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Solitons in optical metamaterials with anti-cubic nonlinearity

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Abstract. This paper studies soliton perturbation in optical metamaterials, with anti-cubic nonlinearity, by implementing three integration schemes. Bright, dark and singular soliton solutions are retrieved. The existence criteria of these solitons in metamaterials are also presented.

1 Introduction

The study of solitons in optical metameterials is trending as a hotspot in the field of optical materials. There has been a substantial amount of results that are reported in this field. However, there is still a long way to go. There are more unanswered questions than answers. This paper will quench this thirst partially. In the past, solitons in optical metamaterials have been studied with various forms of non-Kerr laws of nonlinearity where several integration schemes have been implemented [1–20]: this paper is going to revisit the study of solitons in optical metamaterials for a specific form of nonlinear medium: this is of anti-cubic (AC) type. There are three forms of integration algorithms that will be applied to extract soliton solutions to metamaterials with AC nonlinearity. These schemes will retrieve bright, dark and singular soliton solutions that will be very important in the study of optical materials. These solitons will appear with constraint conditions that are otherwise referred to as existence criteria of the soliton parameters. After a quick introduction to the model, the integration techniques will be applied and the details are enumerated in the subsequent sections.

2 Governing model

The nonlinear dynamics that describes the propagation of pulses in optical metamaterials (MMs) is given by the nonlinear Schrödinger equation (NLSE). In the presence of parabolic law nonlinearity, with an additional anti-cubic nonlinear term and perturbation terms that include inter-modal dispersion (IMD), self-steepening (SS) as well as nonlinear dispersion (ND), the governing equation reads [2,19]

$$iq_t + aq_{xx} + \left(b_1|q|^{-4} + b_2|q|^2 + b_3|q|^4\right) q = i\left\{\alpha q_x + \beta\left(|q|^2 q\right)_x + \nu\left(|q|^2\right)_x q\right\} + \theta_1\left(|q|^2 q\right)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q^*_{xx}. \quad (1)$$

In eq. (1), the unknown or dependent variable $q(x,t)$ represents the wave profile, while x and t are the spatial and temporal variables, respectively. The first and second terms are the linear temporal evolution term and group velocity dispersion (GVD), while the third term introduces the anti-cubic nonlinear term, the fourth and fifth terms account for the parabolic law nonlinearity, and the sixth, seventh and eighth terms represent IMD, SS and ND, respectively. Finally, the last three terms with θ_l for $l = 1, 2, 3$ appear in the context of metamaterials [4,5].

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2.1 Mathematical analysis

In order to solve eq. (1), the starting hypothesis is [4,5]

$$q(x,t) = U(\zeta)\exp\left[i\phi(x,t)\right], \tag{2}$$

where

$$\zeta = k(x - vt), \tag{3}$$

and the phase component ϕ is given by

$$\phi(x,t) = -\kappa x + \omega t + \theta. \tag{4}$$

In eqs. (2) and (3), $U(x,t)$ represents the amplitude portion of the soliton, and k and v are the inverse width and velocity of the soliton. From (4), κ is the frequency of the soliton, ω is the wave number of the soliton and finally θ is the phase constant. Inserting (2) into (1) and then decomposing into real and imaginary parts yields a pair of relations. The imaginary part gives

$$v = -\alpha - 2a\kappa \tag{5}$$

and

$$3\beta + 2\nu - 2\kappa(3\theta_1 + \theta_2 - \theta_3) = 0, \tag{6}$$

while the real part leads to

$$\begin{aligned} &ak^2U'' - \left(\omega + a\kappa^2 + \alpha\kappa\right)U + b_1U^{-3} + \left(b_2 - \beta\kappa + \kappa^2\theta_1 + \kappa^2\theta_2 + \kappa^2\theta_3\right)U^3 \\ &+ b_3U^5 - \left(3k^2\theta_1 + k^2\theta_2 + k^2\theta_3\right)U^2U'' - 6k^2\theta_1 U(U')^2 = 0. \end{aligned} \tag{7}$$

