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New solitary wave solutions of the time-fractional Cahn-Allen equation via the improved $(\frac{G'}{G})$-expansion method

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Abstract. An improved $(\frac{G'}{G})$-expansion method is proposed for extracting more general solitary wave solutions of the nonlinear fractional Cahn-Allen equation. The temporal fractional derivative is taken in the sense of Jumarie's fractional derivative. The results of this article are generalized and extended version of previously reported solutions.

1 Introduction

The subject of fractional calculus shows glorious developments in the various phenomena in many natural and social science fields, like economics, engineering, geology, meterology, chemistry, control theory and physics. In recent years, it turned out that many physical phenomena are successfully modeled by the use of fractional-order derivatives. Fractional differential equations have become important in recent decades as mathematical models of processes that exhibit such properties as long-term memory and self-similarity. Finding a variety of exact solutions of nonlinear fractional partial differential equations is an important work because these equations explain the exact description of nonlinear phenomena of various real life problems. Problems in fractional calculus are not only important but quite challenging.

In recent decades, with the developments in symbolic computations, a variety of new techniques have been proposed to solve the nonlinear partial differential equations (PDEs) exactly. Some of those techniques including the solitary wave Ansatz approach [1–3], the first integral method [4,5], Exp-function method [6], the $(\frac{G'}{G})$-expansion method [7–9], Lie symmetry analysis [10], Kudryashov method [11], residual power method [12], sine-Gordon expansion method [13–15], extended Fan sub-equation method [16], $\exp(-\phi(\xi))$-expansion method [17], power series method [18], fractional sub-equation method [19], complex envelop function ansatz [20,21], and so on, have been applied to construct the exact solutions of not only integer-order PDEs but also fractional-order PDEs as well.

In this article, exact travelling wave solutions are obtained for time-fractional Cahn-Allen equation using the improved $(\frac{G'}{G})$-expansion method [22,23] via fractional complex transform. The fractional complex transform often is used to convert fractal time-space to continuous time-space. We consider the fractional complex transform in the sense of Jumarie's modified Riemann-Liouville derivative, which can convert fractional partial differential equations into its equivalent ordinary differential equations.

The rest of the article is arranged as follows: In sect. 2, some preliminaries and notations dealing with the fractional calculus theory are briefly described. Section 3 presents the algorithmic approach to the improved $(\frac{G'}{G})$-expansion method. The exact solutions of the nonlinear Cahn-Allen equation with temporal fractional evolution via proposed method are constructed, along with graphical representation, in sect. 4.

2 Preliminaries and notations

Riemann-Liouville, Grunwald-Letnikov and Caputo derivatives are basic definitions and notations of the fractional calculus theory. Jumarie [24] presented a modification of the Riemann-Liouville definition which appears to provide

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a framework for fractional calculus. In this article, the fractional evolution terms of the evolution equation are in the form of Jumarie's point of view. In this section, we give the definitions and some properties of the Jumarie's modified Riemann-Liouville derivative which are further used in this paper.

Definition. Assume that $f : R \to R$, $x \to f(x)$ denote a continuous (but not necessarily differentiable) function. The Jumarie's modified Riemann-Liouville derivative of order β is defined, as

$$\frac{\partial^\beta f(x)}{\partial x^\beta} = \begin{cases} \frac{1}{\Gamma(-\beta)} \int_0^x (x-\zeta)^{-\beta-1}[f(\zeta)-f(0)]\mathrm{d}\zeta, & \beta < 0, \\ \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-\zeta)^{-\beta}[f(\zeta)-f(0)]\mathrm{d}\zeta, & 0 < \beta < 1, \\ (f^{(n)}(x))^{(\beta-n)}, & n \le \beta \le n+1,\ n \ge 1. \end{cases} \tag{1}$$

