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Quantum Markov processes: From attractor structure to explicit forms of asymptotic states

Asymptotic dynamics of quantum Markov processes

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Abstract. Markov processes play an important role in physics and in particular in the theory of open systems. In this paper we study the asymptotic evolution of trace-nonincreasing homogeneous quantum Markov processes (both types, discrete quantum Markov chains and continuous quantum Markov dynamical semigroups) equipped with a subinvariant faithful state in the Schrödinger and the Heisenberg picture. We derive a fundamental theorem specifying the structure of the asymptotics and uncover a rich set of transformations between attractors of quantum Markov processes in both pictures. Moreover, we generalize the structure theorem derived earlier for quantum Markov chains to quantum Markov dynamical semigroups showing how the internal structure of generators of quantum Markov processes determines attractors in both pictures. Based on these results we provide two characterizations of all asymptotic and stationary states, both strongly reminding in form the well-known Gibbs states of statistical mechanics. We prove that the dynamics within the asymptotic space is of unitary type, *i.e.* quantum Markov processes preserve a certain scalar product of operators from the asymptotic space, but there is no corresponding unitary evolution on the original Hilbert space of pure states. Finally simple examples illustrating the derived theory are given.

1 Introduction

Any physical system is inevitably in contact with its surrounding. This may be intended or even designed with the purpose to achieve external control over quantum phenomena (quantum control theory [1]). However, it is always accompanied by unavoidable mutual interactions between the system and its surrounding typically leading to information and energy system-environment exchange. While the former is responsible for decoherence —the loss of quantum coherences [2], the latter may cause dissipation [3]. In all cases it breaks the unitary evolution of individual subsystems and the resulting irreversible open dynamics of the system of interest becomes highly involved and in most cases escapes the possibility for analytical solutions. Exceptions are rare and often assume additional conditions like the representation of the surrounding as a thermal bath of harmonic oscillators [4]. A recent review mapping the scarcely occupied space of exactly solvable quantum many-body models can be found in [5].

In order to avoid the overall complexity of composed system-environment dynamics some simplifying assumptions are often applied [6]. One of the most convenient approaches focuses on quantum systems whose evolution can be described to sufficient extent by Markovian dynamics [7]. In such a case the system stays uncorrelated with the environment and changes arising in the surrounding environment can be neglected. Basically two large classes of Markovian processes are at hand. Both classes of quantum Markov processes (QMPs), continuous quantum Markov dynamical semigroups (QMDSs) [8,9] and discrete quantum Markov chains (QMCHs) [10] are frequently employed to investigate a broad class of processes like the equilibration of quantum systems [11], interaction of matter with electromagnetic radiation [6,7], decoherence effects in noisy environment [12]. As a complement, engineered Markovian dynamics provides a tool allowing for example the preparation of a system in the desired quantum state [13–15], its storage and manipulation [16,17], verification of quantum programs [18], the synchronization of subsystems clocks [19] or environment-assisted quantum transport [20–22].

Despite the simplifications made towards the Markovian regime, a full solution of quantum Markovian evolution constitutes in general a challenging task. However, a large class of problems of interest, like the already mentioned

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problem of decoherence free states [23], equilibrium states, transport efficiency or synchronization of subsystems, may be resolved at the level of asymptotic dynamics. Unlike the rest the of dynamics, the asymptotic part is fully determined by the asymptotic (peripheral) spectrum of the relevant Markovian generator and its associated eigenvectors (attractors) forming the attractor (asymptotic) space. Understanding what structures attractors form, their mutual algebraic properties or how they are determined via the internal structure of Markovian generators is of central interest. Due to their relevance for the growing field of quantum thermodynamics and quantum cryptography, intensive research is devoted to the analysis of stationary states and related fixed points of QMPs. The existence of a unique invariant state for irreducible trace-preserving QMPs was investigated within the Perron-Frobenius theory in infinite-dimensional (respectively, in finite-dimensional) Hilbert spaces [24] (respectively [25]). Based on irreducible decomposition, a characterization of asymptotic states was provided in [26] for trace-preserving QMPs in finite-dimensional Hilbert spaces and recently it was generalized for stationary states of QMCHs in infinite-dimensional Hilbert spaces [27]. How the internal structure of QMPs determines the set of fixed points was investigated for unital channels in infinite-dimensional (respectively, finite-dimensional) Hilbert spaces [28] (respectively [29]), for trace-preserving QMDSs equipped with a faithful invariant state [30, 31] and lately for any quantum channel [32] and quantum Markov dynamical semigroup [33]. The structure analysis of the whole attractor space emerging from QMPs was carried out for quantum operations equipped with a faithful invariant state [34] and for trace-preserving QMDSs [35].

Another equally important issue concerns the mutual relationship between attractors in the Schrödinger and Heisenberg picture. As the former contains all stationary states and the latter all conserved quantities of the evolution, it plays a crucial role in our understanding how conserved quantities determine the resulting stationary state [36]. In closed, unitary dynamics both sets of attractors coincide and the same applies to QMCHs generated by unital channels [37]. However, attractors in the Schrödinger and the Heisenberg picture of QMCHs equipped with a general faithful invariant state are in general different and a simple algebraic relation between them was presented in [34].

In this paper we reveal that this relation [34] is just a particular example from a whole family of mutual relations among the two sets of attractors and we show that it applies to all quantum Markov processes (including trace-nonincreasing processes and quantum Markov dynamical semigroups) equipped with a subinvariant faithful state (a generalization of faithful invariant states for trace-nonincreasing QMPs). This family is generated by operator monotone functions and each its instance provides a dual basis of the attractor space allowing to express the asymptotic evolution for any initial state. We present two important examples of operator monotone functions, each giving rise to a useful (and convenient) characterization of stationary states (respectively, asymptotic states) via integrals of motion (respectively, Hermitian attractors) of given QMP. Their importance is twofold. First, it avoids problems with positivity of the dynamics induced density operator which arises if we construct stationary or asymptotic states directly from attractors. Second, these states strongly resemble Gibbs states well known from statistical physics [38]. We expect that this link might have further applications in quantum processing.

The second goal of the paper is to examine how the inner structure of Markovian generator determines attractor spaces of QMPs in the Schrödinger and the Heisenberg picture. Using relations between attractors in both pictures we generalize previously known results for QMCHs [34] and derive algebraic equations determining attractors of continuous QMDSs in both pictures in terms of their Hamiltonian, Lindblad operators and eventually an optical potential [11]. The obtained results apply to all discrete QMCHs as well as continuous QMDSs, which are equipped with a subinvariant faithful state. We stress that these QMPs may not be trace-preserving and therefore allow for analysis of physical situations where part of the dynamics is not known [11] or is gradually lost, *e.g.* evolution with a sink [22, 39, 40]. Based on the obtained algebraic equations we analyze the algebraic structure of attractors in both pictures and specify the type of evolution running inside the asymptotic space. In both, discrete and continuous, cases it is shown that the asymptotic dynamics is reversible. With a properly redefined Hilbert-Schmidt scalar product it might be seen as an unitary evolution. In particular, we show that a quantum channel capable to reverse the evolution inside the attractor space of QMCH is the so-called Petz recovery map [41]. For QMDSs, we derive a master equation driving the dynamics inside the attractor space.

Both emphasized goals are motivated by the effort to develop techniques capable to provide an explicit form of asymptotic dynamics for important classes of QMPs, *e.g.* many-body quantum systems with repeated local interactions. An advantage of such an approach is the possibility to analyze asymptotic evolution in the limit of large number of interacting subsystems.

We briefly describe the structure of the paper. In sect. 2 we provide settings and important definitions that are used throughout the whole paper. The aim of sect. 3 is to introduce the concept of quantum Markov processes, describe them in terms of their generators and specify their asymptotic regime. Section 4 is devoted to studies of the mutual relationships between attractors of QMPs in the Schrödinger and the Heisenberg picture. Employing operator monotone functions we construct dual basis for both, continuous and discrete, QMPs. Finally two important cases of operator monotone functions are discussed. In sect. 5 we prove the structure theorem for attractors of QMPs in both pictures and reveal their algebraic properties. Two useful characterizations of asymptotic and stationary states are given in sect. 6. The description of the dynamics inside the attractor space is investigated in sect. 7. Finally, we examine two examples in sect. 8 and conclude in sect. 9.

2 Preliminaries and definitions

Throughout the whole paper we assume a quantum system associated with a *finite* N-dimensional Hilbert space $\mathscr{H}$ equipped with a scalar product $\langle\cdot,\cdot\rangle$. Let $\mathsf{B}(\mathscr{H})$ be the associated Hilbert space of all operators acting on $\mathscr{H}$ and we denote its corresponding Hilbert-Schmidt product as $(A,B)=\mathrm{Tr}\{A^\dagger B\}$ and corresponding Hilbert-Schmidt norm $\|A\|_{HS}=\sqrt{(A,A)}$ with $A,B\in\mathsf{B}(\mathscr{H})$ ($A^\dagger$ denotes the adjoint operator of A).

A state of a quantum system is described by a density operator, a positive operator with unit trace. Let us denote the set of all states as $\mathcal{S}(\mathscr{H})$. The most general physical state change is given by a quantum operation $\mathcal{T}:\mathsf{B}(\mathscr{H})\to\mathsf{B}(\mathscr{H})$, a linear completely positive map (CP) which admits decomposition into Kraus operators $\{A_j\}_{i=1}^k\subseteq\mathsf{B}(\mathscr{H})$

$$\mathcal{T}(\cdot)=\sum_{j=1}^k A_j(\cdot)A_j^\dagger. \tag{1}$$

Moreover, any quantum operation is supposed to be trace-nonincreasing and thus satisfying $\mathcal{T}^\dagger(I)\le I$ or equivalently expressed in terms of Kraus operators as $\sum_{i=1}^k A_i^\dagger A_i\le I$. Note that the adjoint map $\mathcal{T}^\dagger$ of a quantum operation $\mathcal{T}$ is also a completely positive map with Kraus operators $\{A_j^\dagger\}_{i=1}^k$, but it does not represent a quantum operation in general because the map $\mathcal{T}^\dagger$ is not necessarily trace-nonincreasing. Let us introduce a little more terminology used in the context of quantum operations. If $\mathcal{T}^\dagger(I)=\sum_j A_j^\dagger A_j=I$ we call the quantum operation $\mathcal{T}$ a channel or a trace-preserving quantum operation. A quantum operation which leaves the maximally mixed state unchanged is called unital and satisfies $\mathcal{T}(I)=\sum_j A_jA_j^\dagger=I$. A prominent example of unital channels are random unitary operations (random external fields). In the less restrictive cases when $\mathcal{P}(I)=\sum_j A_jA_j^\dagger\le I$ the quantum operation is called sub-unital.

