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Nonlocal symmetry and explicit solutions from the CRE method of the Boussinesq equation

Zhonglong Zhao[1,2,a] and Bo Han[1]

[1] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
[2] Department of Mathematics, University of British Columbia, Vancouver V6T 1Z2, Canada

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Abstract. In this paper, we analyze the integrability of the Boussinesq equation by using the truncated Painlevé expansion and the CRE method. Based on the truncated Painlevé expansion, the nonlocal symmetry and Bäcklund transformation of this equation are obtained. A prolonged system is introduced to localize the nonlocal symmetry to the local Lie point symmetry. It is proved that the Boussinesq equation is CRE solvable. The two-solitary-wave fusion solutions, single soliton solutions and soliton-cnoidal wave solutions are presented by means of the Bäcklund transformations.

1 Introduction

The symmetry analysis method is one of the most effective methods to investigate the partial differential equations (PDEs) [1–4]. With the development of the theory of symmetry, the classical Lie symmetry method has been extended to nonclassical symmetry [5], Lie-Bäcklund [6], high-order symmetry [2], which are all local symmetries. To find more generalized symmetries of the PDEs, the nonlocal symmetries have been proposed. It is proved that the nonlocal symmetries can be constructed by means of Darboux transformation [7], Bäcklund transformation [8], Lax pair [9,10] and so on. Bluman *et al.* used the nonlocally related system to find nonlocal symmetries and nonlocal conservation laws [2]. The Painlevé technique [11,12] is very useful for investigating the integrable properties of PDEs. One can yield auto-Bäcklund transformations and Lax pairs for the Painlevé integrable equations by using the generalized Weiss algorithm [13]. Painlevé analysis has been identified as an effective method for searching new integrable systems [14]. Recently, Lou proposed a simple technique to obtain nonlocal symmetries of PDEs by using the truncated Painlevé expansion [15]. These types of nonlocal symmetries are also called residual symmetries. Then, based on the truncated Painlevé expansion, Lou developed a new consistent Riccati expansion (CRE) method [16] to find interaction solutions between soliton and other types of waves. These solutions cannot easily be obtained with the aid of the Lie symmetry method [17–19]. Inspired by Lou's work, many integrable equations' nonlocal symmetries and various interaction solutions, such as the $(2+1)$-dimensional Broer-Kaup-Kupershmidt system [20], the $(2+1)$-dimensional AKNS equation [21], the $(2+1)$-dimensional modified KdV-Calogero-Bogoyavlenkskii-Schiff equation [22], the Gardner equation [23], the $(2+1)$-dimesional Konopelchenko-Dubrovsky equation [24] and the generalized KP equation [25] have been obtained, respectively.

In this paper, we shall focus on a Boussinesq equation

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta(u^2)_{xx} = 0, \tag{1}$$

which is an important physical model to describe long waves in shallow water [26]. The Boussinesq equation can be reduced to the ill-posed Boussinesq equation [27,28]

$$u_{tt} - u_{xx} - u_{xxxx} - (u^2)_{xx} = 0. \tag{2}$$

This equation plays an important role in fluid mechanics. Research shows that this equation is integrable in the sense of N-soliton solutions [29], Painlevé integrable [30], Bäcklund transformation [31], etc. Clarkson and Kruskal developed a direct method for investigating some new similarity reductions of the Boussinesq equation [32]. New

[a] e-mail: zhaozlhit@163.com (corresponding author)

nonclassical symmetry reductions and Painlevé analysis of a generalized Boussinesq equation were investigated [33]. Many new exact traveling wave solutions of the Boussinesq equation were constructed by using the Riccati expansion method [34]. Rational solutions of the Boussinesq equation were derived [35]. The potential symmetries method was applied to determine the symmetry reductions of the Boussinesq equation [36]. The Boussinesq equation was used for describing the convection phenomenon in the viscous incompressible flows and it arises in geophysics when $\alpha = -1$, $\beta = k$ [37]. Xin *et al.* derived the nonlocal symmetries of an equivalent form of Boussinesq equation from Lax pair [38]. The paper [39] investigated the rational and semirational solutions of the Boussinesq equation. To our knowledge, the nonlocal symmetries obtained from the truncated Painlevé expansion and CRE solvability of eq. (1) have not been presented.