Some useful formulas and properties of the modified Riemann-Liouville derivative can be summarized, as

$$\frac{\partial^\beta x^\gamma}{\partial x^\beta} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)} x^{\gamma-\beta}, \quad \gamma > 0, \tag{2}$$

$$\frac{\partial^\beta (u(x)v(x))}{\partial x^\beta} = v(x)\frac{\partial^\beta u(x)}{\partial x^\beta} + u(x)\frac{\partial^\beta v(x)}{\partial x^\beta}, \tag{3}$$

$$\frac{\partial^\beta f[u(x)]}{\partial x^\beta} = f'_u[u(x)]\frac{\partial^\beta u(x)}{\partial x^\beta}, \tag{4}$$

$$\frac{\partial^\beta f[u(x)]}{\partial x^\beta} = \frac{\partial^\beta f[u(x)]}{\partial u^\beta}(u'_x)^\beta. \tag{5}$$

The function $u(x)$ is non-differentiable in eqs. (3) and (4) and is differentiable in eq. (5). Thus, formulas (3)–(5) should be applied carefully.

The fractional complex transformation [25], defined as

$$\begin{aligned} &u(x_1,x_2,x_3,t) = u(\zeta), \\ &\zeta = \frac{kx_1^\beta}{\Gamma(1+\beta)} + \frac{lx_2^\gamma}{\Gamma(1+\gamma)} + \frac{mx_3^\delta}{\Gamma(1+\delta)} + \frac{ct^\beta}{\Gamma(1+\beta)}, \end{aligned} \tag{6}$$

reduces the nonlinear partial differential equation of fractional order to a nonlinear ordinary differential equation (ODE), where k, l, m are arbitrary constants, the localized wave solution $u = u(\zeta)$ travels with speed c and ζ is the amplitude of the travelling wave.

2.1 The improved $(\frac{G'}{G})$-expansion method

Suppose that a nonlinear fractional partial differential equation has the form

$$P\left(u, D_t^{\beta_1}u, D_{x_1}^{\beta_2}u, D_{x_2}^{\beta_3}u, D_{x_3}^{\beta_4}u, \cdots, D_{x_n}^{\beta_{n+1}}u, D_t^{\beta_1}D_t^{\beta_1}u, D_t^{\beta_1}D_{x_1}^{\beta_2}u, \cdots\right) = 0, \tag{7}$$

where $u = u(t, x_1, x_2, x_3, \cdots, x_n)$ is an unknown function, $0 < \beta_1, \beta_2, \beta_3, \cdots, \beta_{n+1} \le 1$ and P is a polynomial in u and its various partial fractional derivatives in which highest-order derivatives and nonlinear terms are involved. The summary of the improved $(\frac{G'}{G})$-expansion method can be presented in the following steps.

Step 1

The fractional complex transformation defined in eq. (6) permits us to transform eq. (7) into the following nonlinear ordinary differential equation:

$$Q(u, u', u', u'', \cdots) = 0, \tag{8}$$

where prime denotes the derivatives with respect to ζ and Q is a polynomial in variable $u(\zeta)$ and its derivatives. Equation (8) can be integrated as many times as possible and taking the constants of integration as zero for simplicity.

Step 2

Assume that the solution of eq. (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as

$$u(\zeta) = \sum_{i=-m}^{i=m} a_i \left(\frac{G'}{G}\right)^i. \tag{9}$$

The function $G = G(\zeta)$ is the solution of auxiliary linear ordinary differential equation,

$$G''(\zeta) + \nu G'(\zeta) + \sigma G(\zeta) = 0, \tag{10}$$

where $G' = \frac{\mathrm{d}G}{\mathrm{d}\zeta}$, ν and σ are real constants. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and the nonlinear terms appearing in eq. (8).

Step 3

Substituting eq. (9) together with eq. (10) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero gives a system of algebraic equations for a_i $(i = 0, 1, \cdots, m)$, k_i $(i = 0, 1, 2, \cdots, n)$ and L.