From a different perspective quantum operations are linear maps acting on operators and for this reason they are called superoperators. Other examples of superoperators, we exploit in our analysis, are the left (respectively, right) multiplication operator defined as $L_P(X)=PX$ (respectively, $R_P(X)=XP$), where P is a positive operator from $\mathsf{B}(\mathscr{H})$. From these two superoperators one can construct the relative modular operator

$$\Delta_{Q,P}=L_QR_P^{-1},$$

where P is assumed to be strictly positive and following $R_P^{-1}=R_{P^{-1}}$ [42].

A convenient tool for our analysis of mutual relationships between attractors in Schrödinger and Heisenberg picture are operator monotone functions, which can be introduced as follows. Assume a function f defined on a real interval I and a self-adjoint operator A with its spectrum lying in the interval I. Invoking its spectral decomposition $A=UDU^\dagger$ with diagonal matrix $D=\mathrm{diag}(\lambda_1,\lambda_2,\ldots)$ one can introduce the operator $f(A)=Uf(D)U^\dagger$, where $f(D)$ is the diagonal matrix $f(D)=\mathrm{diag}(f(\lambda_1),f(\lambda_2),\ldots)$. A function $k:[0,+\infty)\to[0,+\infty)$ is called an operator monotone if any two operators A,B such that $A\ge B\ge 0$ implies $k(A)\ge k(B)$. Interestingly, Löwner has shown that all operator monotones can be characterized by an integral representation [43].

Theorem 1. $k:[0,+\infty)\to[0,+\infty)$ *is an operator monotone if and only if there is a positive finite measure* μ *such that*

$$k(y)=\alpha+\beta y+\int_0^{+\infty}\frac{y(1+s)}{s+y}\mathrm{d}\mu(s), \tag{2}$$

with $\alpha=k(0)\ge 0$ *and* $\beta=\lim_{t+\infty}\frac{f(t)}{t}\ge 0$.

3 Homogeneous quantum Markov processes

Let us recall the concept of Markovian evolution and specify its form in quantum domain. In general, a stochastic process is called Markov if the future evolution of any present state is independent on its past. Its future is given solely by the action of a propagating map onto the present state. In the context of quantum mechanics it follows that the Markov evolution during any possible finite time interval $\langle t_1,t_2\rangle$ is given by some quantum operation $\mathcal{T}(t_1,t_2)$. Here we stress that we allow changes which are trace-nonincreasing. If any state change driven by quantum Markov process depends solely on the length of the time interval $\triangle t=t_2-t_1$, *i.e.* $\mathcal{T}(t_1,t_2)=\mathcal{T}(\triangle t)$, the quantum Markov process is called homogenous. Thus to describe a state change under a homogenous quantum Markov process one does not even need to know the initial time of interval in which the state change takes place. In this work we investigate only homogenous quantum Markov processes and thus, to simplify notation, we use the notion quantum Markov process (QMP) to identify homogenous quantum Markov process.

We distinguish two classes of quantum Markov processes. The first one are discrete quantum Markov chains (QMCHs). Their one step of evolution is governed by a generating quantum operation $\mathcal{T}$ taking the state $\rho(n)$ emerging from previous n iterations to the state $\rho(n+1) = \mathcal{T}(\rho(n))$. Thus, within n iterations the system initially prepared in the state $\rho(0)$ evolves into the state $\rho(n) = \mathcal{T}^n(\rho(0))$. In the Heisenberg picture we have, instead of evolving states, evolving observables. A consistence of both descriptions requires that each initial observable $A(0)$ and initial state $\rho(0)$ fulfil mean value condition at any step of QMCH

$$\langle A(0)\rangle_{\mathcal{T}^n(\rho(0))} = \langle \mathcal{T}'_n(A(0))\rangle_{\rho(0)}. \tag{3}$$

$\mathcal{T}'_n$ denotes the propagator describing the n steps of QMCH evolution in the Heisenberg picture and mean values are defined with respect to the Hilbert-Schmidt scalar product, *i.e.* $\langle A\rangle_\rho = \mathrm{Tr}[A^\dagger \rho]$. From (3) it follows that QMCH evolution in the Heisenberg picture is generated by the adjoint map $\mathcal{T}^\dagger$ with respect to the Hilbert-Schmidt scalar product, which is completely positive, unital and takes the form

$$\mathcal{T}^\dagger(\cdot) = \sum_{j=1}^{k} A_j^\dagger(\cdot)A_j. \tag{4}$$

The second class of QMPs is comprised of quantum Markov dynamical semigroups (QMDSs) *i.e.* quantum operations $\mathcal{T}_t$ transforming the state $\rho(t_1)$ at time t_1 into the state $\rho(t_2) = \mathcal{T}_{t_2-t_1}\rho(t_1)$ at time t_2. Assuming uniformly continuous QMDSs, $\mathcal{T}_t$ takes the form

$$\mathcal{T}_t = \exp(\mathcal{L}t). \tag{5}$$

Due to Lindblad original work [8] the generator $\mathcal{L} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ of QMDS can be written as

$$\mathcal{L}(X) = \mathcal{V}(X) - KX - XK^\dagger, \tag{6}$$

with $\mathcal{V}$ being a completely positive map with Kraus operators $\{L_j\}$ and K is an element of $\mathsf{B}(\mathscr{H})$. Splitting K into its Hermitian and anti-Hermitian part $K = iH + \frac{1}{2}\mathcal{V}^\dagger(I) + G$ with Hamiltonian H and an optical potential G [11], the generator of QMDS takes the form

$$\mathcal{L}(X) = i[X,H] + \sum_j L_j X L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, X\} - GX - XG. \tag{7}$$

Here, the Lindblad operators L_i, Hamiltonian H and the optical potential G can be chosen arbitrarily, provided that G is positive to ensure that the generated QMDS is trace-nonincreasing. The corresponding evolution under a QMDS is described by the Markov master equation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \mathcal{L}(\rho). \tag{8}$$

In case that QMDS $\mathcal{T}_t$ is trace-preserving, we find $\mathcal{L}^\dagger(I) = 0$ implying $G = 0$ and the generator may be cast into the well-known Lindblad form for trace-preserving QMDSs

$$\mathcal{L}(X) = i[X,H] + \sum_j L_j X L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, X\}. \tag{9}$$

Analogously as in the discrete case, the evolution of observables is given by the dynamical semigroup $\mathcal{T}_t^\dagger$ whose generator takes the form

$$\mathcal{L}^\dagger(A) = \mathcal{V}^\dagger(A) - K^\dagger A - AK. \tag{10}$$

If QMDS is trace-preserving the generator (10) can be written with the help of the Hamiltonian as

$$\mathcal{L}^\dagger(A) = -i[X,H] + \sum_j L_j^\dagger X L_j - \frac{1}{2}\{L_j^\dagger L_j, X\}. \tag{11}$$

QMPs are frequently used to model a simplified or effective evolution of open quantum systems. In comparison with the class of closed quantum evolutions an additional severe obstacle arises, if we start to analyze its evolution. This is due to the fact that both generators of QMPs generally do not commute with their adjoint map, *i.e.* they are neither Hermitian nor normal. Consequently, this makes the generated dynamics much more involved and harder to solve as the standard method of spectral decomposition is not available usually. Let us list the implicit unpleasant consequences in detail. First, such generator of a QMP may not be diagonalizable and then we are left with Jordan normal form given only in the basis of some generalized eigenvectors [44]. Second, we miss a relationship between eigenvectors or generalized eigenvectors of the generator and its adjoint map. Consequently, corresponding (generalized) eigenvectors may be nonorthogonal and a construction of the important dual basis becomes a hard task.

However, if we are interested in the asymptotic dynamics of QMPs solely, the solution of its behavior is one step less complicated. Indeed, it has been shown [34] that the corresponding part of the generator responsible for asymptotic dynamics of QMP can always be diagonalized. The space $\mathsf{B}(\mathscr{H})$ can be decomposed into two parts, *i.e.* $\mathsf{B}(\mathscr{H}) = \mathsf{Atr} \oplus \mathsf{Y}$ with attractor space Atr supporting the asymptotic dynamics of a given QMP and a dying operator space Y whose elements gradually vanish during the QMP. Both operator spaces Atr and Y are closed under the action of a given QMP. As the spectra of generators of discrete and continuous QMPs differ, definitions of their attractor spaces differ slightly as well. Let us start with discrete QMCHs in the Schrödinger picture, where the attractor space is given as

$$\mathsf{Atr}(\mathcal{T}) = \bigoplus_{\lambda\in\sigma_{as}} \mathsf{Ker}(\mathcal{T} - \lambda I), \tag{12}$$

with asymptotic spectrum σ_{as} containing all eigenvalues of the generator $\mathcal{T}$ with modulo one. Assuming $X_{\lambda,i}$ form basis of each individual eigenspace $\mathsf{Ker}(\mathcal{T} - \lambda I)$ and $X^{\lambda,i}$ form the corresponding dual basis, *i.e.* $(X_{\lambda_1,i}, X^{\lambda_2,j}) = \delta_{\lambda_1\lambda_2}\delta_{ij}$, we can write down the asymptotic dynamics of the given QMCH

$$\rho(n \gg 1) = \sum_{\lambda\in\sigma_{as},i} \lambda^n X_{\lambda,i} \,\mathrm{Tr}\{\rho(0)(X^{\lambda,i})^\dagger\}. \tag{13}$$

In the Heisenberg picture the attractor space is given analogously as

$$\mathsf{Atr}(\mathcal{T}^\dagger) = \bigoplus_{\lambda\in\sigma_{as}} \mathsf{Ker}(\mathcal{T}^\dagger - \lambda I), \tag{14}$$

and the asymptotic evolution of observables is written in the corresponding dual basis of the attractor space

$$A(n \gg 1) = \sum_{\lambda\in\sigma_{as},i} \lambda^n (X^{\lambda,i})^\dagger \,\mathrm{Tr}\{A(0)X_{\lambda,i}\}. \tag{15}$$

A special attention belongs to attractors associated with eigenvalue one. They are also called fixed points because they do not evolve during a given evolution. While fixed points of QMCH in the Schrödinger picture $\mathsf{Fix}(\mathcal{T})$ contain all stationary states, fixed points of QMCH in the Heisenberg picture $\mathsf{Fix}(\mathcal{T}^\dagger)$ contain all integrals of motion. Both sets are nonempty and contain at least one positive operator.