To obtain an analytic solution, the transformations $\theta_1 = 0$ and $\theta_2 = -\theta_3$ are applied in eq. (7), and give

$$ak^2U'' - \left(\omega + a\kappa^2 + \alpha\kappa\right)U + b_1U^{-3} + (b_2 - \kappa\beta)U^3 + b_3U^5 = 0, \tag{8}$$

where

$$3\beta + 2\nu + 4\kappa\theta_3 = 0. \tag{9}$$

In order to obtain closed-form solutions, we employ the transformation given by

$$U = V^{\frac{1}{2}}, \tag{10}$$

that will reduce eq. (8) into the ODE

$$ak^2\left\{2VV'' - (V')^2\right\} + 4b_1 - 4\left(\omega + a\kappa^2 + \alpha\kappa\right)V^2 + 4(b_2 - \kappa\beta)V^3 + 4b_3V^4 = 0. \tag{11}$$

The extended G'/G-expansion method [8,10,11,20], the extended Jacobi's elliptic function expansion scheme [1,3,7, 11,12,15,17] and the $\exp(-\Phi(\zeta))$-expansion approach [9,13,16,18] will now be applied, in the subsequent sections, to eq. (11) to retrieve bright, dark and singular soliton solutions to the NLSE with AC nonlinearity (1).

3 Extended G′/G-expansion method

Suppose that the solution to eq. (11) can be expressed by

$$V(\zeta) = \alpha_0 + \sum_{i=1}^{M}\left\{\alpha_i\left(\frac{G'}{G}\right)^i + \beta_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\sigma\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} + \gamma_i\left(\frac{G'}{G}\right)^{-i} + \delta_i\frac{(\frac{G'}{G})^{-i+1}}{\sqrt{\sigma(1 + \frac{1}{\mu}(\frac{G'}{G})^2)}}\right\}, \tag{12}$$

where α_0, α_i, β_i, γ_i, δ_i $(i = 1, \ldots, M)$ are constants to be determined later, $\sigma = \pm1$, M is a positive integer, and $G = G(\zeta)$ satisfies the following second-order linear ordinary differential equation:

$$G'' + \mu G = 0, \tag{13}$$

where μ is a constant to be determined later. According to the homogeneous balance method, eq. (11) has the solution in the form

$$V(\zeta) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \beta_1\sqrt{\sigma\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} + \gamma_1\left(\frac{G'}{G}\right)^{-1} + \delta_1\frac{1}{\sqrt{\sigma(1 + \frac{1}{\mu}(\frac{G'}{G})^2)}}. \tag{14}$$

Substituting (14) along with eq. (13) into eq. (11), and equating the coefficients of $(\frac{G'}{G})^j$ and $(\frac{G'}{G})^j\sqrt{\sigma\{1 + \frac{1}{\mu}(\frac{G'}{G})^2\}}$ to zero, we obtain a set of over-determined algebraic equations and, by solving it, we find the following results.

Set 1:

$$\begin{aligned}
\alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3}\,,\\
\alpha_1 &= \pm\frac{ik\sqrt{3a}}{2\sqrt{b_3}}\,,\\
\beta_1 &= \gamma_1 = \delta_1 = 0,\\
\omega &= \frac{16b_3[ak^2\mu - 2\kappa(\alpha + a\kappa)] - 9(b_2 - \beta\kappa)^2}{32b_3}\,,\\
b_1 &= -\frac{3[3(b_2 - \beta\kappa)^2 - 16ab_3k^2\mu]^2}{4096b_3^3}\,.
\end{aligned} \tag{15}$$

Set 2:

$$\begin{aligned}
\alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3}\,,\\
\gamma_1 &= \pm\frac{ik\mu\sqrt{3a}}{2\sqrt{b_3}}\,,\\
\alpha_1 &= \beta_1 = \delta_1 = 0,\\
\omega &= \frac{16b_3[ak^2\mu - 2\kappa(\alpha + a\kappa)] - 9(b_2 - \beta\kappa)^2}{32b_3}\,,\\
b_1 &= -\frac{3[3(b_2 - \beta\kappa)^2 - 16ab_3k^2\mu]^2}{4096b_3^3}\,.
\end{aligned} \tag{16}$$

