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A new anisotropic solution by MGD gravitational decoupling

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Abstract. This paper is focused on the search for new anisotropic analytic solutions to Einstein's field equations for a spherically symmetric and static stellar distribution by means of the gravitational decoupling realized via the Minimal Geometric Deformation (MGD) approach. Firstly, a Buchdahl perfect fluid inside the stellar distribution is considered and the Einstein's field equations are used in order to obtain the explicit form of the pressure and density for the perfect fluid. Then, the matching conditions for stellar distributions are used to find the constants involved in the Buchdahl solution in order to ensure the geometric continuity at the stellar surface. Finally, the Buchdahl solution is deformed to obtain the anisotropic solution and the matching conditions are used to find the constants involved in the new solution. The result is a new analytic and well-behaved anisotropic solution, in which all their physical parameters, such as the effective density, the effective radial and tangential pressure, fullfill each of the requirements for the physical acceptability available in the literature. Therefore, this solution can give a satisfactory description of realistic astrophysical compact objects like stars.

1 Introduction

Stellar distributions have been studied ever since the first solution of Einstein's field equations for the interior of an astrophysical compact object was obtained by Schwarzschild in 1916. He solved the field equations for the interior region by assuming a perfect fluid, and also for the outside vacuum region. The interior solution has the geometry of a three-sphere, which was noticed in 1919 by Weyl [1]. The search for the exact isotropic and anisotropic solutions describing static and spherically symmetric stellar distributions has attracted the interest of physicist. However, there are very few exact interior isotropic and anisotropic solutions of the field equations satisfying the required general physical conditions inside the star. The study of the interior of stellar distributions via finding exact solutions of the field equations is still a field of research. Fodor [2] has proposed an algorithm to generate any number of interior isotropic solutions for spherically symmetric and static distributions. Also, Schmidt and Homann [3] discussed numerical solutions of Einstein's field equations describing a static and spherically symmetric photon star.

Since the pioneering work of Bowers and Liang [4] there is extensive literature devoted to the study of anisotropic solutions of Einstein's field equations. Harko and Mak [5,6] have shown that nuclear matter may be anisotropic in high density ranges, or from the point of view of the Newtonian gravity spherical galaxies can have anisotropic matter distribution. In addition, Harko and Mak argue that the interior of a star must fulfills the general physical conditions that describes a well-behaved isotropic or anisotropic solution. The theoretical investigations of Ruderman [7] about more realistic stellar models show that the nuclear matter may be anisotropic at least in certain very high density ranges ($\rho > 10^{17}\,\mathrm{kg/m^3}$), where the nuclear interactions must be treated relativistically. The anisotropic behaviour occurs when the pressure is split in two different contributions, the radial and tangential pressure.

Anisotropies in fluid usually arise due to the presence of a mixture of fluids of different types, rotation, viscosity, the existence of a solid core, the presence of type $3A$ superfluid [8], different kinds of phase transitions [9], a magnetic field or by other physical phenomena. The sources of anisotropies have been widely studied, particularly for different highly compact astrophysical objects such as compact stars or black holes, either in 4 dimensions [10] as well as in the context of braneworld solution in higher dimensions [11]. In a recent paper [12], the first simple, systematic and direct approach to decoupling gravitational sources in general relativity (GR) was developed from the so-called Minimal Geometric Deformation (MGD) approach. The MGD was originally proposed [13,14] in the context of the

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Randall-Sundrum braneworld [15,16] (for some earlier works on MGD, see, for instance, refs. [17–20], and for some recent applications refs. [21–30]).

Regarding decoupling gravitational sources, Ovalle proposed to solve the Einstein's field equations by solving the field equations for each gravitational source individually (see ref. [12]), namely,

$$\hat{G}_{\mu\nu} = -k^2\,\hat{\Psi}_{\mu\nu}, \quad \text{to find } \{\hat{g}_{\mu\nu}, \hat{\Psi}_{\mu\nu}\} \tag{1}$$

and then

$$G^{\star}_{\mu\nu} = -k^2\,\Phi^{\star}_{\mu\nu}, \quad \text{to find } \{g^{\star}_{\mu\nu}, \Phi^{\star}_{\mu\nu}\}, \tag{2}$$

where $k^2 = 8\pi$. Once the two metrics are found by eqs. (1) and (2), namely, $\hat{g}_{\mu\nu}$ and $g^{\star}_{\mu\nu}$, they can be combined to derive the complete solution for the total system. The remarkable of the MGD decoupling is that one isotropic solution is deformed and it is produces a new solution that preserves spherical symmetry. Therefore, we could choose isotropic well-behaved and spherically symmetric solutions and then research if the new solution is still well-behaved. Regarding this, from 127 published isotropic solutions analyzed in Delgaty and Lake [31] only 16 are well-bahaved. In particular, one interesting isotropic well-behaved and spherically symmetric solution is the Buchdahl solution [32]. Since Einstein field equations are non-linear, the MGD decoupling proposed by J. Ovalle represents an advance to search for new anisotropic solutions especially for situations beyond the trivial cases, such as self-gravitating systems with gravitational sources more realistics than perfect fluid [33,34].