Similarly, the attractor space of QMDSs is composed of the corresponding kernels of the generator $\mathcal{L}$

$$\mathsf{Atr}(\mathcal{T}) = \bigoplus_{\lambda\in\sigma_{as}} \mathsf{Ker}(\mathcal{L} - \lambda I), \tag{16}$$

where, in the continuous case, the asymptotic spectrum σ_{as} contains only purely imaginary eigenvalues of the generator $\mathcal{L}$. This can be deduced from the fact that X is an eigenoperator of the generator $\mathcal{L}$, *i.e.* $\mathcal{L}(X) = aX$, if and only if X is an eigenoperator of the quantum operation $\mathcal{T}_t = \exp(\mathcal{L}t)$ associated with the eigenvalue $\exp(at)$ for any positive time t. Hence, eigenoperator X of $\mathcal{L}$ associated with eigenvalue a is an attractor of QMDS generated by $\mathcal{L}$ iff $|\exp(at)| = 1$ for any $t \geq 0$. Provided we find the dual basis $X^{\lambda,i}$ for some chosen basis $X_{\lambda,i}$ of the attractor space, the asymptotic dynamics in the Schrödinger picture takes the form

$$\rho(t \gg 1) = \sum_{\lambda\in\sigma_{as},i} \exp(\lambda t) X_{\lambda,i} \,\mathrm{Tr}\{\rho(0)(X^{\lambda,i})^\dagger\}. \tag{17}$$

As we have already mentioned, compare to closed unitary evolution we miss a relationship between eigenvectors of the generator and its adjoint map. Therefore we cannot be sure whether the operator spaces Atr and Y are mutually orthogonal and also different kernels forming the attractor space may be in general nonorthogonal. In addition, finding the dual basis of the attractor space becomes a nontrivial problem. Thus, without such relation we lose a connection between the asymptotic dynamics of QMP in the Schrödinger and the Heisenberg pictures. One of our aims is to show that for a broad class of QMPs these obstacles can be removed and we can enjoy the benefits of an analogous theory we have got to use for closed unitary evolutions. Some relation between eigenvectors of the generator and its adjoint map were already revealed for QMCHs [34]. Here we intend to show that there is a deep connection between eigenvectors of evolution generators in both pictures. The previously found results are an example of this connection and we generalize findings presented in [34] and extend their validity also to QMDSs. As will be shown, it helps to uncover algebraic properties of attractors, especially algebraic properties of integrals of motion and stationary states of quantum Markov processes. Such results are essential for the complete understanding of equilibria and their formation.

4 Relations between eigenvectors of QMPs in the Schrödinger and the Heisenberg picture

This part is devoted to spectral properties of the generator responsible for the asymptotic dynamics of a given QMP. We first focus on QMCHs, a generalization to QMDSs is straightforward.

4.1 Quantum Markov chains

Throughout the rest of paper we assume a generating quantum operation $\mathcal{T}$ equipped with a subinvariant faithful $\mathcal{T}$-state $\sigma \in \mathsf{B}(\mathscr{H})$ which means that σ is strictly positive and satisfies $\mathcal{T}(\sigma) \leq \sigma$. We call such state $\mathcal{T}$-state. This definition generalizes the concept of faithful invariant states [28,45] for QMPs which are not trace-preserving, *i.e.* they may decrease trace of input operators. The starting point of the following considerations is the theorem which establishes a basic relation between eigenvectors of quantum operation $\mathcal{T}$ and its adjoint map (for proofs, exploiting different techniques, see [32,34,46]).

Theorem 2. *If λ is from the asymptotic spectrum of quantum operation $\mathcal{T}$ then for any attractor X associated with the eigenvalue λ we have*

1) $X \in \mathsf{Ker}(\mathcal{T} - \lambda I) \Leftrightarrow R_\sigma^{-1}(X) \in \mathsf{Ker}(\mathcal{T}^\dagger - \overline{\lambda} I)$,
2) $X \in \mathsf{Ker}(\mathcal{T} - \lambda I) \Leftrightarrow L_\sigma^{-1}(X) \in \mathsf{Ker}(\mathcal{T}^\dagger - \overline{\lambda} I)$.

Employing this theorem we can find a broad family of linear bijections mapping individual kernels $\mathsf{Ker}(\mathcal{T} - \lambda I)$ of the attractor space onto itself. It can be formulated as follows.

Theorem 3. *Let λ be an element of asymptotic spectrum of quantum operation $\mathcal{T}$ and σ_1, σ_2 two, not necessarily different, $\mathcal{T}$-states. Then any operator monotone function k establishes a well-defined bijection $k(\Delta_{\sigma_1,\sigma_2})$ onto attractors (in both the Schrödinger and the Heisenberg picture) associated with the eigenvalue λ.*

Proof. From theorem 2 we can deduce that the relative modular operator $\Delta_{\sigma_1,\sigma_2}$ is strictly positive and defines a bijection onto the attractor subspace $\mathsf{Ker}(\mathcal{T} - \lambda I)$ as well as onto the attractor subspace $\mathsf{Ker}(\mathcal{T}^\dagger - \lambda I)$. This is true also for the map $sI + \Delta_{\sigma_1,\sigma_2}$ and its existing inverse, where s is nonnegative and I stands for the identity map. Moreover, all these maps commute mutually and thus for any finite measure $\mu(s)$ we observe that also the map

$$k(\Delta_{\sigma_1,\sigma_2}) = \int_0^{+\infty} \frac{\Delta_{\sigma_1,\sigma_2}(1+s)}{sI + \Delta_{\sigma_1,\sigma_2}} \mathrm{d}\mu(s) + \alpha I + \beta \Delta_{\sigma_1,\sigma_2} \tag{18}$$

with $\beta \geq 0$ is a strictly positive bijection onto the individual attractor subspaces $\mathsf{Ker}(\mathcal{T} - \lambda I)$ and $\mathsf{Ker}(\mathcal{T}^\dagger - \lambda I)$. According to theorem 1 we conclude that any operator monotone function k defines strictly positive bijection $k(\Delta_{\sigma_1,\sigma_2})$ onto these attractor subspaces in both pictures. □

We have found a broad and important family of linear bijections of attractor spaces of quantum operation $\mathcal{T}$ and its adjoint map. In sect. 6 we employ two particular examples of these bijections to reveal two inequivalent forms of asymptotic states of QMPs. An apparent advantage of our construction is the fact that each operator monotone function defines a linear bijection of attractor spaces independently of the underlying Hilbert space, *i.e.* its dimension. In a similar way one can construct maps which are not bijections but map attractor spaces back into the same attractor space. Moreover by combining theorems 2 and 3, we receive a general relationship between attractors of QMCH in the Schrödinger and Heisenberg picture.

Corollary 1. *Let λ be an element of the asymptotic spectrum of a quantum operation $\mathcal{T}$ generating QMCH and σ_1, σ_2 its two, not necessarily different, $\mathcal{T}$-states. If k is an operator monotone function then*

$$X \in \mathsf{Ker}(\mathcal{T} - \lambda I) \Leftrightarrow R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X) \in \mathsf{Ker}(\mathcal{T}^\dagger - \overline{\lambda} I). \tag{19}$$

Note that in corollary 1 we could use also the superoperator $(R_{\sigma_1})^{-1} k(\Delta_{\sigma_1,\sigma_2})$, which apparently does the same job. However, our choice is due to the advantage that maps $R_{\sigma_2}^{-1}$ and $k(\Delta_{\sigma_1,\sigma_2})$ commute and consequently their product is again a strictly positive operator. This allows us to define a new scalar product on the space $\mathsf{B}(\mathscr{H})$ for any choice of operator monotone function k

$$(X, Y)_k = (X, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(Y)). \tag{20}$$

Then we say that two operators X and Y are k-orthogonal if $(X, Y)_k = 0$. Similarly, two sets in $\mathsf{B}(\mathscr{H})$ are k-orthogonal if any two elements from these two sets are mutually k-orthogonal. The direct consequence of the corollary 1 are the following k-orthogonality relations.

