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Covariant theory of gravitation in the framework of special relativity

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Abstract. In this work, we study the magnetic effects of gravity in the framework of special relativity. Imposing covariance of the gravitational force with respect to the Lorentz transformations, we show from a thought experiment that a magnetic-like force must be present whenever two or more bodies are in motion. The exact expression for this gravitomagnetic force is then derived purely from special relativity and the consequences of such a covariant theory are developed. For instance, we show that the gravitomagnetic fields satisfy a system of differential equations similar to the Maxwell equations of electrodynamics. This implies that the gravitational waves spread out with the speed of light in a flat spacetime, which is in agreement with the recent results concerning the gravitational waves detection. We also propose that the vector potential can be associated with the interaction momentum in the same way as the scalar potential is usually associated with the interaction energy. Other topics are also discussed, for example, the transformation laws for the fields, the energy and momentum stored in the gravitomagnetic fields, the invariance of the gravitational mass and so on. We remark that is not our intention here to propose an alternative theory of gravitation but, rather, only a first approximation for the gravitational phenomena, so that it can be applied whenever the gravitational force can be regarded as an ordinary effective force field and special relativity can be used with safety. To make this point clear we present briefly a comparison between our approach and that based on the (linearized) Einstein's theory. Finally, we remark that although we have assumed nothing from the electromagnetic theory, we found that gravity and electricity share many properties in common —these similarities, in fact, are just a requirement of special relativity that must apply to any physically acceptable force field.

1 The magnetism of gravity: From Faraday to the present days

Attempts to measure some kind of gravitational magnetism can be dated back to the pioneer experiments of Faraday [1], performed about 1840. In these experiments, Faraday was wondering if gravity would produce some effect analogous to the electromagnetic induction that himself discovered before (see fig. 1). If Faraday had been successful, the existence of gravitational magnetic fields would have been established from the very beginning but, unfortunately, he did not find any positive result. The first theoretical mention of a possible analogy between electromagnetism and gravitation was probably given by Maxwell in [2], although he did not take that matter further. Some time later, Holzmüller [3] and Tisserand [4, 5] independently proposed that the Sun could exercise a gravitational magnetic force on the surrounding planets (this was a naive attempt to explain the advance of Mercury's perihelion through Weber's electrodynamics). It was only at the end of the nineteenth century, however, that Heaviside presented a direct and complete analogy between gravitation and electromagnetism [6]. Since then, the possibility of a gravitational magnetism had been proposed by several authors, for instance, in the first years of the special relativity, this theme appeared in the works of Lorentz [7], Poincaré [8], Minkowski [9] and also Einstein [10].

The way of looking at the magnetic effect of gravity changed, however, after Einstein's formulation of general relativity and his theory of gravitation [11]. In Einstein's theory, the gravitational interaction is no longer regarded as a force, but only as an effect of the spacetime curvature. By this reason, gravitation and electromagnetism seemed to be very different theories and the earlier analogies between these two force fields were almost despised. Actually,

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Fig. 1. Faraday attempts to detect the magnetism of gravity (illustration reproduced with permission of A. de Rújula).

Einstein did show in his seminal work that in the weak field regime Newton's theory of gravitation can be recovered from a linearized form of his field equations. However, he considered in that work only the case where the speed of the gravitating bodies was very small when compared to that of the light. If Einstein had allowed the bodies to move fast, he probably would have rediscovered those aforementioned similarities between gravitation and electromagnetism. Indeed, further developments of the weak field method showed that gravity does present interesting gravitomagnetic effects, for example, the frame-dragging effects of de Sitter [12] and of Lense and Tirring [13,14] as well as some effects analogous to the Thomas precession [15,16].

In modern times, the theoretical possibility of a magnetic gravitational effect was considered from various perspectives. Some authors, notably Jefimenko [17,18], suggested again a pure analogy with electromagnetic theory. Benci and Fortunato [19] presented a mathematical argument (based on a variational principle) that shows some connection between gravity and electromagnetism as well. In the scope of special relativity, the existence of magnetic gravitational fields were discussed by Salisbury [20], Lorrain [21] and then Bedford [22] and Kolbenstvedt [23] (although these works discussed only particular cases and different results were obtained, so that no consensus was achieved regarding the exact formulæ describing the gravitational magnetic field [24,25]). Other more recent considerations include the works of Behera and Naik [26], of Sattinger [27] and also that of Ummarino and Gallerati [28,29], where the influence of gravitomagnetic fields on superconductors were studied. It is nevertheless through the general theory of relativity that the gravitomagnetic effects have been extensively studied in the recent years. These methods are commonly based on the linearized Einstein's field equations or other higher-order expansions as the post-Newtonian approximations. These approaches were developed by several authors, notably Jantzen, Bini, Mashhoon, Iorio, among others —see the reviews [30–35] and references therein. Recently, exact methods have been also presented, for instance, through the analysis of the tidal forces [36,37] or yet in the scope of gauge formulations of general relativity [38–40].