MGD decoupling does not only give physically acceptable interior solutions for different isotropic perfect fluid in GR, but it could be applied in a large number of relevant cases, such as the Einstein-Maxwell [35] and Einstein-Klein-Gordon system [36–39], for higher derivative gravity [40,41], $f(R)$ theories of gravity [42–48], Hořava-aether gravity [49–51], and polytropic spheres [52–54]. In this respect, the simplest practical application of the MGD decoupling consists in extending known isotropic and physically acceptable interior solutions for spherically symmetric self-gravitating systems into the anisotropic domain, at the same time preserving physical acceptability, which is a highly non-trivial problem [55]. For obtaining anisotropic solutions in a generic way, see refs. [56,57].

This paper is organized as follows: In sect. 2, we review the MGD gravitational decoupling of Einstein field equations and the matching procedure for spherically symmetric and static stellar distributions. In sect. 3, we derive the explicit form of the physical parameters (p, ρ) for the Buchdahl perfect fluid solution. In sect. 4, the MGD decoupling is implemented in order to extend the perfect fluid solution in the anisotropic domain. In sect. 5 the concluding remarks are outlined with the perspectives.

2 MGD gravitational decoupling of Einstein field equations

To begin with, we must write the well-known Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} = -k^2\,\Psi^{(\mathrm{tot})}_{\mu\nu}, \tag{3}$$

with

$$\Psi^{(\mathrm{tot})}_{\mu\nu} = (\rho + p)\,u_\mu\,u_\nu - p\,g_{\mu\nu} + \beta\,\Phi_{\mu\nu}, \tag{4}$$

where β is a coupling constant and u^μ is the four-velocity of a perfect fluid with density ρ and isotropic pressure p. The term $\Phi_{\mu\nu}$ in eq. (4) describes any additional source like scalar, vector and tensor fields which can produce anisotropies in the fluid [58]. Divergence-free if one of the features of the Einstein tensor, hence the total energy-momentum tensor (4) must satisfy the conservation equation $\nabla_\nu\,\Psi^{(\mathrm{tot})\mu\nu} = 0$. In standard coordinates $x_\mu = (t, r, \theta, \phi)$ the general line element for a spherically symmetric space-time takes the form

$$\mathrm{d}s^2 = e^{\nu(r)}\,\mathrm{d}t^2 - e^{\lambda(r)}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{5}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$ are functions of the radial coordinate r, ranging from the star center $(r = 0)$ to the star surface $(r = R)$, and the fluid four-velocity is given by $u^\mu = e^{-\nu/2}\,\delta^\mu_0$. The general metric (5) must satisfy the Einstein field equations (3), which can be broken down as follows:

$$k^2(\rho + \beta\,\Phi^0_0) = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{6}$$

$$k^2(p - \beta\,\Phi^1_1) = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{7}$$

$$k^2(p - \beta\,\Phi^2_2) = \frac{e^{-\lambda}}{4}\left(2\,\nu'' + \nu'^2 - \lambda'\,\nu' + 2\,\frac{\nu' - \lambda'}{r}\right). \tag{8}$$

From the system (6)–(8), we can identify an effective density $\tilde{\rho} = \rho + \beta\,\Phi_0^0$, an effetive radial pressure $\tilde{p}_r = p - \beta\,\Phi_1^1$, and an effective tangential pressure $\tilde{p}_t = p - \beta\,\Phi_2^2$. These physical parameters shows that the source $\Phi_{\mu\nu}$ generates an anisotropy inside the stellar distribution given by

$$\Delta \equiv \tilde{p}_t - \tilde{p}_r = \beta(\Phi_1^1 - \Phi_2^2). \tag{9}$$

The system of eqs. (6)–(8) describes an anisotropic fluid [59], which requires to consider the two metric functions, $\nu(r)$ and $\lambda(r)$, and the effective physical parameters $\tilde{\rho}$, $\tilde{p}_r$, and $\tilde{p}_t$. A method to solve eqs. (6)–(8) is the Minimal Geometric Deformation (MGD) gravitational decoupling [60]. The method produces corrections to perfect fluid solutions providing physically plausible and spherically symmetric stellar distributions. Firstly, let us start by considering a perfect fluid solution $\{\xi, \mu, \rho, p\}$ to eqs. (6)–(8), namely, the solution corresponding to $\beta = 0$. The metric (5) now has the following form:

$$ds^2 = e^{\xi(r)}\,dt^2 - \big(\mu(r)\big)^{-1}\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \tag{10}$$

where

$$\mu(r) = 1 - \frac{2\,m(r)}{r} \tag{11}$$

is the standard general relativity (GR) expression for the radial metric component, being $m(r) = \frac{k^2}{2}\int_0^r x^2\rho\,dx$ the GR mass function. In the stellar surface the mass function $m(R)$ is the total mass of the stellar distribution (M_0). For the metric of eq. (5) be a solution of eq. (3), a geometric deformation over the perfect fluid geometry $\{\xi, \mu\}$ in eq. (10) is proposed (see ref. [12]), namely, $\xi \mapsto \nu = \xi + \beta\gamma$ and $\mu \mapsto e^{-\lambda} = \mu + \beta\eta$, where γ and η are functions parametrizing the geometric deformation. From all possible deformations γ and η, the so-called minimal geometric deformation corresponds to $\gamma = 0$ and $\eta = \eta^\star$. Then, in this particular case, the metric in eq. (10) is minimally deformed by the source $\Phi_{\mu\nu}$ and its radial metric component acquires the form

$$\mu(r) \mapsto e^{-\lambda(r)} = \mu(r) + \beta\,\eta^\star(r), \tag{12}$$

whereas the temporal metric component e^ν does not change. Now, by introducing eq. (12) in the Einstein equations (6)–(8), the system is divided as follows:

I) the Einstein equations for a perfect fluid, whose metric is given by eq. (10), with $\xi(r) = \nu(r)$

$$k^2\rho = \frac{1}{r^2} - \frac{\mu}{r^2} - \frac{\mu'}{r}, \tag{13}$$

$$k^2\,p = -\frac{1}{r^2} + \mu\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{14}$$

$$k^2\,p = \frac{\mu}{4}\left(2\nu'' + \nu'^2 + \frac{2\nu'}{r}\right) + \frac{\mu'}{4}\left(\nu' + \frac{2}{r}\right); \tag{15}$$

II) the equations that contain the source $\Phi_{\mu\nu}$

$$k^2\,\Phi_0^0 = -\frac{\eta^\star}{r^2} - \frac{{\eta^\star}'}{r}, \tag{16}$$

$$k^2\,\Phi_1^1 = -\eta^\star\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{17}$$

$$k^2\,\Phi_2^2 = -\frac{\eta^\star}{4}\left(2\nu'' + \nu'^2 + 2\frac{\nu'}{r}\right) - \frac{{\eta^\star}'}{4}\left(\nu' + \frac{2}{r}\right). \tag{18}$$

According to [60], each sector is separately conserved, which means that there is no exchange of energy momentum between them. Hence, the interaction between the two sectors is gravitational only. Also, it is important to remark that once a perfect fluid solution is chosen, the first sector $\{\nu;\mu;\rho;p\}$ is solved, then once the second sector $\{\eta^\star;\Phi_0^0;\Phi_1^1;\Phi_2^2\}$ is solved, we can to obtain directly the effective physical variables $\tilde{\rho}$, $\tilde{p}_r$, and $\tilde{p}_t$. In general, MGD decoupling can be extended for more than one additional sources $\Phi_{\mu\nu}$ (more details in reference [12]), but in this paper we considered one extra gravitational source only.

2.1 Matching procedure for spherically symmetric self-gravitating systems

In the study of stellar distributions the matching conditions at the star surface between the interior and the exterior space-time geometries plays an important role [61,62]. In our case, the interior space-time geometry is given by the

MGD metric (eq. (5) along with eq. (12)):

$$\mathrm{d}s^2 = e^{\nu^-(r)}\,\mathrm{d}t^2 - \left(1 - \frac{2\,\tilde{m}(r)}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{19}$$

where the interior mass function is given by $\tilde{m}(r) = m(r) - \frac{r}{2}\beta\eta^\star(r)$, with m given by the GR mass function and $\eta^\star$ the minimal geometric deformation. The internal metric (19) must be matched with the external metric which is assumed to be the vaccum Schwarzschild metric due to there is no isotropic fluid and such a region just contains the field $\Phi_{\mu\nu}$. The external metric have the general form

$$\mathrm{d}s^2 = e^{\nu^+(r)}\,\mathrm{d}t^2 - e^{\lambda^+(r)}\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{20}$$

where the metric functions ν^+ and λ^+ are obtained by solving the exterior Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -k^2\Phi_{\mu\nu}$. The continuity of the first fundamental form at the star surface Ω (defined by $r = R$) is given by

$$[\mathrm{d}s^2]_\Omega = 0, \tag{21}$$

which yields

$$\nu^-(R) = \nu^+(R) \tag{22}$$

and

$$1 - \frac{2\,M_0}{R} + \beta\,\eta^\star_R = e^{-\lambda^+(R)}, \tag{23}$$

where M_0 is the total mass of the stellar distribution and $\eta^\star_R$ is the minimal geometric deformation at the star surface. The superindices stand for the region from where we approach the stellar surface, either from inside with a minus sign, or from outside using the plus sign.