Step 4

Substituting a_i $(i = 0, 1, 2, \cdots, m)$, k_i $(i = 0, 1, 2, \cdots, n)$, L and general solution of eq. (10) into eq. (9). Depending on the sign of the discriminant $(\nu^2 - 4\sigma)$, the solution of eq. (7) can be obtained.

3 Exact solution of the nonlinear fractional Cahn-Allen equation

The famous models of phase fields arising from the theory of phase transitions, namely the Cahn-Allen equation, Cahn-Hilliard equation, the Penrose-Fife system and the Caginalp system, have gained much interest [26]. The most familiar employed phase-fields models are the Cahn-Allen and Cahn-Hilliard theories. The equation that is going to be discussed in this article is the Cahn-Allen equation. It describes the process of phase separation in iron alloys [27] and is commonly used in solidification and nucleation problems. The nonlinear Cahn-Allen equation with temporal evolution is given as

$$D_t^{\beta} u - u_{xx} + u^3 - u = 0, \quad t > 0,\ 0 < \beta \leq 1, \tag{11}$$

where β is a parameter describing the order of the temporal fractional derivative. Hariharan [28] has obtained the analytical solution of Cahn-Allen equation with the Haar wavelet method. Tascan and Bekir [29] constructed the periodic and solitary wave solutions of the Cahn-Allen equation using the first integral method. The fractional sub-equation method was used to obtained the exact solutions for the spacetime fractional Cahn-Allen equation [30]. New exact solutions of Cahn-Allen equation were obtained using the modified simple equation method and results were also compared with Tascans results [31]. Bekir [32] implemented the double exp-function method and constructed the one-soliton and two-soliton solutions of the Cahn-Allen equations. Ozkan Güner *et al.* [33] determined a variety of exact solutions for the time-fractional Cahn-Allen equation.

In this article, the improved $(\frac{G'}{G})$-expassion method is used to extract the soliton solutions of Cahn-Allen equation. The following traveling wave variable is considered to transform eq. (11) into an ordinary differential equation:

$$u(x,t) = u(\zeta), \qquad \zeta = kx - \frac{Lt^{\beta}}{\Gamma(1+\beta)}, \tag{12}$$

which leads to

$$-Lu' - k^2 u'' + u^3 - u = 0. \tag{13}$$

Suppose that the solutions of eq. (13) can be expressed by a polynomial,

$$u(\zeta) = a_{-1}\left(\frac{G'}{G}\right)^{-1} + a_0 + a_1\left(\frac{G'}{G}\right). \tag{14}$$

Substituting eq. (14) into eq. (13) along with eq. (10), setting the coefficients of $(\frac{G'}{G})^i$, $(i = -1, -2, -3, 0, 1, 2, 3)$ to zero, the following algebraic equations are obtained:

$$\begin{aligned}
a_1^3 - 2a_1k^2 &= 0,\\
La_1 - 3k^2a_1\nu + 3a_0a_1^2 &= 0,\\
La_1\nu - 3a_1a_0^2 - 2k^2a_1\sigma - a_1k^2\nu^2 + 3a_1^2a_{-1} - a_1 &= 0,\\
La_1\sigma - La_{-1} - a_1k^2\nu\sigma - a_{-1}k^2\nu + a_0^3 + 6a_0a_1a_{-1} - a_0 &= 0,\\
-La_{-1}\nu - a_{-1}k^2\nu^2 - 2a_1k^2\sigma + 3a_1a_{-1}^2 + 3a_{-1}a_0^2 - a_{-1} &= 0,\\
-La_{-1}\sigma - 3k^2a_{-1}\nu\sigma + 3a_0a_{-1}^2 &= 0,\\
a_{-1}^3 + 2k^2a_{-1}\sigma^2 &= 0.
\end{aligned}$$

After solving these algebraic equations the following sets of solutions are obtained.