Many experiments were performed as well with the aim of detecting the magnetic effects of gravity. By the end of the fifties, Pugh [41] and then Schiff [42,43] already proposed a way of measuring the gravitational frame-dragging effects with spatial gyroscopes and, after soon, Forward [44] speculated about some gravitational devices analogues to electromagnetic machines; no results however were found with these attempts either. The failure of detecting any gravitational magnetic effect by these experiments can be attributed to the very weakness of the gravitational interaction. Currently, however, our technology has been greatly improved and we are living an exciting moment where those gravitomagnetic effects are beginning to be experimentally verified. For instance, scientists behind the PROBE B group have been conducted experiments that confirm the existence of gravitomagnetic effects predicted by general relativity, such as the geodetic and frame-dragging effects [45–48] and, very recently, we become aware of the astonishing news regarding the first detection of gravitational waves, as reported by scientists of the LIGO group [49–52]. This was very promising for all physicists and we expect that many other welcome surprises will appear in a near future.

The purpose of this work is to show that the gravitomagnetic effects can be also contemplated in the scope of special relativity. This provides an alternative to those methods based on linearized Einstein theory, which has the advantage of being simpler and that can be applied whenever the gravitational field is weak, regardless of the speed of the gravitating bodies. Furthermore, our approach is also very enlightening, since it clarifies the nature and the origin of the gravitomagnetic effects. Employing only the special theory of relativity, we construct a fully covariant theory of gravitation w.r.t. the Lorentz transformations. Although our theory is only an approximation of Einstein's theory, it can be regarded as exact in a flat spacetime, in the same sense as the theory of electromagnetism can be regarded as exact in this case as well. In fact, starting uniquely from the special theory of relativity (without using any result whatsoever of the electromagnetic theory), we found that this covariant theory of gravitation shares many properties in common with the electromagnetic theory. These similarities are, although, not a matter of coincidence, but only a consequence of the covariance of both theories regarding the Lorentz transformations. Since this covariance is required for any physical theory, these properties must hold to any acceptable relativistic force field as well.

This article is organized as follows: in sect. 2 we show from a thought experiment that gravity necessarily must have a magnetic counterpart. In sect. 3 we derive the exact formulæ describing the gravitomagnetic fields purely from covariance requirements of special relativity theory. The question relying on the invariance of the gravitational mass is analyzed in sect. 5, where we show that the gravitational mass should be regarded as an invariant quantity, in the same footing as the electric charge. The transformation laws for the gravitomagnetic fields w.r.t. inertial reference frames are presented in sect. 6, and in sect. 7 we prove that these fields satisfy a set of differential equations which have the same form as the Maxwell equations. The gravitomagnetic potentials are presented in sect. 9, where we argue that the vector potential can be associated with the momentum of interaction between matter and fields in the same manner as the scalar potential is associated with the potential energy. The momentum and energy stored in the gravitational fields are considered in sect. 10 and a manifestly covariant formulation of the theory is presented in sect. 11. The similarities and differences between gravity and electricity are analyzed in sect. 12 and, finally, in sect. 13, we present our conclusions and we discuss some topics that can be exploited further (*e.g.*, the possibility of formulating a quantum version of this theory). Conventions, notations and the vector identities used in this work are presented in the appendices.

2 A thought experiment

We begin our approach with a thought experiment in order to show that matter in motion must generate, besides the gravitational field, also a magnetic field, which is in many ways analogous to the ordinary magnetic field of electromagnetic theory.

This thought experiment can be described as follows: Suppose a system composed of two parallel massive straight lines, say A and B, and also a particle of mass m placed exactly at the midpoint of these lines. For a reference frame S', let those parallel lines be in the X'-direction and suppose that both the particle and line A are at rest w.r.t. S', while line B is moving with a velocity $-v$ in the same direction of the straight lines.

If the densities λ'_A and λ'_B of lines A and B, respectively, are set to be the same in the reference frame S', that is, if we take $\lambda'_A = \lambda'_B = \lambda'$, then the total gravitational force acting on the particle will be zero. In fact, assuming the validity of Newton's law of gravitation in S', the force on the particle due to line A will be given by

$$\boldsymbol{F}'_A = -\frac{2gm\lambda'}{r'}\hat{\boldsymbol{y}}, \tag{1}$$

while the force due to line B will be

$$\boldsymbol{F}'_B = \frac{2gm\lambda'}{r'}\hat{\boldsymbol{y}}, \tag{2}$$

where g is the Newton gravitational constant and r' is the distance between the particle and the lines. This is illustrated in fig. 2.