The organization of the paper is as follows. In sect. 2, we obtain the Bäcklund transformation and nonlocal symmetry of eq. (1) with the aid of the truncated Painlevé expansion. To localize the nonlocal symmetry to the local Lie point symmetry, a prolonged system of eq. (1) shall be introduced. The two-solitary wave fusion solutions are presented by using the Bäcklund transformation. In sect. 3, the CRE solvability of (1) is investigated. Furthermore, the interaction solutions between the soliton and the cnoidal periodic wave are explicitly presented. The last section contains a summary and discussion.

2 Nonlocal symmetry and fusion of solitary waves of eq. (1)

In this section, we shall investigate the nonlocal symmetry of eq. (1) from the truncated Painlevé expansion. According to the truncated Painlevé analysis, eq. (1) can be expanded to the Laurent series

$$u = u_0 + \frac{u_1}{f} + \frac{u_2}{f^2}, \tag{3}$$

where u_0, u_1, u_2 and f are the functions of x and t. Substituting (3) into the (1) and vanishing all the coefficients of the powers of $\frac{1}{f}$, we have

$$u_0 = -\frac{1}{2}\frac{-f_t^2 - 4\alpha f_{xxx} f_x + 3\alpha f_{xx}^2 + f_x^2}{\beta f_x^2}, \qquad u_1 = -\frac{6\alpha f_{xx}}{\beta}, \qquad u_2 = \frac{6\alpha f_x^2}{\beta}, \tag{4}$$

where f satisfies the equation

$$f_{xxxx} = \frac{f_{xx} f_t^2 - 3\alpha f_{xx}^3 + 4\alpha f_x f_{xxx} f_{xx} - f_x^2 f_{tt}}{\alpha f_x^2}, \tag{5}$$

which is equivalent to the Schwarzian form

$$\frac{1}{\alpha} C_t + \left(S + \frac{1}{2\alpha} C^2\right)_x = 0, \tag{6}$$

by introducing notations as

$$C = \frac{f_t}{f_x}, \qquad S = \frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{f_{xx}^2}{f_x^2}.$$

From the above analysis, we have the following Bäcklund transformation theorem.

Theorem 1 (Bäcklund transformation 1). *If function f is a solution of eq.* (6)*, then*

$$u = -\frac{1}{2}\frac{-{f_t}^2 - 4\alpha f_{xxx} f_x + 3\alpha f_{xx}^2 + f_x^2}{\beta f_x^2} - 6\frac{\alpha f_{xx}}{\beta f} + 6\frac{\alpha f_x^2}{\beta f^2}, \tag{7}$$

is a solution of eq. (1).

Theorem 1 provides a method to construct the new solutions of eq. (1). Once the solutions f of eq. (6) are known, the solutions of u can be obtained by using formula (7).

Based on the definition of residual symmetry, the nonlocal symmetry can be obtained from the truncated Painlevé analysis

$$\sigma^u = -\frac{6\alpha f_{xx}}{\beta}. \tag{8}$$

This nonlocal symmetry can also be derived form the Schwarzian form (6). It is known that the Schwarzian form is invariant under the transformation of Möbius

$$f \to \frac{a+bf}{c+df}\,, \quad (ad \neq bc) \tag{9}$$

which means eq. (6) has the point symmetry

$$\sigma^f = -f^2. \tag{10}$$

The symmetry (10) can be simply derived from (9) by making $a = 0$, $b = c = 1$, $d = \varepsilon$. Thus, we have the following nonlocal symmetry theorem.

Theorem 2. *The Boussinesq equation* (1) *has the nonlocal symmetry*

$$\sigma^u = -\frac{6\alpha f_{xx}}{\beta}\,, \tag{11}$$

where u *and* f *satisfy the Bäcklund transformation 1.*

Proof. The symmetry equation for eq. (1) is

$$\sigma^u_{tt} - \sigma^u_{xx} + \alpha\sigma^u_{xxxx} - 4\beta u_x\sigma^u_x - 2\beta\sigma^u u_{xx} - 2\beta u\sigma^u_{xx} = 0. \tag{12}$$

Theorem 2 can be proved by substituting the nonlocal symmetry (11) into (12) with the aid of (6) and (7).