Also, we must take into account the Israel-Darmois [61,62] matching condition at the stellar surface Ω that gives the continuity of the second fundamental form

$$[G_{\mu\nu}\,r^\nu]_\Omega = 0, \tag{24}$$

where r^ν is a unit radial vector. Using eq. (24) and eq. (3), we find

$$\left[\Psi^{(\mathrm{tot})}_{\mu\nu}\,r^\nu\right]_\Omega = 0, \tag{25}$$

which leads to

$$p_R - \beta\,(\Phi_1^{\ 1})^-_R = -\beta\,(\Phi_1^{\ 1})^+_R, \tag{26}$$

where $p_R \equiv p^-(R)$. By using eq. (17) for the internal and external geometry in the condition (26), the second fundamental form can be written as

$$p_R + \beta\,\frac{\eta^\star_R}{k^2}\left(\frac{1}{R^2} + \frac{\nu'_R}{R}\right) = \beta\,\frac{h^\star_R}{k^2}\left[\frac{1}{R^2} + \frac{2\,\mathcal{M}_\mathcal{S}}{R^3}\left(1 - \frac{2\,\mathcal{M}_\mathcal{S}}{R}\right)^{-1}\right], \tag{27}$$

where $\nu'_R \equiv \partial_r\nu^-|_{r=R}$, $h^\star$ is the geometric deformation for the external Schwarzschild solution due to the source $\Phi_{\mu\nu}$ and $\mathcal{M}_\mathcal{S}$ is the Schwarzschild mass (see ref. [60]). At this point, we can remark that eqs. (22), (23) and (27) are the necessary and sufficient conditions for the matching of the interior MGD metric (19) to an external deformed Schwarzschild metric (see ref. [60]). Moreover, by considering an external geometry describes by the exact Schwarzschild metric (see ref. [60]), eq. (27) leads to the condition

$$\tilde{p}_R \equiv p_R + \beta\,\frac{\eta^\star_R}{k^2}\left(\frac{1}{R^2} + \frac{\nu'_R}{R}\right) = 0. \tag{28}$$

This expression suggests that the stellar distribution will be in equilibrium if the effective radial pressure at the surface vanishes.

3 Interior isotropic solution: The Buchdahl solution

In order to solve eqs. (6)–(8) for the interior of a self-gravitating system by the MGD decoupling, the physical parameters $\{\tilde{\rho}, \tilde{p}_r, \tilde{p}_t\}$ and the two metric functions $\{\nu, \lambda\}$ in eq. (5) will be derived. The first step is to turn off β and find a solution for the perfect fluid Einstein equations (13)–(15). In particular, we can choose a known solution with physical relevance, like the well-known Buchdahl solution $\{\nu, \mu, \rho, p\}$ for perfect fluids (see ref. [32]), namely,

$$e^{\nu(r)} = A\left[(1+Cr^2)^{3/2} + B\sqrt{2-Cr^2}\left(5+2Cr^2\right)\right]^2, \tag{29}$$

$$\mu(r) = \frac{2-Cr^2}{2(1+Cr^2)}, \tag{30}$$

$$\rho(r) = \frac{3C(3+Cr^2)}{16\pi(1+Cr^2)^2} \tag{31}$$

and

$$p(r) = \frac{9C}{16\pi(1+Cr^2)}\left[\frac{\sqrt{(1+Cr^2)(2-Cr^2)}(1-Cr^2) - B(2+3Cr^2-2C^2r^4)}{\sqrt{(1+Cr^2)(2-Cr^2)}(1+Cr^2) + B(10-Cr^2-2C^2r^4)}\right]. \tag{32}$$

The constants A, B, and C in eqs. (29)–(32) are determined from the matching conditions in eqs. (21) and (24) between the above interior solution and the exterior metric which we choose to be the Schwarzschild space-time. This yields

$$C = \frac{4M_0}{R^2(3R-4M_0)}, \tag{33}$$

$$B = \frac{\sqrt{(1+CR^2)(2-CR^2)}(1-CR^2)}{2+3CR^2-2C^2R^4}, \tag{34}$$

$$A = \frac{R-2M_0}{R\left[\sqrt{1+CR^2}(1+CR^2) + B\sqrt{2-CR^2}(5+2CR^2)\right]^2}, \tag{35}$$

with the compactness $M_0/R < 4/9$, and $M_0 = m(R)$ the total mass of the stellar distribution. The expressions in eqs. (33)–(35) ensure the geometric continuity at $r = R$ and will change when we add the source $\Phi_{\mu\nu}$.

The conditions of physical acceptability for isotropic solutions are well known [63]. In order that both pressure (p) and density (ρ) be positive definite at the origin and monotonically decreasing to the boundary, the constant C must be equals 1 and the constant B must be equals 1/2(see ref. [31]). When choosing $C = 1$ and $B = 1/2$, we obtain from eq. (34), $R = 1/\sqrt{6}$. Then, from eq. (33), we obtain $M_0 = 0.044$. Finally, from eq. (35), $A = 0.03$. Using these values, fig. 1 shows the behaviour of all physical parameters for the inner space, as the density ρ, the pressure p, the speed of sound $v_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}$ and the pressure-density ratio p/ρ. The results are in agreement with the criterion listed in refs. [31, 63] for a well-behaved isotropic fluid sphere. Also, since the final aim in finding the interior solutions is to model the astrophysical configurations, it is also necessary to check the four energy conditions (weak, null, strong and dominant). The energy conditions are a set of constraints which are usually imposed on the energy-momentum tensor in order to avoid exotic matter sources (see ref. [22]). The energy conditions are shown as follows: a) the Null Energy Condition (NEC), $T_{\mu\nu}K^\mu K^\nu \geq 0$ for any null vector K^μ. For a perfect fluid, this condition implies $\rho + p \geq 0$. b) the Weak Energy Condition (WEC), $T_{\mu\nu}X^\mu X^\nu \geq 0$ for any time-like vector X^μ, which, for a perfect fluid, yields $\rho \geq 0$ and $\rho + p \geq 0$. c) the Dominant Energy Condition (DEC), $T^\mu{}_\nu X^\nu = -Y^\mu$, where Y^μ must be a future-pointing causal vector. For a perfect fluid, this means $\rho > |p|$. Finally, d) the Strong Energy Condition (SEC), $(T_{\mu\nu}\frac{1}{2}T\,g_{\mu\nu})X^\mu X^\nu \geq 0$, or, for a perfect fluid, $\rho + p \geq 0$ and $\rho + 3p \geq 0$. All of the above conditions are satisfied.

