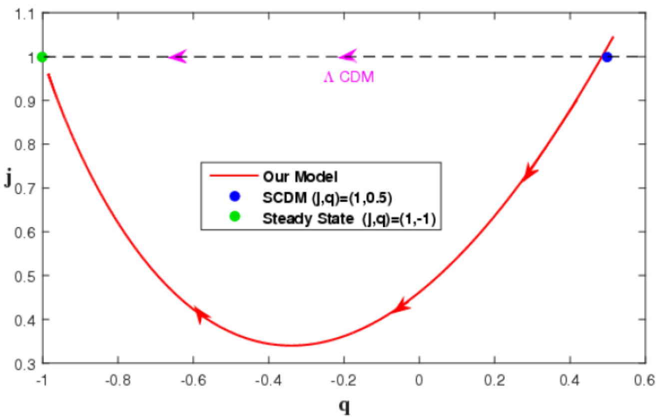

Fig. 6. The evolution of the MHRDE model in the j-q plane for $\alpha_1 = 0.66$ and $\alpha_2 = 0.06$.

scalar-tensor theory corresponds to the quintessence era of the universe which is a favourable sign to the scalar-field conjecture. The work of Sarkar [55,56] on Bianchi-type-I and -V interacting DE models of the universe with variable deceleration parameter also supports the behaviour of the EoS parameter of our model. Das and Sultana [57] have studied the Bianchi-type-VI$_0$ MHRDE model in general relativity with sign-changeable interaction and obtained a similar result from an analysis of the EoS parameter. Rao and Prasanthi [44] have discussed Bianchi-type-I and -III MHRDE models in Saez-Ballester scalar-tensor theory of gravitation, and they have obtained a model which starts the evolution from the phantom region and ultimately reaches the quintessence region. Also, Mishra *et al.* [58] found a quite similar result from the analysis of the EoS parameter in their study of Bianchi-type-V string cosmological model with anisotropic distribution of DE. It can be observed that the behaviour of the MHRDE model obtained, here, is in good agreement with the current observational data [59–61].

Figure 5 depicts the variation of the deceleration parameter with respect to cosmic time t. It can be seen that the deceleration parameter is a decreasing function of time, and varies in both positive and negative regions. Modern observational data [62] indicates that the universe is accelerating and the value of the deceleration parameter lies within the range $-1 < q < 0$. In our MHRDE model the constants $\alpha_1 = 0.66$ and $\alpha_2 = 0.06$ have been chosen so that there is a smooth transition of the model from early deceleration to late time acceleration.

Figure 6 shows the evolution of our MHRDE model in the j-q plane. Here, the blue and green dots represent the fixed points $\{j, q\} = \{1, -1\}$ and $\{j, q\} = \{1, 0.5\}$ for Steady State (SS) and standard cold dark matter ($SCDM$) models, respectively. The dotted line at $j = 1$ explains the time evolution of the ΛCDM model. It can be seen from the j-q trajectory that there is a sign change in the deceleration parameter (q), *i.e.*, from positive to negative in the quintessence region. Our MHRDE model starts from the $SCDM$ model and approaches to the SS model at late times. It can also be observed from the j-q trajectory that our MHRDE model in BD theory exhibits almost similar behaviour like the quintessence scalar field model [63,64], which approaches asymptotically to the ΛCDM model at late times.

6 Final remarks

The main motive of this paper is to discuss the behavior of the modified holographic Ricci dark energy model in axially symmetric anisotropic Bianchi-type space-time within the framework of Brans-Dicke scalar-tensor theory of gravitation. We have presented a deterministic cosmological model by solving the field equations of BD theory in the presence of matter and anisotropic MHRDE using hybrid expansion law for the scale factor of the universe. The dynamical parameters of the MHRDE model are determined and their physical interpretation is presented through a graphical representation, and summarized them in the following.

The BD scalar-field increases with cosmic time t (fig. 1). The energy densities of matter and MHRDE are positive decreasing functions of cosmic time and satisfy the null energy condition, *i.e.*, $\rho_M \geq 0$ and $\rho_\Lambda \geq 0$ (fig. 2). The skewness parameter is always positive, increases with the increase in BD scalar field and also vanishes for $k = 1$. The EoS parameter of our MHRDE model starts in the matter-dominated ($\omega > -1/3$) region at the early epoch, crosses dust dominated era ($\omega = 0$) of the universe and ultimately attains a constant value in the quintessence region ($-1 < \omega < -1/3$). Also, it is observed that the EoS parameter never crosses the phantom divided line ($\omega = -1$) and hence we have a quintessence model. But at late times it approaches to the ΛCDM model ($\omega = -1$) (fig. 4). This behaviour of the EoS parameter is in good agreement with the recent observational data [57–59]. We observe that there is a phase transition of our universe from the early deceleration to the present acceleration, since the deceleration parameter is time-dependent and exhibits a signature change from positive to negative (fig. 5). In our model, the jerk parameter is also a function of cosmic time t. The trajectory of the j-q plane is plotted to discriminate our MHRDE model with the existing DE models. It is observed that the j-q trajectory starts from $SCDM$ model at an early time and approaches to the SS model at the later epoch. We noticed that the j-q trajectory of our MHRDE model exhibits a similar behaviour of quintessence model [63,64]. It is interesting to mention, here, that our results of the EoS parameter and the j-q trajectory favour the quintessence behaviour of the model. This type of behaviour of the MHRDE model favours the scalar-field hypothesis. Also, it is observed, from the above results, that our MHRDE model approaches the ΛCDM model at late times since $\omega \to -1$ and $(j, q) = (1, -1)$.

A very interesting feature of our model is the following: usually, in the discussion of DE models we obtain the quintessence ($-1 < \omega < -1/3$) or $\Lambda CDM(\omega = -1)$ models. But, in our case, it can be observed from the j-q and EoS trajectories that we are obtaining both the quintessence model in the present epoch and the ΛCDM model at late times.

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