Set a) $L = -\frac{3k}{\sqrt{2}}, \quad a_0 = -\frac{(1+\sqrt{2}k\nu)}{2}, \quad a_1 = -\sqrt{2}k, \quad \sigma = \frac{-1+2k^2\nu^2}{8k^2}, \quad a_{-1} = 0$

Using eq. (14) and the solution of eq. (10), three types of traveling wave solutions of eq. (11) can be calculated.

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_1(\zeta) = -\frac{1}{2} - \frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]. \tag{15}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_2(\zeta) = -\frac{1}{2} - \frac{\iota}{2}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]. \tag{16}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_3(\zeta) = -\frac{1}{2} - \sqrt{2}k\left(\frac{C_2}{C_1 + C_2\zeta}\right), \tag{17}$$

where $\zeta = k(x + \frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

In particular, if $A_1 \neq 0$ and $A_2 = 0$ in eq. (15), it turns out that the kink wave solutions are

$$u_4(\zeta) = -\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} + \frac{3t^\beta}{4\Gamma(1+\beta)}\right).$$

Set b) $L = -\frac{3k}{\sqrt{2}}, \quad a_0 = \frac{1+\sqrt{2}k\nu}{2}, \quad a_1 = \sqrt{2}k, \quad \sigma = \frac{-1+2k^2\nu^2}{8k^2}, \quad a_{-1} = 0$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_5(\zeta) = \frac{1}{2} + \frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]. \tag{18}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_6(\zeta) = \frac{1}{2} + \frac{\iota}{2}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]. \tag{19}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_7(\zeta) = \frac{1}{2} + \sqrt{2}k\left(\frac{C_2}{C_1 + C_2\zeta}\right), \tag{20}$$

where $\zeta = k(x + \frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

In particular, if $A_1 \neq 0$ and $A_2 = 0$ in eq. (18), it turns out that the kink wave solutions are

$$u_8(\zeta) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} + \frac{3t^\beta}{4\Gamma(1+\beta)}\right).$$

Set c) $L = \frac{3k}{\sqrt{2}}$, $a_0 = \frac{-(1+\sqrt{2}k\nu)}{2}$, $a_1 = -\sqrt{2}k$, $\sigma = \frac{-1+2k^2\nu^2}{8k^2}$, $a_{-1} = 0$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_9(\zeta) = \frac{1}{2} - \frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]. \tag{21}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_{10}(\zeta) = \frac{1}{2} - \frac{\iota}{2}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]. \tag{22}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_{11}(\zeta) = \frac{1}{2} - \sqrt{2}k\left(\frac{C_2}{C_1 + C_2\zeta}\right), \tag{23}$$

where $\zeta = k(x - \frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

In particular, if $A_1 \neq 0$ and $A_2 = 0$ in eq. (21), it turns out that the kink wave solutions are

$$u_{12}(\zeta) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\beta}{4\Gamma(1+\beta)}\right).$$

Set d) $L = \frac{3k}{\sqrt{2}}$, $a_0 = \frac{1-\sqrt{2}k\nu}{2}$, $a_1 = -\sqrt{2}k$, $\sigma = \frac{-1+2k^2\nu^2}{8k^2}$, $a_{-1} = 0$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_{13}(\zeta) = -\frac{1}{2} + \frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]. \tag{24}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_{14}(\zeta) = -\frac{1}{2} + \frac{\iota}{2}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]. \tag{25}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_{15}(\zeta) = -\frac{1}{2} + \sqrt{2}k\left(\frac{C_2}{C_1 + C_2\zeta}\right), \tag{26}$$

where $\zeta = k(x - \frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

In particular, if $A_1 \neq 0$ and $A_2 = 0$ in eq. (24), it turns out that the kink wave solutions are

$$u_{16}(\zeta) = -\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\beta}{4\Gamma(1+\beta)}\right).$$