4 Interior anisotropic solution by MGD decoupling

Now lets turn on the coupling constant β in the interior. The temporal and radial metric components are given by eqs. (29) and (12) respectively, where the interior deformation $\eta^\star(r)$ and the source $\Phi_{\mu\nu}$ are related through eqs. (16)–(18). In the following, one new, exact and physically acceptable interior solution will be generated.

From the matching condition (28) we see that the exterior geometry describes by the exact Schwarzschild exterior metric will be compatible with the interior geometry describes by the MGD metric as long as $\beta\,(\Phi_1{}^1)_R^- \sim p_R$. The choice should be

$$\Phi_1{}^1(r) = p(r), \tag{36}$$

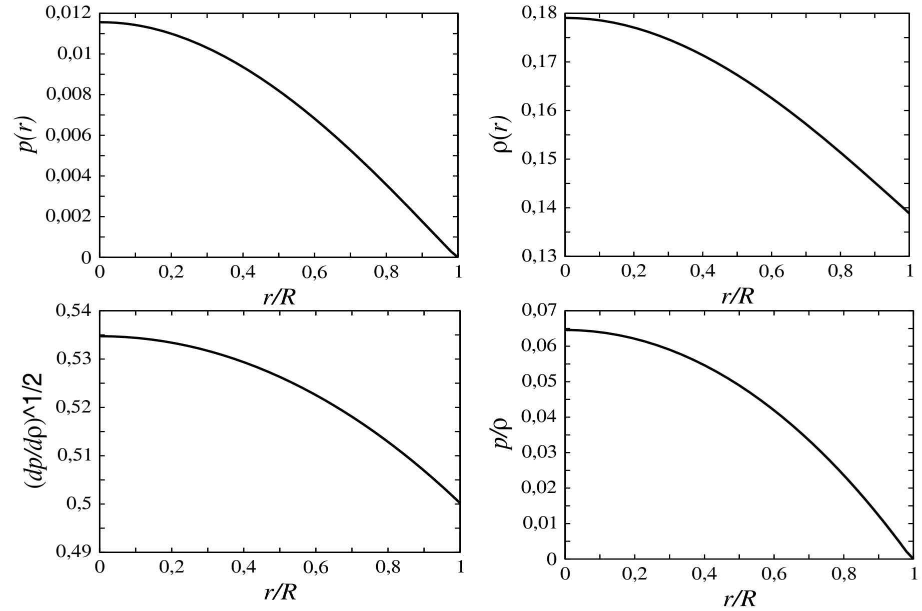

Fig. 1. Pressure p and density ρ for the Buchdahl perfect fluid solution as an gravitational source in a spherically symmetric space-time. Any perfect fluid solution can be consistently extended to the anisotropic domain via the MGD decoupling.

which, according to eq. (14), can be written as

$$k^2\,\Phi_1^{\,1} = -\frac{1}{r^2} + \mu(r)\left(\frac{1}{r^2} + \frac{\nu'}{r}\right). \tag{37}$$

From eq. (17) we can see that the mimic constraint in eq. (36) is equivalent to

$$\eta^\star(r) = -\mu(r) + \frac{1}{1 + r\,\nu'(r)}\,. \tag{38}$$

Hence the radial metric component reads

$$\begin{aligned} e^{-\lambda(r)} &= (1-\beta)\mu(r) \\ &\quad + \beta\left(\frac{\sqrt{(1+Cr^2)(2-Cr^2)}(1+Cr^2) + B(2-Cr^2)(5+2Cr^2)}{\sqrt{(1+Cr^2)(2-Cr^2)}(1+6C+Cr^2) + B[(2-Cr^2)(5+2Cr^2)+6C(1-2Cr^2)]}\right), \end{aligned} \tag{39}$$

where the expressions in eqs. (12) and (29) have been used. The interior metric functions given by eqs. (29) and (39) represent the Buchdahl solution minimally deformed by the source $\Phi_{\mu\nu}$. We can see that the limit $\beta \to 0$ in eq. (39) leads to the standard Buchdahl solution for a perfect fluid.

Now we want to match our interior metric in eq. (5) with metric functions (29) and (39) with the exterior Schwarzschild solution (see ref. [60]). We can see that, for a given distribution of mass M_0 and radius R, we have four unknown parameters $\{A, B, C\}$ from the interior solution in eqs. (29) and (39), and the Schwarzschild mass $\mathcal{M}_\mathcal{S}$.