To find out the group of symmetry

$$u \to \tilde{u}(\varepsilon) = u + \varepsilon\sigma^u, \tag{13}$$

where σ^u is given by (11), we should solve the following initial value problem:

$$\frac{\mathrm{d}\tilde{u}(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{6\alpha f_{xx}}{\beta}\,, \qquad \tilde{u}(\varepsilon)|_{\varepsilon=0} = u,$$

where ε is an infinitesimal parameter. Due to σ^u in (13) contains the extra variable f_{xx} other than x, t and u, the group (13) is a group of nonlocal symmetry. However, it is difficult to solve the above initial value problem directly. In order to solve the above value problem simply, one can localize the nonlocal symmetry to the local Lie point symmetry for an enlarged system. We introduce the following new variables:

$$f_1 = f_x, \qquad f_2 = f_{1x}, \tag{14}$$

then we obtain a prolonged system including (1), (6), (7) and (14). The Lie point symmetry of the prolonged system is

$$\begin{pmatrix} \sigma^u \\ \sigma^f \\ \sigma^{f_1} \\ \sigma^{f_2} \end{pmatrix} = \begin{pmatrix} -\frac{6\alpha}{\beta} f_2 \\ -f^2 \\ -2ff_1 \\ -2f_1^2 - 2ff_2 \end{pmatrix} \tag{15}$$

Based on the Lie's first theorem, the corresponding initial value problem of the Lie point symmetry is

$$\begin{aligned} &\frac{\mathrm{d}\tilde{u}(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{6\alpha}{\beta}\tilde{f}_2(\varepsilon), \quad \tilde{u}(0) = u, \\ &\frac{\mathrm{d}\tilde{f}(\varepsilon)}{\mathrm{d}\varepsilon} = -\tilde{f}(\varepsilon)^2, \quad \tilde{f}(0) = f, \\ &\frac{\mathrm{d}\tilde{f}_1(\varepsilon)}{\mathrm{d}\varepsilon} = -2\tilde{f}(\varepsilon)\tilde{f}_1(\varepsilon), \quad \tilde{f}_1(0) = f_1, \\ &\frac{\mathrm{d}\tilde{f}_2(\varepsilon)}{\mathrm{d}\varepsilon} = -2\tilde{f}_1(\varepsilon)^2 - 2\tilde{f}(\varepsilon)\tilde{f}_2(\varepsilon), \quad \tilde{f}_2(0) = f_2. \end{aligned} \tag{16}$$

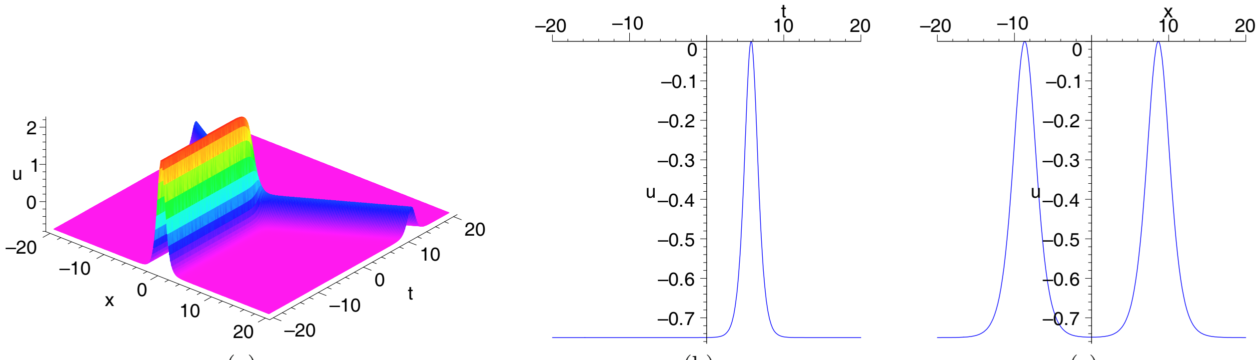


Fig. 1. The fusion of two bell solitary waves (19) with $p_1 = 1$, $p_2 = -1$, $\alpha = 1$, $\beta = -2$ (a) Perspective view of the wave u. (b) The wave along the t-axis with $x = 10$. (c) The wave along the x-axis with $t = 5$.

Solving the above initial problem, we obtain the transformation group that corresponds to the symmetry (15) of the prolonged system

$$\begin{aligned}\tilde{f}(\varepsilon) &= \frac{f}{1+\varepsilon f},\\ \tilde{f}_1 &= \frac{f_1}{(1+\varepsilon f)^2},\\ \tilde{f}_2 &= \frac{f_2}{(1+\varepsilon f)^2} - \frac{2\varepsilon f_1^2}{(1+\varepsilon f)^3},\\ \tilde{u} &= u - \frac{6\alpha}{\beta}\frac{\varepsilon f_2}{1+\varepsilon f} + \frac{6\alpha}{\beta}\frac{\varepsilon^2 f_1^2}{(1+\varepsilon f)^2}.\end{aligned}$$

where the ε is an arbitrary group parameter. Thus, if $\{u, f, f_1, f_2\}$ is the solution of the prolonged system (1), (6), (7) and (14), $\{\tilde{u}, \tilde{f}, \tilde{f}_1, \tilde{f}_2\}$ is also the solution of this prolonged system.