Theorem 4. *Let* λ_1 *and* λ_2 *be different elements of the asymptotic spectrum of quantum operation* $\mathcal{T}$ *generating QMCH and* σ_1, σ_2 *its two, not necessarily different* $\mathcal{T}$*-states. If* k *is an operator monotone function then*

1) *attractor subspaces* $\mathsf{Ker}(\mathcal{T}-\lambda_1 I)$ *and* $\mathsf{Ker}(\mathcal{T}-\lambda_2 I)$ *are* k*-orthogonal,*
2) *attractor subspace* $\mathsf{Ker}(\mathcal{T}-\lambda_1 I)$ *and range* $\mathsf{Ran}(\mathcal{T}-\lambda_1 I)$ *are* k*-orthogonal.*

Proof. Assume $X_i \in \mathsf{Ker}(\mathcal{T}-\lambda_i I)$. Then

$$\begin{aligned}(X_1,X_2)_k &= \frac{1}{\lambda_2}(X_1, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(\mathcal{T}(X_2)))\\ &= \frac{1}{\lambda_2}(\mathcal{T}^\dagger(R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_1)), X_2)\\ &= \frac{\lambda_1}{\lambda_2}(R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_1), X_2) = \frac{\lambda_1}{\lambda_2}(X_1,X_2)_k\end{aligned}$$

following from the strict positivity of the operator $R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})$ and we find $(X_1,X_2)_k = 0$. In order to prove the second statement consider $X \in \mathsf{Ker}(\mathcal{T}-\lambda_1 I)$ and $Y \in \mathsf{Ran}(\mathcal{T}-\lambda_1 I)$. Hence there exists $0 \neq Z \in \mathsf{B}(\mathscr{H})$ such that $Y = \mathcal{T}(Z) - \lambda Z$ and one can check

$$\begin{aligned}(X,Y)_k &= (X, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(\mathcal{T}(Z)-\lambda_1 Z))\\ &= (\mathcal{T}^\dagger(R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X)), Z) - \lambda(X, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(Z))\\ &= \lambda[(R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X), Z) - (X, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(Z))] = 0.\end{aligned}$$

□

Theorem 4 simply tells that attractors associated with different eigenvalues from asymptotic spectrum of generating quantum operation $\mathcal{T}$ are k-orthogonal and they are also k-orthogonal to the in time dying space Y. In turn it means that we have found a dual basis $X^{\lambda,i}$ of eigenvectors $X_{\lambda,i}$. Assuming k is a given operator monotone function, the dual basis $X^{\lambda,i}$ reads

$$X^{\lambda,i} = \frac{R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_{\lambda,i})}{(X_{\lambda,i}, R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X_{\lambda,i}))}\,. \tag{21}$$

4.2 Quantum Markov dynamical semigroups

So far we have considered asymptotic evolution of discrete QMCHs. Let us show that the same theory applies to continuous QMDSs as well. Apparently, it is sufficient to prove theorem 2 for QMDSs. In order to proceed we need to define an analogy of $\mathcal{T}$-state for QMDSs. We call a faithful state σ, *i.e.* $\sigma > 0$, $\mathcal{T}$-state if $\mathcal{T}_t(\sigma) = \exp(\mathcal{L}t)(\sigma) \le \sigma$ holds for any positive time t. This definition is more involved, especially when it comes to the point if a given QMDS satisfies it. However, in fact it is sufficient to check whether this condition is fulfilled for times from some right neighborhood of zero. Moreover, if the studied QMDS is trace-preserving, the definition of a $\mathcal{T}$-state σ simply reduces to the condition $\mathcal{L}(\sigma) = 0$. With this modification, theorem 2 and all of the follow-up theory applies to QMDSs.

Theorem 5. *Let* σ *be a* $\mathcal{T}$*-state of QMDS* $\mathcal{T}_t = \exp(\mathcal{L}t)$*. If* λ *is from the asymptotic spectrum of a QMDS* $\mathcal{T}_t$ *then for any attractor* X *associated with the eigenvalue* λ *we have*

1) $X \in \mathsf{Ker}(\mathcal{L}-\lambda I) \Leftrightarrow R_\sigma^{-1}(X) \in \mathsf{Ker}(\mathcal{L}^\dagger - \overline{\lambda} I)$,
2) $X \in \mathsf{Ker}(\mathcal{L}-\lambda I) \Leftrightarrow L_\sigma^{-1}(X) \in \mathsf{Ker}(\mathcal{L}^\dagger - \overline{\lambda} I)$.

Proof. If $X \in \mathsf{Ker}(\mathcal{L}-\lambda I)$ then $X \in \mathsf{Ker}(\mathcal{T}_t - \exp(\lambda t) I)$ for any positive t. Due to theorem 2 we have $R_\sigma^{-1}(X) \in \mathsf{Ker}((\mathcal{T}_t)^\dagger - \overline{\exp(\lambda t)} I)$ for any positive t which in turn, by differentiation with respect to time at $t = 0$, means that $R_\sigma^{-1}(X) \in \mathsf{Ker}(\mathcal{L}^\dagger - \overline{\lambda} I)$. Other implications can be proven in the same way. □

We have established a general theory for analyzing the attractor spaces of discrete and continuous QMPs in both pictures. In the following we employ two examples of operator monotone functions which provide an additional insight into the inverse evolution restricted onto the asymptotic space and into the structure of asymptotic and stationary states.

4.3 Operator monotone function $k(y) = y^\alpha$

One of the well-known operator monotones is $k(y) = y^\alpha$ for $\alpha \in (0,1]$. Its integral representation [47] is given as

$$k(y) = y^\alpha = \int_0^{+\infty} \frac{ys^{\alpha-1}}{y+s}\frac{\sin(\alpha\pi)}{\pi}\mathrm{d}s. \tag{22}$$

Taking into account theorem 2 (or theorem 5 in the case of QMDSs) the action of linear bijection onto individual subspaces composing the whole attractor space is given as $k(\Delta_{\sigma_1,\sigma_2})(X) = \sigma_1^\alpha X \sigma_2^{-\alpha}$ for $\alpha \in (0,1]$. Since the considered map is a bijection, we can generalize this result by iteration and inverse to any real α. Consequently, for any real α the superoperator

$$R_{\sigma_2}^{-1} k(\Delta_{\sigma_1,\sigma_2})(X) = \sigma_1^\alpha X \sigma_2^{-\alpha-1} \tag{23}$$

defines a one-to-one correspondence between attractors $\mathsf{Ker}(\mathcal{T} - \lambda I)$ and $\mathsf{Ker}(\mathcal{T}^\dagger - \overline{\lambda} I)$ (analogously for QMDSs). Attractors associated with different eigenvalues are mutually k-orthogonal with respect to the scalar product $(X,Y)_k = (X, \sigma_1^\alpha Y \sigma_2^{-\alpha-1}) \equiv (X,Y)_\alpha$.

Moreover, each of these new scalar products defines an associated adjoint map of the quantum operation $\mathcal{T}$. The case $\alpha = 1/2$ deserves a special attention. Indeed, choosing σ_1 and σ_2 equal to σ we find that the adjoint map to quantum operation $\mathcal{T}$ with respect to the scalar product $(X,Y)_{1/2} = (X, \sigma^{-1/2} Y \sigma^{-1/2})$ takes the form

$$\mathcal{T}^\ddagger(X) = \sum_k \sigma^{1/2} A_k^\dagger \sigma^{-1/2} X \sigma^{-1/2} A_k \sigma^{1/2}. \tag{24}$$

Apparently, this is again a completely positive trace-nonincreasing map. As will be shown later, it is a quantum operation capable to reverse the evolution running inside the attractor space.

This type of bijections reveal an interesting structure of attractor spaces. One could naively infer that any $\mathcal{T}$-state must commute with all attractors, which would significantly decrease the complexity of the attractor structure. However this is not true as we demonstrate in sect. 8.

4.4 Operator monotone function $k(y) = \log(1 + y)$

As another example we apply the derived theory to the operator monotone function $k(y) = \log(1+y)$ with its integral representation [47]

$$k(y) = \log(1+y) = \int_1^{+\infty} \frac{ys^{-1}}{y+s}\mathrm{d}s. \tag{25}$$

There follows that both operators $\log(I + \Delta_{\sigma_1,\sigma_2})$ and $\log(I + (\Delta_{\sigma_1,\sigma_2})^{-1})$ are bijections which map individual attractor spaces, in both pictures, associated with a given eigenvalue onto itself. Employing the following identity:

$$\log(I + \Delta_{\sigma_1,\sigma_2}) = \log(I + (\Delta_{\sigma_1,\sigma_2})^{-1}) + \log(\Delta_{\sigma_1,\sigma_2}) \tag{26}$$

we find that the operator $\log(\Delta_{\sigma_1,\sigma_2})$ is not necessary a bijection but it also maps individual attractor spaces, in both pictures, back to the original individual attractor space. A straightforward calculation reveals that $\log(\Delta_{\sigma_1,\sigma_2}) = L_{\log(\sigma_1)} - R_{\log(\sigma_2)}$, which proves the following interesting statement.

Corollary 2. *Let λ be an element of the asymptotic spectrum of quantum operation $\mathcal{T}$ and σ_1, σ_2 two, not necessarily different, $\mathcal{T}$-states. Then the map $L_{\log(\sigma_1)} - R_{\log(\sigma_2)}$ is an endomorphism onto the attractor space $\mathsf{Ker}(\mathcal{T} - \lambda I)$ of the quantum operation $\mathcal{T}$.*

This statement provides a key ingredient for a characterization of all asymptotic states of QMPs (for details see sect. 6).

5 Structure theorems for quantum Markov processes

This part is devoted to the analysis how an inner structure of a generator governs the attractors of the resulting QMPs. Thus in this part we presume that either Kraus operators $\{A_i\}$ of quantum operation generating QMCH or operators $\{L_i, H, G\}$ in (7) defining QMDS are known. The ultimate goal is to uncover how these operators determine attractors of QMPs. The structure theorem for quantum Markov chains was already derived in [34].