Set e) $L=\frac{-3k}{\sqrt{2}}$, $a_0=\pm\frac{1}{2}$, $a_1=0$, $a_{-1}=\pm\frac{3\sqrt{2}}{8k}$, $\sigma=\frac{3}{8k^2}$, $\nu=-\frac{\sqrt{2}}{k}$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{17}(\zeta)=\pm\frac{1}{2}\pm\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{27}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution is obtained as

$$u_{18}(\zeta)=\pm\frac{1}{2}\pm\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{28}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{19}(\zeta)=\pm\frac{1}{2}\pm\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{C_2}{C_1+C_2\zeta}\right]^{-1}. \tag{29}$$

Set f) $L=\frac{3k}{\sqrt{2}}$, $a_0=\pm\frac{1}{2}$, $a_1=0$, $a_{-1}=\mp\frac{3\sqrt{2}}{8k}$, $\sigma=\frac{3}{8k^2}$, $\nu=-\frac{\sqrt{2}}{k}$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{20}(\zeta)=\pm\frac{1}{2}\mp\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{30}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution is obtained as

$$u_{21}(\zeta)=\pm\frac{1}{2}\mp\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{31}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{22}(\zeta)=\pm\frac{1}{2}\mp\frac{3}{4\sqrt{2}k}\left[-\frac{1}{\sqrt{2}k}+\frac{C_2}{C_1+C_2\zeta}\right]^{-1}. \tag{32}$$

Set g) $L=\frac{-3k}{\sqrt{2}}$, $a_0=\frac{-1}{2}$, $a_1=0$, $a_{-1}=\pm\frac{1}{4\sqrt{2}}$, $\sigma=-\frac{1}{8k^2}$, $\nu=0$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{23}(\zeta)=-\frac{1}{2}\pm\frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]^{-1}. \tag{33}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution is obtained as

$$u_{24}(\zeta)=-\frac{1}{2}\pm\frac{1}{2\iota}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]^{-1}. \tag{34}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{25}(\zeta)=-\frac{1}{2}\pm\frac{1}{4\sqrt{2}k}\left(\frac{C_2}{C_1+C_2\zeta}\right)^{-1}. \tag{35}$$

Set h) $L = \frac{3k}{\sqrt{2}}\,, \quad a_0 = \pm\frac{1}{2}\,, \quad a_1 = 0, \quad a_{-1} = \mp\frac{1}{4\sqrt{2}}\,, \quad \sigma = -\frac{1}{8k^2}\,, \quad \nu = 0$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_{26}(\zeta) = \pm\frac{1}{2} \mp \frac{1}{2}\left[\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right]^{-1}. \tag{36}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_{27}(\zeta) = \pm\frac{1}{2} \mp \frac{1}{2\iota}\left[\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right]^{-1}. \tag{37}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_{28}(\zeta) = \pm\frac{1}{2} \mp \frac{1}{4\sqrt{2}k}\left(\frac{C_2}{C_1 + C_2\zeta}\right)^{-1}, \tag{38}$$

where $\zeta = k(x - \frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

Set i) $L = \frac{3k}{\sqrt{2}}\,, \quad a_0 = \pm\frac{1}{2}\,, \quad a_1 = \mp\sqrt{2}k, \quad a_{-1} = \mp\frac{1}{16\sqrt{2}}\,, \quad \sigma = -\frac{1}{32k^2}\,, \quad \nu = 0$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_{29}(\zeta) = \pm\frac{1}{2} \mp \frac{1}{4}\left[\frac{A_1\sinh(\frac{\zeta}{4\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{4\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{4\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{4\sqrt{2}k})}\right]^{-1} \mp \frac{1}{4}\left[\frac{A_1\sinh(\frac{\zeta}{4\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{4\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{4\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{4\sqrt{2}k})}\right]. \tag{39}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_{30}(\zeta) = \pm\frac{1}{2} \pm \frac{\iota}{4}\left[\frac{-A_1\sin(\frac{\iota\zeta}{4\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{4\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{4\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{4\sqrt{2}k})}\right]^{-1} \mp \frac{\iota}{4}\left[\frac{A_1\sinh(\frac{\zeta}{4\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{4\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{4\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{4\sqrt{2}k})}\right]. \tag{40}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_{31}(\zeta) = \pm\frac{1}{2} \mp \frac{1}{16\sqrt{2}k}\left(\frac{C_2}{C_1 + C_2\zeta}\right)^{-1} \mp \frac{1}{16\sqrt{2}k}\left(\frac{C_2}{C_1 + C_2\zeta}\right). \tag{41}$$