Remark 1. The nonlocal symmetry (11) of eq. (1) is actually a Lie point symmetry of a prolonged system (1), (6), (7) and (14). It depends on the new variables f_{xx}, where f satisfies Schwarzian form (6). The Lie point symmetry of the prolonged system has not been investigated. Thus the nonlocal symmetry can be regarded as a new symmetry of eq. (1). Solving the new reduction equations of the prolonged system can yield new solutions.

For eq. (5), we let

$$f = 1 + e^{p_1 x + q_1 t} + e^{p_2 x + q_2 t}. \tag{17}$$

Substituting the ansatz into (5) will produce the following relations:

$$q_1 = \frac{\sqrt{3\alpha}}{3} p_1(p_1 - 2p_2), \qquad q_2 = \frac{\sqrt{3\alpha}}{3} p_2(2p_1 - p_2). \tag{18}$$

Substituting eq. (17) with (18) into (7) gives the two-solitary-wave fusion solution

$$\begin{aligned}u(x,t) = &-\frac{1}{2}\beta^{-1}(p_1 e^{\Delta_1} + p_2 e^{\Delta_2})^{-2}\Bigg(-\left(-\frac{1}{3}\sqrt{3\alpha}p_1(-2p_2+p_1)e^{\Delta_1} + \frac{\sqrt{3\alpha}}{3}(-p_2+2p_1)p_2 e^{\Delta_2}\right)^2\\ &-4\alpha({p_1}^3 e^{\Delta_1} + {p_2}^3 e^{\Delta_2})\left(p_1 e^{\Delta_1} + p_2 e^{\Delta_2}\right) + 3\left({p_1}^2 e^{\Delta_1} + {p_2}^2 e^{\Delta_2}\right)^2\alpha + \left(p_1 e^{\Delta_1} + p_2 e^{\Delta_2}\right)^2\Bigg)\\ &-6\alpha\beta^{-1}\left({p_1}^2 e^{\Delta_1} + {p_2}^2 e^{\Delta_2}\right)\left(1 + e^{\Delta_1} + e^{\Delta_2}\right)^{-1} + 6\alpha\beta^{-1}\left(p_1 e^{\Delta_1} + p_2 e^{\Delta_2}\right)^2\left(1 + e^{\Delta_1} + e^{\Delta_2}\right)^{-2},\\ \Delta_1 = &\; p_1 x - \frac{\sqrt{3\alpha}}{3}(-2p_2+p_1)p_1 t, \qquad \Delta_2 = p_2 x + \frac{\sqrt{3\alpha}}{3}(-p_2+2p_1)p_2 t.\end{aligned} \tag{19}$$

Figure 1 shows the fusion phenomenon of the two bell solitary waves when $p_1 = 1$, $p_2 = -1$, $\alpha = 1$ and $\beta = -2$.

3 CRE solvable and soliton-cnoidal waves of eq. (1)

Based on the CRE method, the CRE solutions of eq. (1) can be written as

$$u = u_0 + u_1 R(w) + u_2 R(w)^2, \quad w \equiv w(x,t), \tag{20}$$

where $R(w)$ is the solution of the following Riccati equation

$$R(w)_w = a_0 + a_1 R(w) + a_2 R(w)^2, \tag{21}$$

where a_0, a_1 and a_2 are arbitrary constants. By substituting the expression (20) with eq. (21) into eq. (1) and collecting all the coefficients of the power of $R(w)$, we can obtain the overdetermined system about u_0, u_1 and u_2. Solving this system, we obtain

$$u_0 = \frac{1}{2}\frac{4\alpha w_x w_{xxx} - 3\alpha {w_{xx}}^2 + 6 w_{xx}\alpha {w_x}^2 a_1 + \alpha({a_1}^2 + 8a_2a_0){w_x}^4 - {w_x}^2 + {w_t}^2}{{w_x}^2\beta}, \tag{22}$$