Set 3:

$$\begin{aligned}
\alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3}\,,\\
\beta_1 &= \pm\frac{ik\sqrt{3a}\sqrt{\mu}}{2\sqrt{b_3}\sqrt{\sigma}}\,,\\
\alpha_1 &= \gamma_1 = \delta_1 = 0,\\
\omega &= -\frac{8b_3[4\kappa(\alpha + a\kappa) + ak^2\mu] + 9(b_2 - \beta\kappa)^2}{32b_3}\,,\\
b_1 &= -\frac{27(b_2 - \beta\kappa)^4 + 144ab_3k^2\mu(b_2 - \beta\kappa)^2}{4096b_3^3}\,.
\end{aligned} \tag{17}$$

Set 4:

$$\begin{aligned}
\alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3}\,,\\
\alpha_1 &= \mp\frac{ik\sqrt{3a}}{2\sqrt{b_3}}\,,\\
\gamma_1 &= \pm\frac{ik\mu\sqrt{3a}}{2\sqrt{b_3}}\,,\\
\beta_1 &= \delta_1 = 0,\\
\omega &= -\kappa(\alpha + a\kappa) + 2ak^2\mu - \frac{9(b_2 - \beta\kappa)^2}{32b_3}\,,\\
b_1 &= -\frac{3[3(b_2 - \beta\kappa)^2 - 64ab_3k^2\mu]^2}{4096b_3^3}\,.
\end{aligned} \tag{18}$$

Set 5:

$$\begin{aligned}
\alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3}, \\
\alpha_1 &= \pm\frac{ik\sqrt{3a}}{4\sqrt{b_3}}, \\
\beta_1 &= \frac{ik\sqrt{3a}\sqrt{\mu}}{4\sqrt{b_3}\sqrt{\sigma}}, \\
\gamma_1 &= \delta_1 = 0, \\
\omega &= \frac{4b_3[ak^2\mu - 8\kappa(\alpha + a\kappa)] - 9(b_2 - \beta\kappa)^2}{32b_3}, \\
b_1 &= -\frac{3[3(b_2 - \beta\kappa)^2 - 4ab_3k^2\mu]^2}{4096b_3^3},
\end{aligned} \tag{19}$$

where κ and μ are arbitrary constants.

As a consequence, substituting the solution sets (15)–(19) along with the general solutions of eq. (13) into eqs. (10) and (14), and inserting the result into the wave transformation (2), we obtain exact solutions to (1) in the following forms.

When $\mu < 0$, hyperbolic traveling wave solutions are

$$q(x,t) = \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \pm \frac{k\sqrt{3a\mu}}{2\sqrt{b_3}}\left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right], \tag{20}$$

$$q(x,t) = \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \mp \frac{k\sqrt{3a\mu}}{2\sqrt{b_3}}\left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]^{-1}\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right], \tag{21}$$

$$q(x,t) = \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \mp \frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\sqrt{1 - \left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]^2}\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right], \tag{22}$$

$$\begin{aligned}
q(x,t) = &\left\{\frac{3(\beta\kappa - b_2)}{8b_3} \mp \frac{k\sqrt{3a\mu}}{2\sqrt{b_3}}\left(\left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]\right.\right. \\
&\left.\left. + \left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]^{-1}\right)\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right],
\end{aligned} \tag{23}$$

$$\begin{aligned}
q(x,t) = &\left\{\frac{3(\beta\kappa - b_2)}{8b_3} + \frac{k\sqrt{3a\mu}}{4\sqrt{b_3}}\left[\pm\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right.\right. \\
&\left.\left. + i\sqrt{1 - \left[\frac{A_1\sinh(\sqrt{-\mu}\zeta) + A_2\cosh(\sqrt{-\mu}\zeta)}{A_1\cosh(\sqrt{-\mu}\zeta) + A_2\sinh(\sqrt{-\mu}\zeta)}\right]^2}\right]\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right],
\end{aligned} \tag{24}$$

where A_1 and A_2 are arbitrary constants, $\zeta = k\{x + (\alpha + 2a\kappa)t\}$ and ω are given by the solution sets (15)–(19), respectively. It should be noted that these solitons are valid for $ab_3\mu > 0$.