The continuity of the first fundamental form given by eqs. (22) and (23) leads to

$$A\left[(1+CR^2)^{3/2} + B\sqrt{2-CR^2}(5+2CR^2)\right]^2 = 1 - \frac{2\mathcal{M}_\mathcal{S}}{R} \tag{40}$$

and

$$\begin{aligned} &(1-\beta)\mu(R) \\ &+ \beta\left(\frac{\sqrt{(1+CR^2)(2-CR^2)}(1+CR^2) + B(2-CR^2)(5+2CR^2)}{\sqrt{(1+CR^2)(2-CR^2)}(1+6C+CR^2) + B[(2-CR^2)(5+2CR^2)+6C(1-2CR^2)]}\right) = 1 - \frac{2\mathcal{M}_\mathcal{S}}{R}\,, \end{aligned} \tag{41}$$

whereas continuity of the second fundamental form in eq. (26) yields

$$p_R - \beta\,(\Phi_1^{\,1})_R^- = 0. \tag{42}$$

By using the mimic constraint in eq. (36) in the condition (42), we obtain

$$p_R = 0, \tag{43}$$

which, according to eq. (32), leads to

$$B = \frac{\sqrt{(1+CR^2)(2-CR^2)}(1-CR^2)}{2+3CR^2-2C^2R^4}. \tag{44}$$

On the other hand, by using the condition in eq. (41), we obtain for the Schwarzschild mass

$$\frac{2\mathcal{M}_{\mathcal{S}}}{R} = \frac{2M_0}{R} + \beta\left(1-\frac{2M_0}{R}\right) - \beta\left(\frac{\sqrt{(1+CR^2)(2-CR^2)}(1+CR^2)+B(2-CR^2)(5+2CR^2)}{\sqrt{(1+CR^2)(2-CR^2)}(1+6C+CR^2)+B[(2-CR^2)(5+2CR^2)+6C(1-2CR^2)]}\right), \tag{45}$$

where the expression in eq. (11) has been used. Finally, by using the expression in eq. (45) in the matching condition (40), we obtain

$$A\left[(1+CR^2)^{3/2}+B\sqrt{2-CR^2}(5+2CR^2)\right]^2 = (1-\beta)\left(1-\frac{2M_0}{R}\right) + \beta\left(\frac{\sqrt{(1+CR^2)(2-CR^2)}(1+CR^2)+B(2-CR^2)(5+2CR^2)}{\sqrt{(1+CR^2)(2-CR^2)}(1+6C+CR^2)+B[(2-CR^2)(5+2CR^2)+6C(1-2CR^2)]}\right), \tag{46}$$

which then allows to determine the constant A.

Equations (44)–(46) are the necessary and sufficient conditions to match the interior solution with the exterior Schwarzschild space-time. By using the mimic constraint in eq. (36), the effective radial pressure $\tilde{p}_r$ reads

$$\tilde{p}_r(r,\beta) = \frac{9(1-\beta)C}{16\pi(1+Cr^2)}\left[\frac{\sqrt{(1+Cr^2)(2-Cr^2)}(1-Cr^2)-B(2+3Cr^2-2C^2r^4)}{\sqrt{(1+Cr^2)(2-Cr^2)}(1+Cr^2)+B(10-Cr^2-2C^2r^4)}\right], \tag{47}$$

where the constant B is given by eq. (44). The expression in eq. (47) shows that the effective radial pressure mimics the perfect fluid pressure $p(r)$ in eq. (32). On the other hand, the effective density and effective tangential pressure are given, respectively, by

$$\tilde{\rho}(r,\beta) = \rho - \beta\left(\frac{p(r^2\nu''-2r\nu'-3)-rp'(1+r\nu')}{(1+r\nu')^2}\right), \tag{48}$$

$$\tilde{p}_t(r,\beta) = \tilde{p}_r(r,\beta) - \frac{\beta rp'(2+3r\nu'+r^2\nu'^2)}{4(1+r\nu')^2}, \tag{49}$$

where $p = p(r)$ is given by eq. (32), $\nu' \equiv \partial_r\nu(r)$ is given by

$$\nu' = \frac{6Cr(\sqrt{(1+Cr^2)(2-Cr^2)}+B(1-2Cr^2))}{\sqrt{(1+Cr^2)(2-Cr^2)}(1+Cr^2)+B(2-Cr^2)(5+2Cr^2)} \tag{50}$$

and $p' \equiv \partial_r p$ is given by

$$p' = \frac{3C}{16\pi}\left(\frac{2(Z_2'+Z_3')(Z_4+Z_5)-2(Z_2+Z_3)(Z_4'+Z_5')-Z_1'(Z_4+Z_5)^2}{(Z_4+Z_5)^2}\right), \tag{51}$$