Set j) $L = \frac{3k}{\sqrt{2}}\,, \quad a_0 = \pm\frac{(1-\sqrt{1+8k^2\sigma})}{2}\,, \quad a_1 = 0, \quad a_{-1} = \pm\sqrt{2}k\sigma, \quad \nu = -\frac{\sqrt{1+8k^2\sigma}}{\sqrt{2}k}$

If $\nu^2 - 4\sigma > 0$, the hyperbolic function solution is obtained as

$$u_{32}(\zeta) = \pm\frac{1 - \sqrt{1+8k^2\sigma}}{2} \pm \sqrt{2}k\sigma\left[\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k} + \frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k}) + A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k}) + A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{42}$$

If $\nu^2 - 4\sigma < 0$, the trigonometric function solution is obtained as

$$u_{33}(\zeta) = \pm\frac{1 - \sqrt{1+8k^2\sigma}}{2} \pm \sqrt{2}k\sigma\left[\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k} + \frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k}) + A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{43}$$

If $\nu^2 - 4\sigma = 0$, the rational function solution is obtained as

$$u_{34}(\zeta) = \pm\frac{1 - \sqrt{1+8k^2\sigma}}{2} \pm \sqrt{2}k\sigma\left(\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k} + \frac{C_2}{C_1 + C_2\zeta}\right)^{-1}. \tag{44}$$

Set k) $L=\frac{3k}{\sqrt{2}}$, $a_0=\pm\frac{(1+\sqrt{1+8k^2\sigma})}{2}$, $a_1=0$, $a_{-1}=\pm\sqrt{2}k\sigma$, $\nu=\frac{\sqrt{1+8k^2\sigma}}{\sqrt{2}k}$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{35}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\pm\sqrt{2}k\sigma\left[-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{45}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution as

$$u_{36}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\pm\sqrt{2}k\sigma\left[-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{46}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{37}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\pm\sqrt{2}k\sigma\left(-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{C_2}{C_1+C_2\zeta}\right)^{-1}. \tag{47}$$

Set l) $L=-\frac{3k}{\sqrt{2}}$, $a_0=\pm\frac{(1+\sqrt{1+8k^2\sigma})}{2}$, $a_1=0$, $a_{-1}=\mp\sqrt{2}k\sigma$, $\nu=-\frac{\sqrt{1+8k^2\sigma}}{\sqrt{2}k}$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{38}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left[\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{48}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution is obtained as

$$u_{39}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left[\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{49}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{40}(\zeta)=\pm\frac{1+\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left(\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{C_2}{C_1+C_2\zeta}\right)^{-1}. \tag{50}$$

Set m) $L=-\frac{3k}{\sqrt{2}}$, $a_0=\pm\frac{(1-\sqrt{1+8k^2\sigma})}{2}$, $a_1=0$, $a_{-1}=\mp\sqrt{2}k\sigma$, $\nu=\frac{\sqrt{1+8k^2\sigma}}{\sqrt{2}k}$

If $\nu^2-4\sigma>0$, the hyperbolic function solution is obtained as

$$u_{41}(\zeta)=\pm\frac{1-\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left[-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{1}{2\sqrt{2}k}\left(\frac{A_1\sinh(\frac{\zeta}{2\sqrt{2}k})+A_2\cosh(\frac{\zeta}{2\sqrt{2}k})}{A_1\cosh(\frac{\zeta}{2\sqrt{2}k})+A_2\sinh(\frac{\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{51}$$