If, however, $\mu > 0$, the trigonometric traveling wave solutions are

$$q(x,t) = \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \pm \frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\left[\frac{A_1\cos(\sqrt{\mu}\zeta) - A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta) + A_2\cos(\sqrt{\mu}\zeta)}\right]\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right], \tag{25}$$

$$q(x,t) = \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \pm \frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\left[\frac{A_1\cos(\sqrt{\mu}\zeta) - A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta) + A_2\cos(\sqrt{\mu}\zeta)}\right]^{-1}\right\}^{\frac{1}{2}} \exp\left[i\left(-\kappa x + \omega t + \theta\right)\right], \tag{26}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\sqrt{1+\left[\frac{A_1\cos(\sqrt{\mu}\zeta)-A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta)+A_2\cos(\sqrt{\mu}\zeta)}\right]^2}\right\}^{\frac{1}{2}}\exp\left[i\left(-\kappa x+\omega t+\theta\right)\right], \tag{27}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\left(\frac{A_1\cos(\sqrt{\mu}\zeta)-A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta)+A_2\cos(\sqrt{\mu}\zeta)}-\left[\frac{A_1\cos(\sqrt{\mu}\zeta)-A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta)+A_2\cos(\sqrt{\mu}\zeta)}\right]^{-1}\right)\right\}^{\frac{1}{2}}\times\exp\left[i\left(-\kappa x+\omega t+\theta\right)\right], \tag{28}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}+\frac{k\sqrt{-3a\mu}}{4\sqrt{b_3}}\left[\pm\frac{A_1\cos(\sqrt{\mu}\zeta)-A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta)+A_2\cos(\sqrt{\mu}\zeta)}+\sqrt{1+\left[\frac{A_1\cos(\sqrt{\mu}\zeta)-A_2\sin(\sqrt{\mu}\zeta)}{A_1\sin(\sqrt{\mu}\zeta)+A_2\cos(\sqrt{\mu}\zeta)}\right]^2}\right]\right\}^{\frac{1}{2}}\times\exp\left[i\left(-\kappa x+\omega t+\theta\right)\right], \tag{29}$$

where A_1 and A_2 are arbitrary constants, $\zeta=k\{x+(\alpha+2a\kappa)t\}$ and ω are given by the solution sets (15)–(19), respectively. These trigonometric traveling wave solutions will exist provided the constraint condition holds: $ab_3\mu<0$.

Finally, when $\mu=0$, the plane wave solutions are the following.

By using the results in eqs. (15) and (18), we have

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{2\sqrt{b_3}}\left(\frac{A_1}{A_1\zeta+A_2}\right)\right\}^{\frac{1}{2}}\exp\left[i\left(-\kappa x+\omega t+\theta\right)\right], \tag{30}$$

where A_1 and A_2 are arbitrary constants, $\zeta=k\{x+(\alpha+2a\kappa)t\}$ and ω is given by the solution set (15).

By using the results in eq. (19), we have

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{4\sqrt{b_3}}\left(\frac{A_1}{A_1\zeta+A_2}\right)\right\}^{\frac{1}{2}}\exp\left[i\left(-\kappa x+\omega t+\theta\right)\right], \tag{31}$$

where A_1 and A_2 are arbitrary constants, $\zeta=k\{x+(\alpha+2a\kappa)t\}$ and ω is given by the solution set (19). The constraint condition for the existence of the solutions (30) and (31) is $ab_3<0$.

The special cases are as follows.