Theorem 6. *Let $\mathcal{T} : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a quantum operation (1) equipped with a $\mathcal{T}$-state σ. If X is an attractor of QMCH in the Schrödinger picture generated by $\mathcal{T}$ associated with eigenvalue λ then it necessary satisfies the following set of equations:*

$$\begin{aligned} A_j X\sigma^{-1} &= \lambda X\sigma^{-1}A_j, \qquad & A_j^\dagger X\sigma^{-1} &= \overline{\lambda} X\sigma^{-1}A_j^\dagger, \\ A_j \sigma^{-1}X &= \lambda \sigma^{-1}XA_j, \qquad & A_j^\dagger \sigma^{-1}X &= \overline{\lambda} \sigma^{-1}XA_j^\dagger, \end{aligned} \tag{27}$$

for all j's. If X is an attractor of QMCH in the Heisenberg picture associated with eigenvalue λ then it necessary satisfies the set of eqs. (27) for $\sigma = I$.

Moreover, if quantum operation $\mathcal{T}$ is either trace-preserving or $\mathcal{T}$-state σ is additionally invariant then the converse statement applies as well.

The importance of theorem 6 is twofold. First, it significantly simplifies the calculation of asymptotic behavior of QMCHs. We should stress that this can be done analytically in many cases, especially if the studied evolution possess some sort of symmetry. Second, it also reveals the algebraic structure of attractors. We shall discuss this point simultaneously for discrete and continuous QMPs later.

In the following we show that QMDSs follow a similar structure theorem for their attractors. Similar formulas for attractors of trace-preserving QMDSs in Heisenberg picture can be found in [35].

Theorem 7. *Let $\mathcal{T}_t : \mathsf{B}(\mathscr{H}) \to \mathsf{B}(\mathscr{H})$ be a quantum Markov dynamical semigroup with generator $\mathcal{L}$ (7) equipped with a $\mathcal{T}$-state σ. If $X \in \mathsf{B}(\mathscr{H})$ is an attractor of QMDS in the Schrödinger picture associated with eigenvalue $\lambda = ia$ then the following set of equations holds:*

$$[L_j, X\sigma^{-1}] = [L_j, \sigma^{-1}X] = [L_j^\dagger, X\sigma^{-1}] = [L_j^\dagger, \sigma^{-1}X] = 0, \tag{28}$$

$$[X\sigma^{-1}, G] = [\sigma^{-1}X, G] = 0, \tag{29}$$

$$[\sigma^{-1}X, H] = a\sigma^{-1}X, \quad [X\sigma^{-1}, H] = aX\sigma^{-1} \tag{30}$$

for all j's. If $X \in \mathsf{B}(\mathscr{H})$ is an attractor of QMDS in the Heisenberg picture associated with eigenvalue $\lambda = ia$ then it must satisfy all eqs. (28), (29) and (30) with $\sigma = I$.

If QMDS $\mathcal{T}_t$ is either trace-preserving or $\mathcal{T}$-state σ is stationary then the converse statement applies as well.

Proof. We first derive the necessary conditions which follow from the assumption that operator X is an attractor of QMDS, *i.e.* $\mathcal{L}(X) = iaX$. The generator $\mathcal{L}$ maps Hermitian operators back to Hermitian operators and thus also $X^\dagger$ is an attractor, *i.e.* $\mathcal{L}(X^\dagger) = iaX^\dagger$. Employing theorem 5 we find that

$$\mathcal{L}^\dagger(\sigma^{-1}X) = -ia\sigma^{-1}X, \quad \mathcal{L}^\dagger(\sigma^{-1}X^\dagger) = ia\sigma^{-1}X^\dagger, \quad \mathcal{L}^\dagger(X^\dagger\sigma^{-1}) = iaX^\dagger\sigma^{-1}, \tag{31}$$

which can be equivalently rewritten as

$$\begin{aligned} \sum_i L_i^\dagger \sigma^{-1}XL_i &= -ia\sigma^{-1}X + K^\dagger\sigma^{-1}X + \sigma^{-1}XK, \\ \sum_i L_i^\dagger \sigma^{-1}X^\dagger L_i &= ia\sigma^{-1}X^\dagger + K^\dagger\sigma^{-1}X^\dagger + \sigma^{-1}X^\dagger K, \\ \sum_i L_i^\dagger X^\dagger\sigma^{-1} L_i &= iaX^\dagger\sigma^{-1} + K^\dagger X^\dagger\sigma^{-1} + X^\dagger\sigma^{-1}K, \\ \sum_i L_i^\dagger L_i &\leq K + K^\dagger. \end{aligned} \tag{32}$$

The last inequality expresses the fact that QMDS is trace-nonincreasing. In order to proceed, an additional inequality is needed. From theorem 5 we have for each positive t

$$\mathcal{T}_t^\dagger(\sigma^{-1}X) = e^{-iat}\sigma^{-1}X, \quad \mathcal{T}_t^\dagger(X^\dagger\sigma^{-1}) = e^{iat}X^\dagger\sigma^{-1}.$$

Using the Schwarz operator inequality [48,49] for subunital quantum operations $\mathcal{T}_t^\dagger$ we obtain

$$\mathcal{T}_t^\dagger(\sigma^{-1}XX^\dagger\sigma^{-1}) \leq \mathcal{T}_t^\dagger(\sigma^{-1}X)\mathcal{T}_t^\dagger(X^\dagger\sigma^{-1}) = \sigma^{-1}XX^\dagger\sigma^{-1},$$

As this applies to all positive t we find $\mathcal{L}^\dagger(\sigma^{-1}XX^\dagger\sigma^{-1}) \leq 0$ or, equivalently,

$$\sum_i L_i^\dagger \sigma^{-1}XX^\dagger\sigma^{-1}L_i \leq K^\dagger\sigma^{-1}XX^\dagger\sigma^{-1} + \sigma^{-1}XX^\dagger\sigma^{-1}K. \tag{33}$$

Let us set $V_i = X\sigma^{-1}L_i - L_iX\sigma^{-1}$. Equipped with relations (32) and (33) we receive

$$\sum_i V_i^\dagger V_i = \sum_i L_i^\dagger \sigma^{-1} X X^\dagger \sigma^{-1} L_i - \left(\sum_i L_i^\dagger \sigma^{-1} X L_i\right) X\sigma^{-1} \\ - \sigma^{-1}X^\dagger \left(\sum_i L_i^\dagger X\sigma^{-1}L_i\right) + \sigma^{-1}X^\dagger \left(\sum_i L_i^\dagger L_i\right) X\sigma^{-1} \leq 0.$$

This inevitably means that all operators V_i are equal to zero and consequently $[L_i, X\sigma^{-1}] = 0$. Due to theorem 5 the operator $\tilde{X} = \sigma^{-1}X\sigma$ is an attractor satisfying $\mathcal{L}(\tilde{X}) = ia\tilde{X}$ which proves the commutation relation $[L_i, \tilde{X}\sigma^{-1}] = [L_i, \sigma^{-1}X] = 0$. Both sets of these commutation relations are also valid for the attractor $X^\dagger$. Taking the adjoint of these equations we obtain the last two commutation relations $[L_i^\dagger, X\sigma^{-1}] = [L_i^\dagger, \sigma^{-1}X] = 0$.

In order to prove commutation relations (29) and (30) we will first derive commutation relations with the operator K for which we have to prove two useful equalities. Using commutation relations (28) we can rewrite equation $\mathcal{L}(X) = iaX$ into the form

$$Z_1 \equiv X\sigma^{-1}KX^\dagger - KX\sigma^{-1}X^\dagger - iaX\sigma^{-1}X^\dagger = X\sigma^{-1}\left[\sum_i K\sigma + \sigma K^\dagger - L_i\sigma L_i^\dagger\right]\sigma^{-1}X^\dagger.$$

As σ is a $\mathcal{T}$-state, it follows that operator Z_1 is positive. On the other hand, using

$$\mathcal{L}^\dagger(X\sigma^{-1}) = \sum_i L_i^\dagger X\sigma^{-1}L_i - K^\dagger X\sigma^{-1} - X\sigma^{-1}K = -iaX\sigma^{-1},$$

we find

$$\operatorname{Tr} Z_1 = \operatorname{Tr}\left\{\left[\sum_i L_i^\dagger L_i - K^\dagger - K\right] X\sigma^{-1}X^\dagger\right\} = \operatorname{Tr}\{\mathcal{L}^\dagger(I)X\sigma^{-1}X^\dagger\} \\ = \operatorname{Tr}\{\mathcal{L}(X\sigma^{-1}X^\dagger)\} \leq 0.$$

As a positive operator with a nonpositive trace must be equal to the zero operator, we have $Z_1 = 0$.

In order to obtain the second required equality we start from the equation $\mathcal{L}^\dagger(X\sigma^{-1}) = -iaX\sigma^{-1}$. Employing (28) we find that

$$Z_2 \equiv \sigma^{-1}X^\dagger KX\sigma^{-1} - \sigma^{-1}X^\dagger X\sigma^{-1}K + ia\sigma^{-1}X^\dagger X\sigma^{-1} = \sigma^{-1}X^\dagger\left[\sum_i K + K^\dagger - L_iL_i^\dagger\right]X\sigma^{-1}.$$

Operator Z_2 is obviously positive but on the other hand its trace can be rewritten using

$$\mathcal{L}^\dagger(\sigma^{-1}X^\dagger) = \sum_i L_i^\dagger\sigma^{-1}X^\dagger L_i - K^\dagger\sigma^{-1}X^\dagger - \sigma^{-1}X^\dagger K = ia\sigma^{-1}X^\dagger,$$

into the inequality

$$\operatorname{Tr} Z_2 = \operatorname{Tr}\left\{\left[\sum_i L_i^\dagger L_i - K^\dagger - K\right]\sigma^{-1}X^\dagger X\sigma^{-1}\right\} = \operatorname{Tr}\{\mathcal{L}^\dagger(I)\sigma^{-1}X^\dagger X\sigma^{-1}\} \\ = \operatorname{Tr}\{\mathcal{L}(\sigma^{-1}X^\dagger X\sigma^{-1})\} \leq 0.$$

Hence we conclude that $Z_2 = 0$.