If $\nu^2-4\sigma<0$, the trigonometric function solution is obtained as

$$u_{42}(\zeta)=\pm\frac{1-\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left[-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{\iota}{2\sqrt{2}k}\left(\frac{-A_1\sin(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\cos(\frac{\iota\zeta}{2\sqrt{2}k})}{A_1\cos(\frac{\iota\zeta}{2\sqrt{2}k})+A_2\sin(\frac{\iota\zeta}{2\sqrt{2}k})}\right)\right]^{-1}. \tag{52}$$

If $\nu^2-4\sigma=0$, the rational function solution is obtained as

$$u_{43}(\zeta)=\pm\frac{1-\sqrt{1+8k^2\sigma}}{2}\mp\sqrt{2}k\sigma\left(-\frac{\sqrt{1+8k^2\sigma}}{2\sqrt{2}k}+\frac{C_2}{C_1+C_2\zeta}\right)^{-1}, \tag{53}$$

where $\zeta=k(x\pm\frac{3t^\beta}{\sqrt{2}\Gamma(1+\beta)})$.

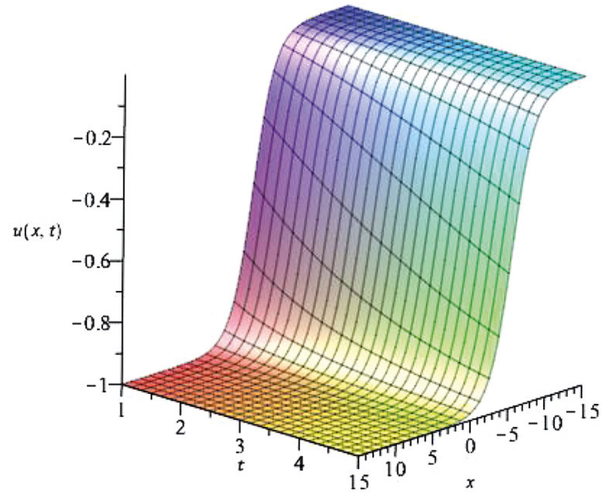

Fig. 1. Plot of the solution u_4 with $\beta = 0.5$.

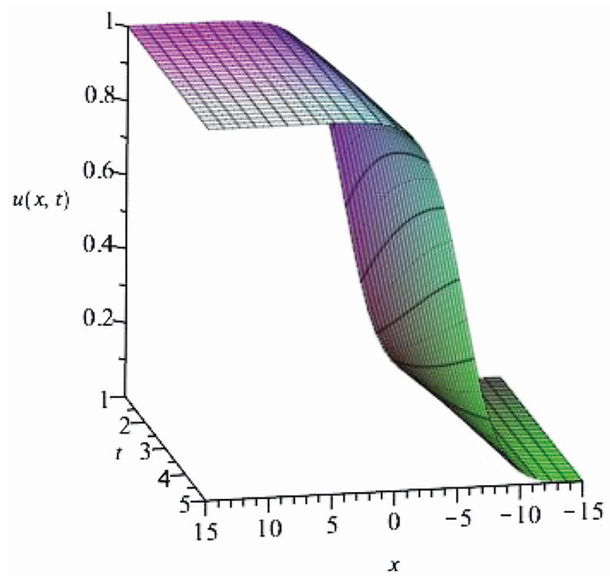

Fig. 2. Plot of the solution u_8 with $\beta = 0.5$.