When $\mu<0$ and $A_1^2>A_2^2$, then we deduce the following optical soliton solutions from (20)–(24), respectively:

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{3a\mu}}{2\sqrt{b_3}}\tanh\left[k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+\zeta_0\right]\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(\frac{16b_3\left[ak^2\mu-2\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right], \tag{32}$$

which is a dark soliton solution. The singular soliton solution is given by

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{3a\mu}}{2\sqrt{b_3}}\coth\left[k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+\zeta_0\right]\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(\frac{16b_3\left[ak^2\mu-2\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right] \tag{33}$$

and the bright soliton is

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\operatorname{sech}\left[k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+\zeta_0\right]\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x-\left(\frac{8b_3\left[4\kappa(\alpha+a\kappa)+ak^2\mu\right]+9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right], \tag{34}$$

together with another singular soliton solution

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{3a\mu}}{\sqrt{b_3}}\coth\left[2k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+2\zeta_0\right]\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(-\kappa(\alpha+a\kappa)+2ak^2\mu-\frac{9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right] \tag{35}$$

and a complexiton solution

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}+\frac{k\sqrt{3a\mu}}{4\sqrt{b_3}}\left(\pm\tanh\left[k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+\zeta_0\right]\right.\right.$$
$$\left.\left.+i\,\mathrm{sech}\left[k\sqrt{-\mu}\left\{x+(\alpha+2a\kappa)t\right\}+\zeta_0\right]\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{4b_3\left[ak^2\mu-8\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{36}$$

where $\zeta_0=\tanh^{-1}(A_2/A_1)$. Also, setting $A_1=0$, $A_2\neq 0$ or $A_2=0$, $A_1\neq 0$ in (20)–(24), we can obtain more solitary wave solutions which are omitted.

If, however, $\mu>0$, we obtain singular periodic waves from (25)–(29), respectively,

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\tan\left[k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-\zeta_0\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{16b_3\left[ak^2\mu-2\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{37}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\cot\left[k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-\zeta_0\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{16b_3\left[ak^2\mu-2\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{38}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a\mu}}{2\sqrt{b_3}}\sec\left[k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-\zeta_0\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{8b_3\left[4\kappa(\alpha+a\kappa)+ak^2\mu\right]+9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{39}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\mp\frac{k\sqrt{-3a\mu}}{\sqrt{b_3}}\cot\left[2k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-2\zeta_0\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-\kappa(\alpha+a\kappa)+2ak^2\mu-\frac{9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{40}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}+\frac{k\sqrt{-3a\mu}}{4\sqrt{b_3}}\left(\mp\tan\left[k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-\zeta_0\right]\right.\right.$$
$$\left.\left.+\sec\left[k\sqrt{\mu}\left\{x+(\alpha+2a\kappa)t\right\}-\zeta_0\right]\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{4b_3\left[ak^2\mu-8\kappa(\alpha+a\kappa)\right]-9(b_2-\beta\kappa)^2}{32b_3}\right)t+\theta\right\}\right],\tag{41}$$

where $\zeta_0=\tan^{-1}(A_1/A_2)$. Also, setting $A_1=0$, $A_2\neq 0$ or $A_2=0$, $A_1\neq 0$ in (25)–(29), we can obtain more periodic wave solutions which are omitted.

4 Extended Jacobi's elliptic function expansion scheme

Suppose that the structure solution of (11) is given by

$$V(x,t)=V(\zeta)=\sum_{j=0}^{N}\alpha_j\,\mathrm{sn}^j\,\zeta+\sum_{j=1}^{N}\beta_j\,\mathrm{sn}^{-j}\,\zeta,\tag{42}$$

where α_0, α_j, β_j $(j=1,2,\ldots,N)$ are constants to be determined. Balancing the order of VV'' and V^4 in eq. (11), we have $N=1$. Therefore, eq. (11) has a solution in the form

$$V(\zeta)=\alpha_0+\alpha_1\,\mathrm{sn}\,\zeta+\beta_1\,\mathrm{sn}^{-1}\,\zeta.\tag{43}$$

Then, substituting eq. (43) into (11) and equating the coefficients of the exponents of $\mathrm{sn}\,\zeta$ to zero, one obtains a system of nonlinear algebraic equations; by solving it, one recovers the following.