$$u_1 = 6\frac{a_2\alpha(w_{x,x} + {w_x}^2 a_1)}{\beta}, \tag{23}$$

$$u_2 = 6\frac{\alpha {a_2}^2 {w_x}^2}{\beta} \tag{24}$$

and the function $w(x,t)$ satisfy

$$w_{x,x,x,x} = \frac{4w_x\alpha w_{x,x}w_{x,x,x} - 3\alpha {w_{x,x}}^3 + (\alpha({a_1}^2 - 4a_2a_0){w_x}^4 + {w_t}^2)w_{x,x} - {w_x}^2 w_{t,t}}{{w_x}^2\alpha}, \tag{25}$$

which is equivalent to the Schwarz-type equation

$$\frac{1}{\alpha}C_t + \left(S + \frac{1}{2\alpha}C^2 - \frac{a_1^2 - 4a_0a_2}{2}w_x^2\right)_x = 0, \tag{26}$$

by introducing notations as

$$C = \frac{w_t}{w_x}, \qquad S = \frac{w_{xxx}}{w_x} - \frac{3}{2}\frac{w_{xx}^2}{w_x^2}.$$

Then we can establish a Bäcklund transformation between the solution u of eq. (1) and $R(w)$ of the Riccati equation (21).

Theorem 3 (Bäcklund transformation 2). *If function w is the solution of Schwarz-type equation* (26), *then*

$$\begin{aligned} u = {} & \frac{1}{2}\frac{4\alpha w_x w_{xxx} - 3\alpha {w_{xx}}^2 + 6 w_{xx}\alpha {w_x}^2 a_1 + \alpha({a_1}^2 + 8a_2a_0){w_x}^4 - {w_x}^2 + {w_t}^2}{{w_x}^2\beta} \\ & + 6\frac{a_2\alpha(w_{x,x} + {w_x}^2 a_1)}{\beta}R(w) + 6\frac{\alpha {a_2}^2 {w_x}^2}{\beta}R(w)^2 \end{aligned} \tag{27}$$

is a Bäcklund transformation between u and $R(w)$.

3.1 Single soliton solutions of eq. (1)

As is known the Riccati equation (21) has the following tanh-function solution:

$$R(w) = -\frac{1}{2a_2}\left[a_1 + \sqrt{a_1^2 - 4a_0a_2}\tanh\left(\frac{1}{2}\sqrt{a_1^2 - 4a_0a_2 w}\right)\right]. \tag{28}$$

We take the

$$w(x,t) = kx + ht + l, \tag{29}$$

and substitute it into (27), then it comes that

$$u = \frac{1}{2\beta}\left((a_1^2 + 8a_0a_2)k^2 - 1 + \frac{h^2}{k^2}\right) + \frac{6\alpha a_1 a_2 k^2}{\beta}R(w) + \frac{6\alpha a_2^2 k^2}{\beta}R(w)^2, \tag{30}$$

is the single soliton solution of eq. (1). (See fig. 2.)

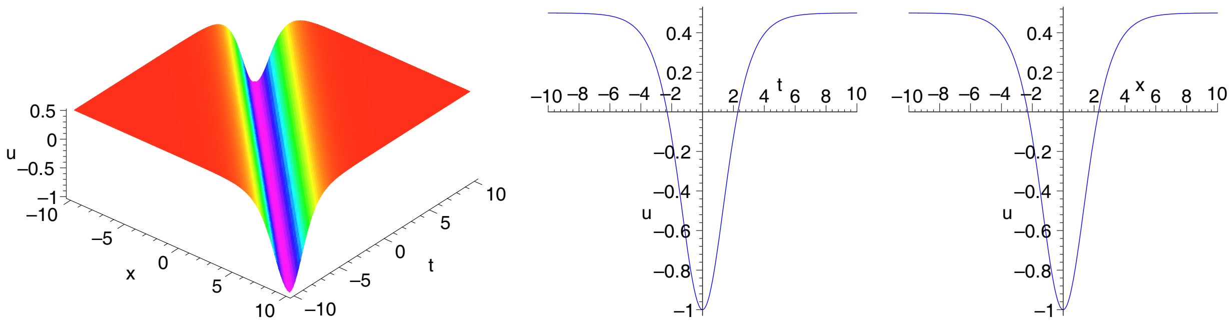


Fig. 2. Single soliton wave (30) with $\alpha = 1$, $\beta = 1$, $a_0 = 0$, $a_1 = -1$, $a_2 = 1$, $k = 1$, $h = 1$, $l = 0$. (a) Perspective view of the wave u. (b) The wave along the t-axis. (c) The wave along the x-axis.