where

$$
\begin{aligned}
Z_1' &\equiv \frac{-2Cr}{(1+Cr^2)^2},\\
Z_2 &\equiv (1+Cr^2)^{1/2}(2-Cr^2)^{3/2},\\
Z_2' &\equiv Cr(1+Cr^2)^{-1/2}(2-Cr^2)^{3/2} - 3Cr(1+Cr^2)^{1/2}(2-Cr^2)^{1/2},\\
Z_3 &\equiv B(1-2Cr^2)(2-Cr^2),\\
Z_3' &\equiv -2BCr(5-4Cr^2),\\
Z_4 &\equiv (1+Cr^2)^{5/2}(2-Cr^2)^{1/2},\\
Z_4' &\equiv 5Cr(1+Cr^2)^{3/2}(2-Cr^2)^{1/2} - Cr(1+Cr^2)^{5/2}(2-Cr^2)^{-1/2},\\
Z_5 &\equiv B(2-Cr^2)(1+Cr^2)(5+2Cr^2),\\
Z_5' &\equiv 2BCr((1-2Cr^2)(5+2Cr^2)+2(1+Cr^2)(2-Cr^2))
\end{aligned}
$$

and $\nu'' \equiv \partial_r \nu'$ is given by

$$
\nu'' = \frac{(Q_1'+Q_5')(Q_2Q_3+Q_4) - (Q_1+Q_5)(Q_2'Q_3+Q_2Q_3'+Q_4')}{(Q_2Q_3+Q_4)^2}\,, \tag{52}
$$

where

$$
\begin{aligned}
Q_1 &\equiv 6C(2r^2+Cr^4-C^2r^6)^{1/2},\\
Q_1' &\equiv \frac{6C(2+2Cr^2-3C^2r^4)}{(2+Cr^2-C^2r^4)^{1/2}}\,,\\
Q_2 &\equiv 1+Cr^2,\\
Q_2' &\equiv 2Cr,\\
Q_3 &\equiv (2+Cr^2-C^2r^4)^{1/2},\\
Q_3' &\equiv \frac{Cr(1-2Cr^2)}{(2+Cr^2-C^2r^4)^{1/2}}\,,\\
Q_4 &\equiv B(10-Cr^2-2C^2r^4),\\
Q_4' &\equiv -2BCr(1+4Cr^2),\\
Q_5 &\equiv 6BCr(1-2Cr^2),\\
Q_5' &\equiv 6BC(1-6Cr^2).
\end{aligned}
$$

Equations (29) and (39) along with eqs. (47)–(49) represent an exact Buchdahl analytic solution to the system of eqs. (6)–(8) minimally deformed by the gravitational source $\Phi_{\mu\nu}$. According to eq. (9), the source $\Phi_{\mu\nu}$ generates an anisotropy given by

$$
\Delta(r,\beta) = -\frac{\beta r p'(2+3r\nu'+r^2\nu'^2)}{4(1+r\nu')^2}\,. \tag{53}
$$

Figure 2 shows the behaviour of the effective quantities for different values of the coupling constant. The isotropic solution corresponds to coupling constant equal to zero. As in the case of the isotropic solution, we will check the four energy conditions for the anisotropic solution (see ref. [22]). The energy conditions are shown as follows: a) the Null Energy Condition (NEC), $T_{\mu\nu}K^\mu K^\nu \geq 0$ where the null vector K^μ can be written as $K^\mu = e^{-\nu/2}\delta^\mu_0 + e^{-\lambda/2}\delta^\mu_1$, yields $T_{\mu\nu}K^\mu K^\nu = e^\nu \tilde{\rho} K^0K^0 + e^\lambda \tilde{p}_r K^1K^1 = \tilde{\rho} + \tilde{p}_r \geq 0$. b) The Weak Energy Condition (WEC), $T_{\mu\nu}X^\mu X^\nu \geq 0$ for any time-like vector X^μ, yields $\tilde{\rho} \geq 0$, $\tilde{\rho} + \tilde{p}_r \geq 0$ and $\tilde{\rho} + \tilde{p}_t \geq 0$. c) the Dominant Energy Condition (DEC), $T^\mu{}_\nu X^\nu = -Y^\mu$, yields $\tilde{\rho} \geq \tilde{p}_r$ and $\tilde{\rho} \geq \tilde{p}_t$. Finally, d) the Strong Energy Condition (SEC), $(T_{\mu\nu}\frac{1}{2}\,T\,g_{\mu\nu})X^\mu X^\nu \geq 0$, leads to

$$
\begin{aligned}
\tilde{\rho} + \tilde{p}_r + 2\tilde{p}_t &= \rho + 3p - \beta\left(\frac{p(r^2\nu'' - 2r\nu' - 3) - rp'(1+r\nu')}{(1+r\nu')^2} + 3p + \frac{rp'(2+3r\nu'+r^2\nu'^2)}{2(1+r\nu')^2}\right)\\
&= \rho + 3p - \beta\left(\frac{rp(r\nu''+4\nu'+3r\nu'^2)}{(1+r\nu')^2} + \frac{rp'}{2}\right) > 0.
\end{aligned}
$$