Set 1:

$$\begin{aligned} \beta_1 &= 0,\\ \alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3},\\ \alpha_1 &= \pm\frac{ikm\sqrt{3a}}{2\sqrt{b_3}},\\ \omega &= -\frac{9(b_2-\beta\kappa)^2 + 8b_3[ak^2(1+m^2)+4\alpha\kappa+4a\kappa^2]}{32b_3},\\ b_1 &= -\frac{27(b_2-\beta\kappa)^4 + 144ab_3k^2(1+m^2)(b_2-\beta\kappa)^2 + 768a^2b_3^2k^4m^2}{4096b_3^3}. \end{aligned} \tag{44}$$

Set 2:

$$\begin{aligned} \alpha_1 &= 0,\\ \alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3},\\ \beta_1 &= \pm\frac{ik\sqrt{3a}}{2\sqrt{b_3}},\\ \omega &= -\frac{9(b_2-\beta\kappa)^2 + 8b_3[ak^2(1+m^2)+4\alpha\kappa+4a\kappa^2]}{32b_3},\\ b_1 &= -\frac{27(b_2-\beta\kappa)^4 + 144ab_3k^2(1+m^2)(b_2-\beta\kappa)^2 + 768a^2b_3^2k^4m^2}{4096b_3^3}. \end{aligned} \tag{45}$$

Set 3:

$$\begin{aligned} \alpha_0 &= \frac{3(\beta\kappa - b_2)}{8b_3},\\ \alpha_1 &= \pm\frac{ikm\sqrt{3a}}{2\sqrt{b_3}},\\ \beta_1 &= \pm\frac{ik\sqrt{3a}}{2\sqrt{b_3}},\\ \omega &= -\frac{9(b_2-\beta\kappa)^2 + 8b_3[ak^2(1+m(6+m))+4\alpha\kappa+4a\kappa^2]}{32b_3},\\ b_1 &= -\frac{27(b_2-\beta\kappa)^4 + 144ab_3k^2(1+m(6+m))(b_2-\beta\kappa)^2 + 3072a^2b_3^2k^4m(1+m)^2}{4096b_3^3}. \end{aligned} \tag{46}$$

Thus, we recover the following Jacobi elliptic function solutions to the model given by (1):

$$\begin{aligned} q(x,t) = {}& \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \pm \frac{km\sqrt{-3a}}{2\sqrt{b_3}}\,\mathrm{sn}\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}\\ & \times\exp\left[i\left\{-\kappa x - \left(\frac{9(b_2-\beta\kappa)^2 + 8b_3\left[ak^2\left(1+m^2\right)+4\alpha\kappa+4a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right], \end{aligned} \tag{47}$$

$$\begin{aligned} q(x,t) = {}& \left\{\frac{3(\beta\kappa - b_2)}{8b_3} \pm \frac{k\sqrt{-3a}}{2\sqrt{b_3}}\,\mathrm{ns}\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}\\ & \times\exp\left[i\left\{-\kappa x - \left(\frac{9(b_2-\beta\kappa)^2 + 8b_3\left[ak^2\left(1+m^2\right)+4\alpha\kappa+4a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right], \end{aligned} \tag{48}$$

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{2\sqrt{b_3}}\left(m\,\mathrm{sn}\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]+\mathrm{ns}\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+8b_3\left[ak^2(1+m(6+m))+4\alpha\kappa+4a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right],\qquad(49)$$

for $ab_3<0$. When the modulus $m\to 1$ in eqs. (47)–(49), the following soliton solutions emerge:

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{2\sqrt{b_3}}\tanh\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+16b_3\left[ak^2+2\alpha\kappa+2a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right],\qquad(50)$$

which is a dark soliton solution and the following two are singular soliton solutions,

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{2\sqrt{b_3}}\coth\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+16b_3\left[ak^2+2\alpha\kappa+2a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right]\qquad(51)$$

and

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{\sqrt{b_3}}\coth 2\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+32b_3\left[2ak^2+\alpha\kappa+a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right].\qquad(52)$$