3.2 Soliton-cnoidal wave solutions of eq. (1)

To find the soliton-cnoidal wave solutions of eq. (1), we let

$$w = k_1 x + h_1 t + \psi(\xi), \quad \xi = k_2 x + h_2 t, \tag{31}$$

$\psi_\xi = \frac{d\psi(\xi)}{d\xi}$ is the solution of the elliptic equation

$$\psi_{\xi\xi}^2 = c_0 + c_1\psi_\xi + c_2\psi_\xi^2 + c_3\psi_\xi^3 + c_4\psi_\xi^4, \tag{32}$$

where c_i $(i = 0 \ldots 4)$ are constants. Substituting (31) with (32) into the (25), we can obtain the following set of constraining equations for c_i:

$$\begin{aligned}
c_0 &= -\frac{1}{3}\frac{4\alpha a_2 {k_2}^2 a_0 {k_1}^4 - {k_2}^2 {h_1}^2 - \alpha {k_2}^2 {a_1}^2 {k_1}^4 + {k_2}^4 \alpha c_2 {k_1}^2 + {h_2}^2 {k_1}^2 - 2\alpha {k_2}^5 k_1 c_1}{\alpha {k_2}^6},\\
c_3 &= \frac{1}{3}\frac{16\alpha a_2 {k_2}^2 a_0 {k_1}^3 - 2k_2 h_1 h_2 + 2{h_2}^2 k_1 - 2{k_2}^4 \alpha c_2 k_1 - 4\alpha {k_2}^2 {a_1}^2 {k_1}^3 + \alpha {k_2}^5 c_1}{\alpha {k_2}^3 {k_1}^2},\\
c_4 &= {a_1}^2 - 4a_0 a_2.
\end{aligned} \tag{33}$$

Then, according to the theorem 1, eq. (1) has the solution

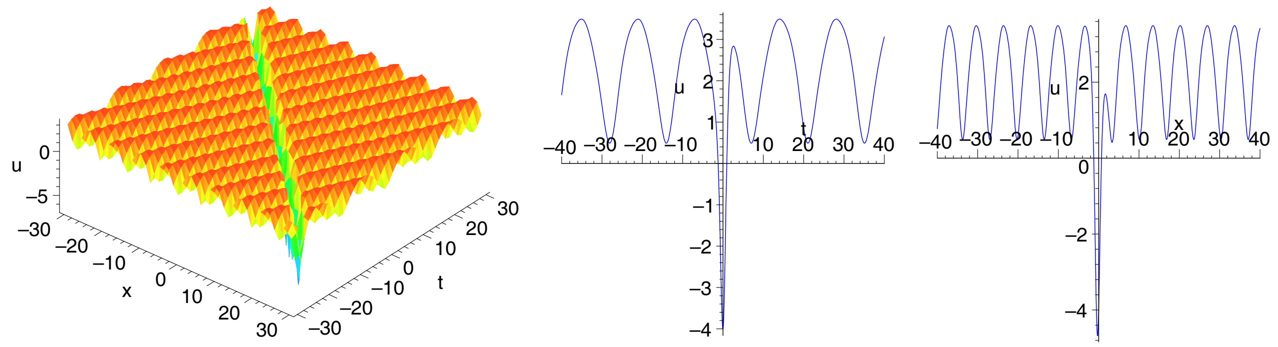
$$\begin{aligned}
u &= \frac{1}{2}\Big[(h_1 + \psi_\xi h_2)^2 + 4\alpha {k_2}^3 (k_1 + \psi_\xi k_2)\psi_{\xi\xi\xi} + 6\alpha a_1 \psi_{\xi\xi} {k_2}^2 (k_1 + \psi_\xi k_2)^2\\
&\quad + (k_1 + k_2)^4 \alpha {a_1}^2 - 3\alpha {k_2}^4 (\psi_{\xi\xi})^2 - (k_1 + \psi_\xi k_2)^2 + 8\alpha a_0 a_2 (k_1 + k_2\psi_\xi)^4\Big] \Big/ (k_1 + k_2\psi_\xi)^2 \beta\\
&\quad - 3\frac{\alpha(\psi_{\xi\xi} {k_2}^2 + (k_1 + \psi_\xi k_2)^2 a_1)(a_1 + \sqrt{{a_1}^2 - 4a_0 a_2}\tanh(1/2\sqrt{{a_1}^2 - 4a_0 a_2}(k_1 x + h_1 t + \psi(k_2 x + h_2 t))))}{\beta}\\
&\quad + 3/2\frac{\alpha(k_1 + k_2\psi_\xi)^2 (a_1 + \sqrt{{a_1}^2 - 4a_0 a_2}\tanh(1/2\sqrt{{a_1}^2 - 4a_0 a_2}(k_1 x + h_1 t + \psi(k_2 x + h_2 t))))^2}{\beta}.
\end{aligned} \tag{34}$$