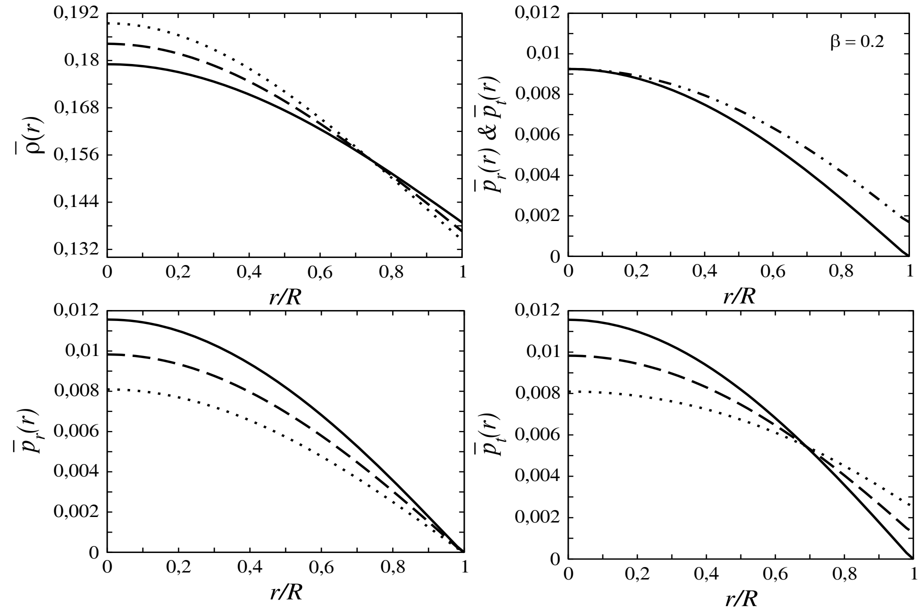

Fig. 2. Effective quantities for different values of β for a stellar distribution with compactness $M_0/R = 0.1$. The solid black line represents the Buchdahl solution for a perfect fluid given by $\beta = 0$; $\beta = 0.15$ (dashed line) and $\beta = 0.3$ (dotted line) represent two anisotropic analytic solutions. The second graph shows a comparision between the effective radial and tangential pressure for $\beta = 0.2$. The anisotropy causes the pressures values to drift apart.

All of the above effective conditions are satisfied. This means that there are no negative effective pressures comparable in magnitude or larger than the effective density $\tilde{\rho}$, and therefore the geometric deformation on the isotropic solution is not strong enough to put in risk the physical acceptability of the system.

5 Conclusions

Using the MGD decoupling, it was shown in detail how to extend an interior isotropic solution for a static and spherically symmetric self-gravitating system in order to include an additional gravitational source. For this purpose, it was shown that the Einstein's field equations in eqs. (6)–(8) can be decoupled in a sector for a perfect fluid $\Psi^{(\mathrm{m})}_{\mu\nu}$ shown in eqs. (13)–(15), and the sector describe by the equations associated with the additional gravitational source $\Phi_{\mu\nu}$ shown in eqs. (16)–(18). There is only gravitational interacction between these two sectors, and there is not exchange of energy momentum between them. The matching conditions at the star surface have been studied in detail for an outer Schwarzschild space-time. In particular, the continuity of the second fundamental form in eq. (28) yields the important result that the effective radial pressure $\tilde{p}_R = 0$. The effective pressure contains both the isotropic pressure of the gravitational source $\Psi^{(\mathrm{m})}_{\mu\nu}$ and the geometric deformation $\eta^\star(r)$ induced by the energy-momentum tensor $\Phi_{\mu\nu}$. The physical acceptability of the found anisotropic solution is inherited from their isotropic parent. In particular, it was shown that the source $\Phi_{\mu\nu}$ always reduces the effective radial pressure $\tilde{p}_R$ inside the self-gravitating system. Variations in the geometric deformation parameter (β) between the isotropic and the anisotropic sector reveals consistent evolution of the effective quantities giving to the MGD gravitational decoupling a prove of validity.

The MGD decoupling could be an efficient way to deal with complex physical problems. For instance, for extending isotropic solutions in General Relativity to solutions of the Einstein-Klein-Gordon system. In such a system, the source $\Phi_{\mu\nu}$ would represent the Klein-Gordon scalar field. In addition, the MGD decoupling could be useful for systems whose spherical symmetry is preserved during a sufficiently slow temporal evolution.

Any known perfect fluid solution can be extended to generate new anisotropic solutions. In principle, for each perfect fluid solution there will be as many anisotropic solutions as independent constraints can be imposed on the system (16)–(18). This approach simplifies the study of self-gravitating systems, which can be developed sector by sector under the MGD decoupling. Finding analytic and physically acceptable solutions for the interior of a self-gravitating system minimally coupled to any gravitational source $\Phi_{\mu\nu}$ seems a difficult task to carry out. Nevertheless, using the MGD decoupling, we can start with an exact and physically acceptable perfect fluid solution, and then focus on the sector represented by the additional gravitational source $\Phi_{\mu\nu}$. Finally, for future work we need to find out the MGD decoupling validity for time-dependent solutions, and extensions beyond the spherical symmetry.

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