Assume now operator $W = KX\sigma^{-1/2} - X\sigma^{-1}K\sigma^{1/2} + iaX\sigma^{-1/2}$. Based on the obtained equalities its Hilbert-Schmidt norm can be expressed as

$$\|W\|_{HS}^2 = \operatorname{Tr}\{X\sigma^{-1}K\sigma K^\dagger\sigma^{-1}X^\dagger - KXK^\dagger\sigma^{-1}X^\dagger - iaXK^\dagger\sigma^{-1}X^\dagger\} \\ + \operatorname{Tr}\{KX\sigma^{-1}KX^\dagger K^\dagger - X\sigma^{-1}KX^\dagger K^\dagger + iaX\sigma^{-1}X^\dagger K^\dagger\} \\ + \operatorname{Tr}\{iaX\sigma^{-1}KX^\dagger - iaKX\sigma^{-1}X^\dagger K^\dagger + a^2X\sigma^{-1}X^\dagger\} \\ = -\operatorname{Tr}\{Z_2\sigma K^\dagger\} - \operatorname{Tr}\{Z_1K^\dagger\} + ia\operatorname{Tr}\{Z_1\} = 0.$$

Thus we get $W = 0$, yielding the commutation relation $[X\sigma^{-1}, K] = iaX\sigma^{-1}$. This commutation relation must be valid also for attractors $\sigma^{-1}X\sigma$, and $X^\dagger$ which provides commutation relations $[\sigma^{-1}X, K] = ia\sigma^{-1}X$, $[X\sigma^{-1}, K^\dagger] = -iaX\sigma^{-1}$ and $[\sigma^{-1}X, K^\dagger] = -ia\sigma^{-1}X$. Now using $K = iH + \frac{1}{2}\mathcal{V}^\dagger(I) + G$ we finally arrive at commutation relations (29) and (30).

If QMDS is trace-preserving the commutation relations (28) and (30) constitute sufficient conditions for X being an attractor. Indeed, a straightforward calculation shows

$$\begin{aligned}\mathcal{L}^\dagger(X\sigma^{-1}) &= \sum_i L_i^\dagger X\sigma^{-1}L_i - K^\dagger X\sigma^{-1} - X\sigma^{-1}K\\ &= X\sigma^{-1}\left[\sum_i L_i^\dagger L_i - K^\dagger - K\right] - iaX\sigma^{-1}\\ &= X\sigma^{-1}\mathcal{L}^\dagger(I) - iaX\sigma^{-1} = -iaX\sigma^{-1}.\end{aligned}$$

Hence $X\sigma^{-1}$ is an attractor in the Heisenberg picture and consequently due to theorem 5 is X an attractor satisfying $\mathcal{L}(X) = iaX$.

Similarly, if $\mathcal{T}$-state σ is stationary, *i.e.* $\mathcal{L}(\sigma) = 0$, we get due to (28) and (30)

$$\mathcal{L}(X) = \sum_i L_i X L_i^\dagger - KX - XK^\dagger = X\sigma^{-1}\mathcal{L}(\sigma) + iaX = iaX,$$

confirming that X is an attractor following $\mathcal{L}(X) = iaX$.

Analogous statements for attractors in the Heisenberg picture follow directly from 5. Assuming $\mathcal{L}^\dagger(X) = iaX$ we have $\mathcal{L}(X\sigma) = -iaX\sigma$ and $\mathcal{L}(\sigma X) = -ia\sigma X$ and thus eqs. (28), (29) and (30) have to be fulfilled for operators $X\sigma$ and σX. □

Note that Lindblad operators and the optical potential are involved in the selection of attractors only. The corresponding asymptotic spectrum is fully determined by the Hamiltonian.

Both structure theorems provide an insight into the algebraic structure of attractor spaces. It is already well known that if X is an eigenvector of QMP associated with eigenvalue λ then $X^\dagger$ is an eigenvector associated with eigenvalue $\overline{\lambda}$ [48]. Employing theorems 6 and 7 it follows for trace-preserving QMP or QMP with invariant $\mathcal{T}$-state σ that if X_1 and X_2 are attractors of QMP associated with eigenvalue λ_1 and λ_2 then $X_1\sigma^{-1}X_2$, or any permutation of these three operators, is also attractor associated with eigenvalue $\lambda_1\lambda_2$. Similarly, we have the same statement for attractors in the Heisenberg picture with $\sigma = I$. Consequently, while attractors in the Schrödinger picture do not form algebra, for attractors in the Heisenberg picture we can formulate the following corollary.

Corollary 3. *Assume a QMP which is either trace-preserving or whose $\mathcal{T}$-state σ is stationary. Then the whole attractor space in the Heisenberg picture and its subspace of fixed points form C^* algebras.*

This statement combined with theorem 3 gives us a useful characterization of all stationary states or even all asymptotic states.

6 Asymptotic and stationary states of trace-preserving QMPs

Due to their structure and algebraic properties, the description of the asymptotics in terms of attractors is elegant. However, we have to face the fact that the attractors are not states. They constitute building blocks, operators, from which asymptotic states (13) and (17) are constructed. The range of coefficients $\mathrm{Tr}\{\rho(0)(X^{\lambda,i})^\dagger\}$ occuring in these formulas is largely unknown and makes a complete characterization of asymptotic states quite involved and in many cases unfeasible. In this part we present two characterizations of asymptotic states and its subset of stationary states of trace-preserving QMPs allowing a glance on the asymptotics.

The first characterization relies on C^* algebraic structure of attractors in the Heisenberg picture. Obviously, the exponential map

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!} \tag{34}$$

maps any Hermitian operator A from the attractor space in the Heisenberg picture (respectively, any integral of motion A) to a strictly positive operator from the attractor space in the Heisenberg picture (respectively, to a strictly positive integral of motion). Using the bijection (23) with $\alpha = -1/2$ between attractors in both pictures we find for any Hermitian operator A from the attractor space in the Heisenberg picture (respectively, any integral of motion A) that the operator

$$\sigma^{1/2}\exp(A)\sigma^{1/2} \tag{35}$$

corresponds, to a strictly positive asymptotic state (respectively, to a strictly positive stationary state). Because the identity operator is an integral of motion we can always choose operator A in a way that (35) is properly normalized.

However, we have a more ambitious inverse task in mind, namely to show that any asymptotic state (respectively, any stationary state) may be written as (35). To proceed we employ the following analytic formula for the logarithmic map

$$\log(I+A)=\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}A^k, \tag{36}$$

which converges for any Hermitian operator with its spectrum within the interval $(-1,1]$. Assume a strictly positive asymptotic state (respectively, a strictly positive stationary state) ρ. Then $\omega=\gamma\sigma^{-1/2}\rho\sigma^{-1/2}$ is a strictly positive operator from the attractor space in the Heisenberg picture (respectively, a strictly positive integral of motion) normalized by $\gamma=1/\operatorname{Tr}(\sigma^{-1/2}\rho\sigma^{-1/2})$. Consequently,

$$\log(\omega)=\log(I+(\omega-I))=\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}(\omega-I)^k \tag{37}$$

is well defined and yields a Hermitian operator from the attractor space in the Heisenberg picture (respectively, an integral of motion). Indeed, the identity I is an integral of motion ensuring trace-preservation of the given QMP and the attractor space as well as the space of fixed points of quantum Markov evolution in the Heisenberg picture is closed under all algebraic operations involved in (37). Thus, any strictly positive asymptotic state takes the form (35). We can generalize this statement to all asymptotic states, because strictly positive asymptotic states constitute a dense set inside all asymptotic states. Indeed, starting with an asymptotic state ρ, we can define a set of strictly positive asymptotic states

$$\omega(s)\equiv(1-s)\rho+s\sigma=\sigma^{1/2}\exp(A(s))\sigma^{1/2} \tag{38}$$

with $s\in(0,1]$. Apparently,

$$\rho=\lim_{s\to 0_+}\omega(s)=\lim_{s\to 0_+}\sigma^{1/2}\exp(A(s))\sigma^{1/2}. \tag{39}$$

A convenient way to express this limit is to choose a Hermitian base $\{Z_i\}$ (with $Z_i^\dagger=Z_i$) of all asymptotic operators. As the attractor space in the Heisenberg picture is enclosed under the adjoint map, such basis always exist. Hence any asymptotic state ρ can be written as

$$\rho=\lim_{s\to 0_+}\sigma^{1/2}\exp\left(\sum_i\beta_i(s)Z_i\right)\sigma^{1/2}. \tag{40}$$

Note that if $\{Z_i\}$ constitute a Hermitian basis of fixed points of quantum Markov evolution in the Heisenberg picture then (40) provides us with all stationary states of the given QMP. In fact, the limiting procedure (40) means that some of these real coefficients $\beta_i(s)$ approach, in the limit $\lim_{s\to 0_+}$, either plus or minus infinity, otherwise the state ρ is strictly positive. This might appear counterintuitive, but in statistical physics we meet this situation frequently. For example, one obtains a ground state of a canonical ensemble by taking the temperature limit $T\to 0_+$ which corresponds to $\beta\equiv 1/(kT)\to+\infty$ [38].

The second characterization of asymptotic and consequently also stationary states provides corollary 2. Assume that σ is an invariant $\mathcal{T}$-state and ρ is some strictly positive stationary state, *i.e.* ρ is an attractor associated with eigenvalue one but it is also a $\mathcal{T}$-state. According to the corollary the operator $\rho\log(\rho)-\rho\log(\sigma)$ is an attractor in the Schrödinger picture associated with eigenvalue one. Hence, the operator $\log(\rho)-\log(\sigma)$ is a Hermitian attractor of evolution in the Heisenberg picture associated with eigenvalue one, *i.e.* it is an integral of motion. By choosing a Hermitian base $\{Y_i\}$ of integrals of motion we can write down any strictly positive stationary state ρ into the form

$$\rho=\exp\left(\log(\sigma)+\sum_i\gamma_iY_i\right), \tag{41}$$

where γ_i are real expansion coefficients of the operator $\log(\rho)-\log(\sigma)$ in the base $\{Y_i\}$.