The general solution of (32) can be written out in terms of Jacobi elliptic functions. Hence if we take the solution of (32) as

$$\psi_\xi = \mu_0 + \mu_1 sn(m\xi, n), \tag{35}$$

we can investigate the interaction solutions, which are under the collision between the soliton wave and cnoidal periodic wave. Substituting (35) with (33) and conditions $cn^2 = 1 - sn^2$ and $dn^2 = 1 - n^2 sn^2$ into (32) and vanishing all the coefficients of powers of sn, we yield

$$\begin{aligned}
c_1 &= 2\mu_0(8{\mu_0}^2 a_2 a_0 - 2{\mu_0}^2 {a_1}^2 + m^2 + m^2 n^2),\\
c_2 &= -24{\mu_0}^2 a_2 a_0 + 6{\mu_0}^2 {a_1}^2 - m^2 n^2 - m^2,\\
h_1 &= \frac{Q_1\sqrt{Q_2}}{Q_3\sqrt{\alpha {a_1}^2 - 4a_0 a_2}}, \qquad h_2 = \frac{P_2\sqrt{\alpha {a_1}^2 - 4\alpha a_0 a_2}}{\sqrt{P_1}}, \qquad \mu_1 = \frac{mn}{\sqrt{{a_1}^2 - 4a_0 a_2}}.
\end{aligned} \tag{36}$$



(a) (b) (c)

Fig. 3. Soliton-cnoidal wave (37) with $m = 1$, $n = \frac{1}{2}$, $k_1 = 1$, $k_2 = 1$, $a_0 = 0$, $a_1 = -1$, $a_2 = 1$, $\mu_0 = 1$. (a) Perspective view of the wave u. (b) The wave along the t-axis. (c) The wave along the x-axis.

Therefore, we obtain the following soliton-cnoidal wave interaction solution of (1)

$$
\begin{aligned}
u(x,t) = {} & \frac{1}{2}\Big[(h_1 + (\mu_0 + \mu_1 S)h_2)^2 + 4\alpha(k_1 + (\mu_0 + \mu_1 S)k_2)(-\mu_1 m^2 D^2 S - \mu_1 m^2 n^2 C^2 S){k_2}^3 \\
& + 6\mu_1 m {k_2}^2 \alpha C D (k_1 + (\mu_0 + \mu_1 S)k_2)^2 a_1 + (k_1 + (\mu_0 + \mu_1 S)k_2)^4 \alpha {a_1}^2\Big] \Big/ \beta(k_1 + (\mu_0 + \mu_1 S)k_2)^2 \\
& -3\frac{\alpha(\mu_1 m {k_2}^2 CD + (k_1 + (\mu_0 + \mu_1 S)k_2)^2 a_1)(a_1 + \sqrt{{a_1}^2 - 4a_0a_2}\tanh(\frac{1}{2}\sqrt{{a_1}^2 - 4a_0a_2}(k_1x + h_1t + \int_0^{k_2x+h_2t}\mu_0 + \mu_1 sn(mY,n)\mathrm{d}Y)))}{\beta} \\
& + \frac{3}{2}\frac{\alpha(k_1 + (\mu_0 + \mu_1 S)k_2)^2(a_1 + \sqrt{{a_1}^2 - 4a_0a_2}\tanh(\frac{1}{2}\sqrt{{a_1}^2 - 4a_0a_2}(k_1x + h_1t + \int_0^{k_2x+h_2t}\mu_0 + \mu_1 sn(mY,n)\mathrm{d}Y)))^2}{\beta},
\end{aligned}
\tag{37}
$$

where a_0, a_1, a_2, μ_0, k_1 and k_2 are arbitrary constants, h_1, h_2 and μ_1 are given by (36), and $S = sn(m(k_2x + h_2t), n)$, $C = cn(m(k_2x + h_2t), n)$ and $D = dn(m(k_2x + h_2t), n)$. Figure 3 exhibits the soliton-cnoidal interaction solution (37) with $m = 1$, $n = \frac{1}{2}$, $k_1 = 1$, $k_2 = 1$, $a_0 = 0$, $a_1 = -1$, $a_2 = 1$ and $\mu_0 = 1$.