However, if $m\to 0$ in eqs. (48) and (49), the following singular periodic solutions emerge:

$$q(x,t)=\left\{\frac{3(\beta\kappa-b_2)}{8b_3}\pm\frac{k\sqrt{-3a}}{2\sqrt{b_3}}\csc\left[k\left\{x+(\alpha+2a\kappa)\,t\right\}\right]\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+8b_3\left[ak^2+4\alpha\kappa+4a\kappa^2\right]}{32b_3}\right)t+\theta\right\}\right].\qquad(53)$$

5 The $\exp(-\Phi(\zeta))$-expansion approach

To start off with the $\exp(-\Phi(\zeta))$-expansion approach, the initial assumption of the solution structure of (11) is made:

$$V(\zeta)=\sum_{i=1}^{N}A_i\left(\exp[-\Phi(\zeta)]\right)^i,\qquad(54)$$

where A_i for $i=0,1,\ldots,N$ are constants to be determined later, such that $A_N\neq 0$, while the function $\Phi(\zeta)$ is the solution of the ordinary differential equation

$$\Phi'(\zeta)=\exp[-\Phi(\zeta)]+\mu\exp[\Phi(\zeta)]+\lambda.\qquad(55)$$

It is well known that eq. (55) has solutions in the following forms.

If $\mu\neq 0$ and $\lambda^2-4\mu>0$,

$$\Phi(\zeta)=\ln\left(-\frac{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\zeta+C)\right)+\lambda}{2\mu}\right).\qquad(56)$$

For $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$\Phi(\zeta) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\zeta + C)\right) - \lambda}{2\mu}\right). \tag{57}$$

However, when $\mu = 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$\Phi(\zeta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\zeta + C)) - 1}\right). \tag{58}$$

Whenever $\mu \neq 0$, $\lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$\Phi(\zeta) = \ln\left(-\frac{2(\lambda(\zeta + C) + 2)}{\lambda^2(\zeta + C)}\right). \tag{59}$$

Finally, if $\mu = 0$, $\lambda = 0$ and $\lambda^2 - 4\mu = 0$,

$$\Phi(\zeta) = \ln(\zeta + C). \tag{60}$$

Here, it is important to note that C is the integration constant. Balancing VV'' with V^4 in eq. (11) yields $N = 1$. The $\exp(-\Phi(\zeta))$-expansion scheme allows us to employ the substitution

$$V(\zeta) = A_0 + A_1 \exp[-\Phi(\zeta)]. \tag{61}$$

Substituting (61) along with (55) into eq. (11) and equating all the coefficients of powers of $\exp(-\Phi(\zeta))$ to be zero, one obtains a system of algebraic equations. Solving this system by Mathematica yields

$$\begin{aligned}
A_0 &= \frac{3\beta\kappa - 3b_2 \pm 2k\lambda\sqrt{-3ab_3}}{8b_3}, \\
A_1 &= \pm\frac{k}{2}\sqrt{-\frac{3a}{b_3}}, \\
\omega &= -\frac{9(b_2 - \beta\kappa)^2 + 4b_3[8\alpha\kappa + a(8\kappa^2 + k^2(\lambda^2 - 4\mu))]}{32b_3}, \\
b_1 &= -\frac{3[3(b_2 - \beta\kappa)^2 + 4ab_3k^2(\lambda^2 - 4\mu)]^2}{4096b_3^3},
\end{aligned} \tag{62}$$

where λ and μ are arbitrary constants. Substituting the solution set (62) into (61), the solution formula of eq. (11) can be written as follows:

$$V(\zeta) = \frac{3\beta\kappa - 3b_2 \pm 2k\lambda\sqrt{-3ab_3}}{8b_3} \pm \frac{k}{2}\sqrt{-\frac{3a}{b_3}}\exp[-\Phi(\zeta)]. \tag{63}$$