A generalization of (41) to all strictly positive asymptotic states follows from the fact that all asymptotic states (13) (respectively, (17) in continuous case) are actually stationary states of the quantum operation

$$\tilde{\mathcal{T}}(\rho) = \sum_{\lambda \in \sigma_{as}, i} X_{\lambda,i} \operatorname{Tr}\{\rho (X^{\lambda,i})^\dagger\}. \tag{42}$$

Indeed, as the asymptotic spectrum σ_{as} of QMP contains a finite number of eigenvalues, we can choose an ascending sequence of natural numbers n_j in such a way [50] that

$$\tilde{\mathcal{T}} = \lim_{j\to+\infty} \mathcal{T}^{n_j}, \quad \text{respectively,} \;\; \tilde{\mathcal{T}} = \lim_{j\to+\infty} \exp(\mathcal{L} n_j). \tag{43}$$

Therefore $\tilde{\mathcal{T}}$ is quantum operation which projects all states onto the set of asymptotic states of the original QMP. All attractors of the original QMP are fixed points of $\tilde{\mathcal{T}}$ and because of positivity of the original QMP we have also $\tilde{\mathcal{T}}(\sigma) \leq \sigma$. Now, let ρ be a strictly positive asymptotic state of the original QMP. It is stationary state of $\tilde{\mathcal{T}}$ and thus it can be written as (41), where $\{Y_i\}$ is chosen as a Hermitian base of fixed points of $\tilde{\mathcal{T}}$ and consequently it forms a Hermitian base of the attractor space of the original QMP. Thus

$$\rho = \exp\left(\log(\sigma) + \sum_i \gamma_i Z_i\right), \tag{44}$$

with a Hermitian basis $\{Z_i\}$ of the attractor space in the Heisenberg picture, describes all strictly positive asymptotic states of the given QMP. Following the same recipe as in the first characterization we finally enlarge (41) to all asymptotic states of the given trace-preserving QMP.

We also stress that while the first characterization (40) is valid only for trace-preserving QMPs, the second characterization of asymptotic states (44) applies to trace-nonincreasing QMPs as well provided there is a stationary strictly positive state σ. We have found two expressions for asymptotic states of QMPs. They are, in general, different as we show in examples 8. We expect that especially the asymptotic form (41) can be further exploited to study thermodynamic properties of QMPs. It has also the advantage that it may apply to trace-nonincreasing QMPs also. Both forms of asymptotic states (40) (41) remind of Gibbs states, the well-known family of macroscopic states in statistical physics [38]. A detailed investigation of their intricate connection will be presented elsewhere.

7 Dynamics within attractor spaces of QMPs

From a different perspective, the attractor space is the part of the total Hilbert space $\mathsf{B}(\mathscr{H})$ which is exempt from effects of decay, decoherence and dissipation in dependence on the detailed features of the process. In principle, any information encoded in this subspace should be fully retrieved. In order to recover this information we need to understand in detail the type of evolution rules the asymptotic states of QMPs. It was shown [51] that asymptotic states of random unitary operations undergo a unitary evolution. In general case of QMPs it is stated [32,26,35] that the evolution inside the attractor space should be of unitary type as well, because all attractors during such evolution acquire only its individual phase (13), (17). Using structure theorems 6 and 7 we show that for trace-preserving QMPs equipped with a faithful invariant state it is in some sense true.

We start with discrete QMCHs. Let X be any operator from the attractor space in the Schrödinger picture, *i.e.* $X = \sum_{\lambda,i} c_{\lambda,i} X_{\lambda,i}$. A straightforward calculation exploiting a quantum operation (24) gives us

$$\begin{aligned}
\mathcal{T}^\ddagger \mathcal{T}(X) &= \mathcal{T}^\ddagger\left(\sum_{\lambda,i} c_{\lambda,i} \sum_k A_k X_{\lambda,i} \sigma^{-1} \sigma A_k^\dagger\right) = \mathcal{T}^\ddagger\left(\sum_{\lambda,i} \lambda c_{\lambda,i} X_{\lambda,i} \sigma^{-1} \mathcal{T}(\sigma)\right) \\
&= \sum_{\lambda,i} \lambda c_{\lambda,i} \sigma^{1/2} \left(\sum_k A_k^\dagger \sigma^{-1/2} X_{\lambda,i} \sigma^{-1/2} A_k\right) \sigma^{1/2} \\
&= \sum_{\lambda,i} |\lambda|^2 c_{\lambda,i} X_{\lambda,i} \sigma^{-1/2} \mathcal{T}^\dagger(I) \sigma^{1/2} = \sum_{\lambda,i} c_{\lambda,i} X_{\lambda,i} = X,
\end{aligned} \tag{45}$$

where we use the fact that if $X_{\lambda,i}$ is an attractor associated with eigenvalue λ of the map $\mathcal{T}$. According to sect. 4.3 $\sigma^{-1/2} X_{\lambda,i} \sigma^{-1/2}$ is an attractor associated with eigenvalue $\bar{\lambda}$ of the map $\mathcal{T}^\dagger$ and thus satisfies theorem 6. Similarly, one can readily find out $\mathcal{T}\mathcal{T}^\ddagger(X) = X$. Hence, trace-preserving quantum operation $\mathcal{T}^\ddagger$ constitute a searched generator of the inverse evolution capable to correct an information inscribed into states from the asymptotic space of a given QMP. In fact this is Petz recovery map [41], which is a special example of the reverse-time quantum Markov operation [52]. Moreover, $\mathcal{T}^{\ddagger}$ is adjoint map of the original generating quantum operation $\mathcal{T}$ with respect to the scalar product $(X,Y)_{1/2} = (X, \sigma^{-1/2} Y \sigma^{-1/2})$. Thus, we can confirm that the asymptotic evolution is unitary, but in a different sense then we are used to. First, it is an unitary evolution on the attractor subspace of operators from the Hilbert space $\mathsf{B}(\mathscr{H})$, *i.e.* there is no underlying unitary evolution on the Hilbert space $\mathscr{H}$. Second, it is an unitary evolution with respect to a different scalar product on the space $\mathsf{B}(\mathscr{H})$.

In the case of trace-preserving QMDSs we derive a master equation governing their asymptotic dynamics. Let X be an operator from the attractor space, *i.e.* according to theorem 7 the operator $X\sigma^{-1}$ commutes with all Lindblad operators L_i's. The effect of the Lindblad generator (9) on X can be simplified as

$$\begin{aligned}\mathcal{L}(X) &= i[X,H] + \sum_j L_j X\sigma^{-1}\sigma L_j^{\dagger} - \frac{1}{2}\{L_j^{\dagger}L_j, X\sigma^{-1}\sigma\}\\ &= i[X,H] + X\sigma^{-1}\left(\sum_j L_j X L_j^{\dagger} - \frac{1}{2}\{L_j^{\dagger}L_j, X\}\right)\\ &= i[X,H] - iX\sigma^{-1}[\sigma,H] = i[HX - X\sigma^{-1}H\sigma],\end{aligned}$$

where we use $\mathcal{L}(\sigma) = 0$. Hence, the master equation takes the form

$$\frac{\mathrm{d}(X\sigma^{-1})}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t}\sigma^{-1} = \mathcal{L}(X)\sigma^{-1} = i[X\sigma^{-1}, H]. \tag{46}$$

Thus, instead of states, their multiplication with the operator σ^{-1} undergo an unitary evolution driven by Hamiltonian H. In the same spirit, its inverse evolution is driven by Hamiltonian $-H$.

8 Examples

In this part we show two examples demonstrating different aspects of the presented theory. We have chosen two simple, but nontrivial examples. The first refers to the creation of entanglement between two qubits and is motivated by the creation of large scale entanglement in a network of many qubits. A full analysis of such a network goes beyond the scope of our paper and is left for a future publication. The other example is motivated by studies of transport of excitation in quantum systems. As in the first example we aim only at illustrating the power of the theory and a complete analysis will be presented elsewhere.

8.1 Discrete random unitary process

Let us assume two qubits and two control NOT operations. U_{12} acts on the first qubit as controlled and on the second qubit as target and U_{21} acts in reverse order

$$U_{12} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \qquad U_{21} = \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}. \tag{47}$$

Suppose these two unitary operations act randomly on both qubits with corresponding probabilities p_{12} and p_{21}. The resulting propagator determining one step of evolution is a random unitary map

$$\mathcal{T}(\rho) = p_{12}U_{12}\rho U_{12}^{\dagger} + p_{21}U_{21}\rho U_{21}. \tag{48}$$

As this is an unital Markov evolution, it has the same six-dimensional attractor space in both pictures [51]. It contains five-dimensional attractor space associated with eigenvalue one, spanned by the identity operator I, $|\phi\rangle\langle\phi|$, $|\psi\rangle\langle\psi|$, $|\phi\rangle\langle\psi|$ and $|\psi\rangle\langle\phi|$ with $|\phi\rangle = |00\rangle$ and $|\psi\rangle = 1/\sqrt{3}(|01\rangle + |10\rangle + |11\rangle)$. Moreover, there is also one-dimensional subspace spanned by operator

$$X_{-1} = \begin{pmatrix} 0&0&0&0\\ 0&0&-1&1\\ 0&1&0&-1\\ 0&-1&1&0 \end{pmatrix} \tag{49}$$

associated with eigenvalue -1.

Let us assume $\mathcal{T}$-state $\sigma = 1/5(I + |\phi\rangle\langle\phi|)$. This $\mathcal{T}$-state does not commute with all attractors, *e.g.* $[\sigma, |\phi\rangle\langle\psi|] = 1/5|\phi\rangle\langle\psi|$. Consequently, two characterizations of stationary or asymptotic states provided in sect. 6 are not equivalent. For example, stationary states $\rho_1 = 1/\mathcal{N}_1 \exp(\log(\sigma) + |\psi\rangle\langle\phi| + |\phi\rangle\langle\psi|)$ and $\rho_2 = 1/\mathcal{N}_2\sqrt{\sigma}\exp(|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi|)\sqrt{\sigma}$, with properly chosen normalizations $\mathcal{N}_{1(2)}$, are different.