4 Conclusions

The nonlocal symmetry of the Boussinesq equation (1) is obtained by using the truncated Painlevé expansion. The form of nonlocal symmetry (11) is quite simple, however, it is difficult to solve its corresponding initial problem. To solve this question, a prolonged system is introduced. Actually the nonlocal symmetry related to the truncated Painlevé expansion is a Lie point symmetry of a prolonged system. Two Bäcklund transformations are presented by means of the truncated Painlevé expansion and CRE method, respectively. The two bell-type solitary wave fusion solutions are obtained with the aid of Bäcklund transformation 1. We also investigate the soliton-cnoidal interaction solutions of (1) by using Bäcklund transformation 2. These solutions are useful for explaining some physical phenomena.

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Appendix A.

$$
\begin{aligned}
Q_1 = \alpha\bigg(& (((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2)((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2n^2){k_2}^4 + \frac{20}{3}\mu_0((8a_2a_0 - 2{a_1}^2){\mu_0}^2 + m^2(1 + n^2)) \\
& \times k_1(-1/4{a_1}^2 + a_2a_0){k_2}^3 + 8/3((-6{a_1}^2 + 24a_2a_0){\mu_0}^2 + m^2(1 + n^2)) \\
& \times {k_1}^2(-1/4{a_1}^2 + a_2a_0){k_2}^2 - 32\mu_0{k_1}^3(-1/4{a_1}^2 + a_2a_0)^2k_2 + 16/3{k_1}^4(-1/4{a_1}^2 + a_2a_0)^2\bigg),
\end{aligned}
$$

$$Q_2 = 3((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2)((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2n^2){k_2}^4 + 24\mu_0((8a_2a_0 - 2{a_1}^2){\mu_0}^2 + m^2(1+n^2))k_1(-1/4{a_1}^2 + a_2a_0){k_2}^3 + 12((-6{a_1}^2 + 24a_2a_0){\mu_0}^2 + m^2(1+n^2)){k_1}^2(-1/4{a_1}^2 + a_2a_0){k_2}^2 - 192\mu_0{k_1}^3(-1/4{a_1}^2 + a_2a_0)^2k_2 + 48{k_1}^4(-1/4{a_1}^2 + a_2a_0)^2,$$

$$Q_3 = ((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2)((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2n^2){k_2}^4 + 8\mu_0((8a_2a_0 - 2{a_1}^2){\mu_0}^2 + m^2(1+n^2))k_1(-1/4{a_1}^2 + a_2a_0){k_2}^3 + 4((-6{a_1}^2 + 24a_2a_0){\mu_0}^2 + m^2(1+n^2)){k_1}^2(-1/4{a_1}^2 + a_2a_0){k_2}^2 - 64\mu_0{k_1}^3(-1/4{a_1}^2 + a_2a_0)^2k_2 + 16{k_1}^4(-1/4{a_1}^2 + a_2a_0)^2$$

$$P_1 = 3((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2)((4a_2a_0 - {a_1}^2){\mu_0}^2 + m^2n^2){k_2}^4 + 24k_1\mu_0((8a_2a_0 - 2{a_1}^2){\mu_0}^2 + m^2(1+n^2))(-1/4{a_1}^2 + a_2a_0){k_2}^3 + 12{k_1}^2(-1/4{a_1}^2 + a_2a_0)((-6{a_1}^2 + 24a_2a_0){\mu_0}^2 + m^2(1+n^2)){k_2}^2 - 192{k_1}^3\mu_0(-1/4{a_1}^2 + a_2a_0)^2k_2 + 48{k_1}^4(-1/4{a_1}^2 + a_2a_0)^2,$$

$$P_2 = \Big(\mu_0((8a_2a_0 - 2{a_1}^2){\mu_0}^2 + m^2(1+n^2)){k_2}^3 + k_1((-6{a_1}^2 + 24a_2a_0){\mu_0}^2 + m^2(1+n^2)){k_2}^2 - 24{k_1}^2\mu_0(-1/4{a_1}^2 + a_2a_0)k_2 + (8a_2a_0 - 2{a_1}^2){k_1}^3\Big)k_2.$$

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