Consequently, one gains exact solutions to the model as follows:

$$\begin{aligned}
q(x,t) = &\left\{\frac{3\beta\kappa - 3b_2 \pm 2k\lambda\sqrt{-3ab_3}}{8b_3} \mp \frac{k}{2}\sqrt{-\frac{3a}{b_3}}\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(k(x + \{\alpha + 2a\kappa\}t) + C)\right) + \lambda}\right)\right\}^{\frac{1}{2}} \\
&\times \exp\left[i\left\{-\kappa x - \left(\frac{9(b_2 - \beta\kappa)^2 + 4b_3\left[8\alpha\kappa + a\left(8\kappa^2 + k^2\left(\lambda^2 - 4\mu\right)\right)\right]}{32b_3}\right)t + \theta\right\}\right],
\end{aligned} \tag{64}$$

which is a singular soliton solution

$$\begin{aligned}
q(x,t) = &\left\{\frac{3\beta\kappa - 3b_2 \pm 2k\lambda\sqrt{-3ab_3}}{8b_3} \pm \frac{k}{2}\sqrt{-\frac{3a}{b_3}}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(k(x + \{\alpha + 2a\kappa\}t) + C)\right) - \lambda}\right)\right\}^{\frac{1}{2}} \\
&\times \exp\left[i\left\{-\kappa x - \left(\frac{9(b_2 - \beta\kappa)^2 + 4b_3\left[8\alpha\kappa + a\left(8\kappa^2 + k^2\left(\lambda^2 - 4\mu\right)\right)\right]}{32b_3}\right)t + \theta\right\}\right],
\end{aligned} \tag{65}$$

and this is a periodic-singular solution along with the following bright-singular combo soliton,

$$q(x,t)=\left\{\frac{3\beta\kappa-3b_2\pm 2k\lambda\sqrt{-3ab_3}}{8b_3}\pm\frac{k}{2}\sqrt{-\frac{3a}{b_3}}\left(\frac{\lambda}{\exp\left(\lambda\left(k(x+\{\alpha+2a\kappa\}t)+C\right)\right)-1}\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+4b_3\left[8\alpha\kappa+a\left(8\kappa^2+k^2\left(\lambda^2-4\mu\right)\right)\right]}{32b_3}\right)t+\theta\right\}\right] \tag{66}$$

and, finally, the remainder ones are plane waves as

$$q(x,t)=\left\{\frac{3\beta\kappa-3b_2\pm 2k\lambda\sqrt{-3ab_3}}{8b_3}\mp\frac{k}{2}\sqrt{-\frac{3a}{b_3}}\left(\frac{\lambda^2\left(k(x+\{\alpha+2a\kappa\}t)+C\right)}{2\left(\lambda\left(k(x+\{\alpha+2a\kappa\}t)+C\right)+2\right)}\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+4b_3\left[8\alpha\kappa+a\left(8\kappa^2+k^2\left(\lambda^2-4\mu\right)\right)\right]}{32b_3}\right)t+\theta\right\}\right], \tag{67}$$

$$q(x,t)=\left\{\frac{3\beta\kappa-3b_2\pm 2k\lambda\sqrt{-3ab_3}}{8b_3}\pm\frac{k}{2}\sqrt{-\frac{3a}{b_3}}\frac{1}{k(x+\{\alpha+2a\kappa\}t)+C}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{9(b_2-\beta\kappa)^2+4b_3\left[8\alpha\kappa+a\left(8\kappa^2+k^2\left(\lambda^2-4\mu\right)\right)\right]}{32b_3}\right)t+\theta\right\}\right]. \tag{68}$$

Here, it should be emphasized that these solitons exist for $ab_3<0$.

6 Conclusions

This paper secured soliton solutions to optical metamaterials that maintained AC nonlinearity. The governing model was studied with a few perturbation terms all of which are of Hamiltonian type. Bright, dark and singular solutions are recovered together with the required constraint conditions for their existence. As a byproduct, singular periodic solutions emerged with the reverse constraints. These periodic solutions are a byproduct of the integration methodologies. The results of this paper are extremely important and carry a lot of weight to the optical materials community where the study of solitons in the context of metamaterials is essentially new. The results of this paper carry a lot of further research prospects in this area. Later, the study will be extended to other types of perturbation terms. These include stochastic perturbation, non-local perturbations and several others. Additional forms of nonlinear metamaterials will also be considered. These are quadratic-cubic type, log-law nonlinearity and several others. Those results are currently awaited and will be reported in the future.

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