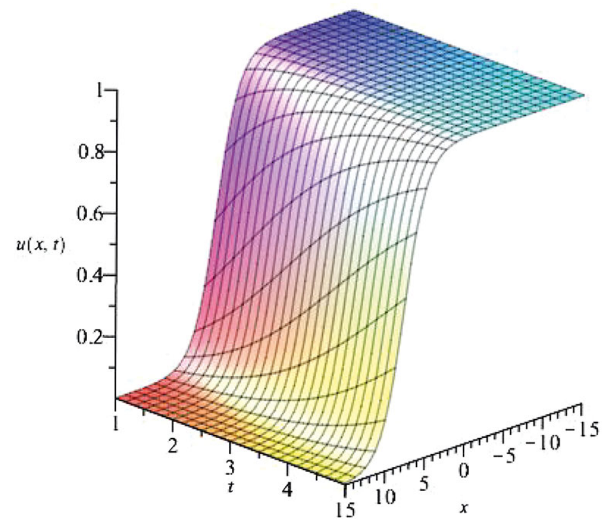

Fig. 3. Plot of the solution u_{12} with $\beta = 0.75$.

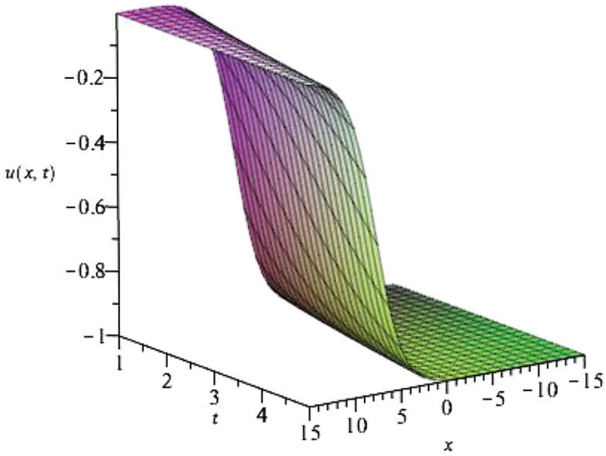

Fig. 4. Plot of the solution u_{16} with $\beta = 0.5$.

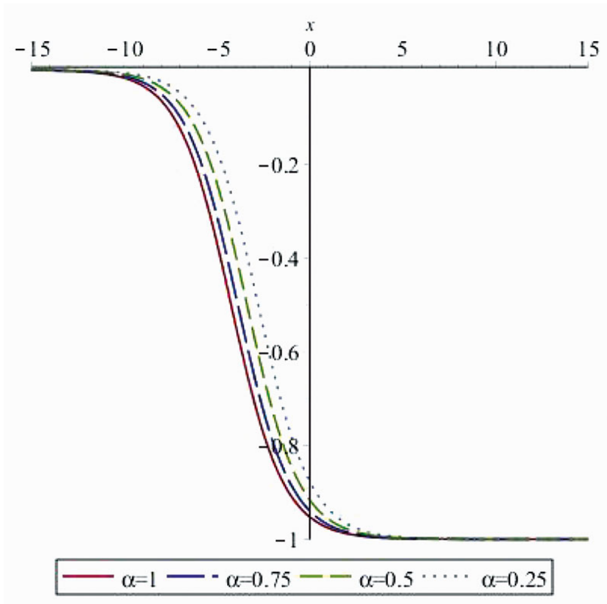

Fig. 5. Comparison of solutions for different values of β.

3.1 Graphical representation of the solutions

In this section, the numerical simulations of time-fractional Cahn-Allen equation are presented. The proposed method provides more general and abundant new kink wave solutions with some free parameters. The kink wave are the type of solitary waves that maintain its shape when travelling down and do not change their shape during the propagation. Kink waves are travelling waves which rise or descend from one asymptotic state to another. Graphical illustrations of the solutions u_4, u_8, u_{12} and u_{16} are illustrated in figs. 1–5 for $\beta = 0.5$ and $\beta = 0.75$. The other exact solutions could be obtained from the remaining solution sets.

4 Conclusion

This study depicts that the improved $(\frac{G'}{G})$-expansion method is quite efficient for extracting the exact travelling wave solutions of the nonlinear Cahn-Allen equation. The reliability of the method and reduction in size of the computational domain give this method a wider applicability. Also it is quite capable and almost well suited for constructing exact travelling wave solutions of other fractional nonlinear evolution equations in mathematical physics.

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