In order to illustrate that asymptotic states can be obtain as a limit (40) let us assume the state

$$\rho = \frac{1}{2}\left(I - |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| + \frac{i}{\sqrt{3}}X_{-1}\right).$$

It is an asymptotic state which is not strictly positive. By choosing a $\mathcal{T}$-state proportional to the identity operator, it is easy to verify that it can be written as a limit of strictly positive asymptotic states

$$\rho = \lim_{s\to+\infty}\frac{1}{\mathcal{N}(s)}\exp\left[\frac{-s}{\sqrt{2}}(I - |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|) + \frac{2is}{\sqrt{6}}X_{-1}\right], \tag{50}$$

with normalization constant $\mathcal{N}(s)$.

8.2 Continuous QMDS with jump Lindblad operators

In this example we assume a quantum system associated with a four-dimensional Hilbert space with orthonormal base $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. Its continuous Markov evolution is governed by Lindbladian $\mathcal{L}$ which acts as

$$\mathcal{L}(\rho) = -i[H,\rho] + 2\left(h_+\rho h_+^\dagger - \frac{1}{2}\{h_+^\dagger h_+, \rho\}\right) + h_-\rho h_-^\dagger - \frac{1}{2}\{h_-^\dagger h_-, \rho\}, \tag{51}$$

with

$$H = \varepsilon(|2\rangle\langle 2| + |3\rangle\langle 3|), \quad h_+ = |0\rangle\langle 1| + |2\rangle\langle 3| = h_-^\dagger. \tag{52}$$

As it is shown below, the system is equipped with a $\mathcal{T}$-state σ and thus our theory applies. The attractor space contains subspaces corresponding to eigenvalues 0 and $\pm i\varepsilon$. The former is two-dimensional, spanned by operators $\{X_1, X_2\}$ and each of the latter is one-dimensional, spanned by operators $X_\pm$. These operators read

$$\begin{aligned} X_1 &= 2|0\rangle\langle 0| + |1\rangle\langle 1|, \qquad & X_2 &= 2|2\rangle\langle 2| + |3\rangle\langle 3|,\\ X_+ &= 2|0\rangle\langle 2| + |1\rangle\langle 3|, \qquad & X_- &= 2|2\rangle\langle 0| + |3\rangle\langle 1|. \end{aligned}$$

As one can easily check, any linear combination $\alpha_1 X_1 + \alpha_2 X_2$ with $\alpha_i \neq 0$ fulfils the requirements of a $\mathcal{T}$-state and our previous considerations are thus correct.

Consequently, the attractor space in the Heisenberg picture is also four-dimensional. It consists of two-dimensional subspace of integrals of motion, which is spanned by the identity operator I and Hamiltonian H and two one-dimensional subspaces corresponding to eigenvalues $\mp i\varepsilon$. These subspaces are spanned by operators $A_\mp$, which read

$$A_- = |0\rangle\langle 2| + |1\rangle\langle 3|, \qquad A_+ = |2\rangle\langle 0| + |3\rangle\langle 1|.$$

It is convenient to define Hermitian operators $A_R = \frac{1}{2}(A_+ + A_-)$ and $A_I = \frac{1}{2i}(A_+ - A_-)$ and to express the results in the Hermitian basis $\{A_1, A_2, A_R, A_I\}$.

Let us have a closer look at the structure of the asymptotic/stationary states. As an example, we assume $\sigma = \frac{1}{6}(X_1 + X_2)$ and one-parameter class of nonstationary asymptotic states $\rho(s) = \frac{1}{6}(X_1 + X_2 + sX_+ + sX_-)$. Apparently $\rho(0) = \sigma$ and $\rho(s)$ is strictly positive for any $s \in (-1, 1)$ and nonstrictly positive for $s = \pm 1$. According to (44) one can thus write for any $s \in (-1, 1)$

$$\rho(s) = \mathcal{N}(s)\exp[\ln\sigma - \beta(s)H - \gamma_R(s)A_R - \gamma_I(s)A_I]. \tag{53}$$

The normalization parameter $\mathcal{N}(s)$ replaces the identity operator (integral of motion) and its corresponding multiplier. Since $\rho(s)$ is balanced in X_1 and X_2, we get $\beta(s) = 0$. Furthermore, $\rho(s)$ is real and thus $\gamma_I(s) = 0$. By a straightforward calculation, we get $\gamma_R(s) = \ln\frac{1+s}{1-s}$. Cases $s = \pm 1$ are resolved via the limit procedure

$$\rho(\pm 1) = \lim_{s\to\pm 1}\mathcal{N}(s)\exp[\ln(\sigma) - \gamma_R(s)A_R] = \lim_{\gamma_R\to\pm\infty}\mathcal{N}(\gamma_R)\exp[\ln(\sigma) - \gamma_R A_R]. \tag{54}$$

Stationary states form a special class of asymptotic states. In this examined case all stationary states can be expressed as a linear combination of operators X_1 and X_2. Thus, all stationary states commute with each other and consequently all strictly positive stationary states can be written in equivalent forms

$$\rho = \mathcal{N}\exp[\ln(\sigma) - \beta_\sigma H] = \mathcal{N}\sigma^{\frac{1}{2}}\exp[-\beta_\sigma H]\sigma^{\frac{1}{2}}, \tag{55}$$

with σ being an arbitrary $\mathcal{T}$-state. As an example, let us take $\sigma = \frac{1}{6}(X_1 + X_2)$. By direct calculation, one can show that strictly positive stationary states can be represented as

$$\exp[\ln(\sigma) - \beta H] = \frac{1}{3 + 3e^{-\beta\varepsilon}}(X_1 + e^{-\beta\varepsilon}X_2), \quad \beta \in \mathbb{R}. \tag{56}$$

Let us parametrize the stationary states as $\rho(s) = \frac{1}{3}((1-s)X_1 + sX_2)$. This linear combination is strictly positive for $s \in (0,1)$ and nonstrictly positive for $s \in \{0,1\}$. For $s \in (0,1)$, we can write

$$\rho(s) = \mathcal{N}(s)\exp[\ln(\sigma) - \beta(s)H], \tag{57}$$

where $\beta(s) = \ln\frac{s}{1-s}$ by direct calculation. Cases $s \in \{0,1\}$ are again resolved via the limit procedure

$$\rho(0) = \lim_{s\to 0}\mathcal{N}(s)\exp[\ln(\sigma) - \beta(s)H] = \lim_{\beta\to-\infty}\mathcal{N}(\beta)\exp[\ln(\sigma) - \beta H],$$
$$\rho(1) = \lim_{s\to 1}\mathcal{N}(s)\exp[\ln(\sigma) - \beta(s)H] = \lim_{\beta\to+\infty}\mathcal{N}(\beta)\exp[\ln(\sigma) - \beta H].$$

Both representations of stationary and asymptotic states (40) and (44) are in this case equivalent. However, if we switch the Hamiltonian (52) off, *i.e.* we set $H = 0$, this statement is not anymore true. The size of the attractor space remains unchanged, however all attractors now correspond to the eigenvalue 0. All asymptotic states are thus stationary states. By making the choice $\sigma = \frac{1}{12}(2X_1 + 2X_2 + X_+ + X_-)$, we can show as an example that the stationary states $\rho_1 = \mathcal{N}_1\exp[\ln(\sigma) - \gamma(A_R + A_I)]$ and $\rho_2 = \mathcal{N}_2\sigma^{\frac{1}{2}}\exp[-\gamma(A_R + A_I)]\sigma^{\frac{1}{2}}$ are not equal. For instance, $\gamma = -\ln(3)$ yields $\rho_1 = \frac{1}{12}(2X1 + 2X2 - i(X_+ + X_-))$ while $\rho_2 = \frac{1}{12}(2X1 + 2X2 + \alpha(X_+ + X_-))$, with $\mathrm{Re}(\alpha) > 0$.

9 Conclusion

Both types of quantum Markov processes (QMPs), discrete quantum Markov chains (QMCHs) and continuous quantum Markov dynamical semigroups (QMDSs) are realistic classes of open-system dynamics describing a wide range of processes of significant importance in physics. After introducing the needed formal frame we derived a number of results describing the properties of the asymptotic dynamics of quantum discrete and continuous, in general trace-nonincreasing, Markov processes. In this way we extend significantly the previously known theory for QMCHs and generalize its application also to QMDSs [34]. We formulate and prove basic fundamental theorems concerning the asymptotic dynamics and point out a number of its interesting properties. In particular, based on operator monotone functions a general set of relations between attractors of QMPs in both pictures are revealed and specified for two important cases. Consequently, it provides a dual basis of attractors in the Schrödinger picture and thus significantly simplifies the task of finding the asymptotic dynamics for any initial state. Furthermore, we derive equations determining attractors of QMPs in both pictures. We showed that the asymptotic evolution of QMPs has a unitary character if we redefine the relevant Hilbert-Schmidt scalar product. However, we stress that this unitary evolution may not correspond to a unitary evolution on the original Hilbert space of pure states. Moreover, based on the developed theory, two characterizations of asymptotic states are provided, both strongly resembling the form of Gibbs states known from statistical physics. This feature will be the subject of further studies as it points out to an intimate relation between the statistical character of the dynamics and the thermodynamic features of the asymptotic dynamics. Finally, we provide two elementary examples to demonstrate how our theory works. The examples describe the simplest possible nontrivial cases. The chosen models have natural extensions. However, they are much more involved and physically more interesting. Their detailed studies will be presented elsewhere as the analysis goes clearly beyond the scope of the present paper.

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