Now, we might ask what happens when this system is observed from another reference frame S, on which the particle and line A have both the velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$, while line B is now at rest. In the reference frame S, the mass densities λ_A and λ_B are no longer the same thanks to the Lorentz contraction effect. In fact, since the mass of line A

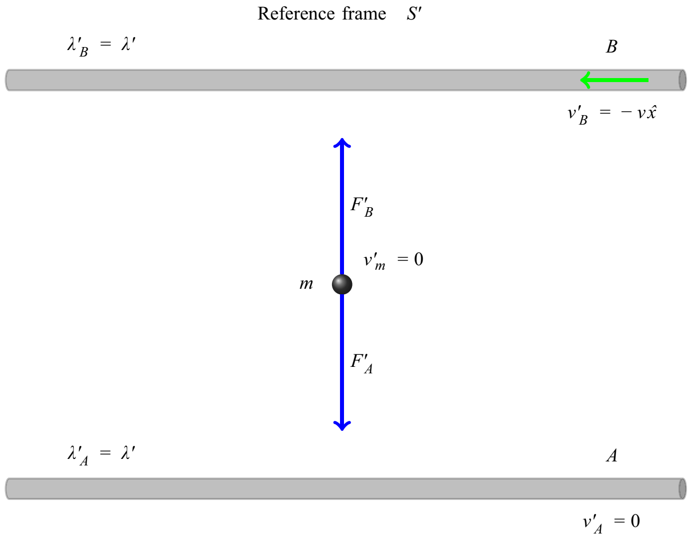

Fig. 2. The thought experiment in S': Two parallel massive straight lines exert a gravitational force in a particle of mass m, placed in the midpoint of them. The densities of the lines are the same so that the resulting force is null. The particle and line A are at rest, while line B is moving with the velocity $\boldsymbol{v} = -v\hat{\boldsymbol{x}}$.

is moving with velocity $\boldsymbol{v}$, the mass density of this line, as measured by S, is given by[1]

$$\lambda_A = \lambda'\gamma, \tag{3}$$

while, for line B, we have, since this line is now at rest,

$$\lambda_B = \lambda'/\gamma. \tag{4}$$

Therefore, in the reference frame S, the force that line A exerts on the particle is given by

$$\boldsymbol{F}_A = -\gamma\frac{2gm\lambda'}{r}\hat{\boldsymbol{y}}, \tag{5}$$

while the force due to line B is

$$\boldsymbol{F}_B = \frac{1}{\gamma}\frac{2gm\lambda'}{r}\hat{\boldsymbol{y}}. \tag{6}$$

Thus, the total gravitational force $\boldsymbol{F} = \boldsymbol{F}_A + \boldsymbol{F}_B$ acting on the particle would be

$$\boldsymbol{F} = \frac{2gm\lambda'}{r}\left(\frac{1}{\gamma} - \gamma\right)\hat{\boldsymbol{y}} = -\frac{2gm\lambda'}{r}\left(\frac{\gamma v^2}{c^2}\right)\hat{\boldsymbol{y}}, \tag{7}$$

which, contrary to what is expected, is not null —rather it is directed towards line A. This odd result however cannot be true, since the principle of relativity ensures that a particle cannot fall towards line A in a given reference frame while staying at rest in another frame. Hence, we are led to the conclusion that, in the reference frame S, there must exist some hidden force acting on the particle, which must depend on its velocity, in order to balance the gravitational forces. This force is nothing but the gravitational analogue of the magnetic force. For the special case we considered here, this magnetic force is given by

$$\boldsymbol{F}^M = \frac{2gm\lambda'}{r}\left(\frac{\gamma v^2}{c^2}\right)\hat{\boldsymbol{y}}. \tag{8}$$

[1] An important question here is about the transformation law for the gravitational mass density: Should the mass density $\lambda = m/V$ transform from S' to S as $\lambda = \gamma\lambda'$ or as $\lambda = \gamma^2\lambda'$? If we regard m as an invariant quantity, then it is clear that the correct transformation law is $\lambda = \gamma\lambda'$, the γ-factor coming from the transformation of the volume: $V = V'/\gamma$. However, if we expressed the mass through Einstein's formula $E = mc^2$ then, since the energy transforms as $E = \gamma E'$, we would be led to the transformation law $\lambda = \gamma^2\lambda'$. Although may appear at first that we have a paradoxical situation here, this is not the case: as will be shown in more details in sect. 5, the correct transformation law is $\lambda = \gamma\lambda'$, so that the mass must be regarded as an invariant quantity in the same manner as the electric charge. In fact, the problem with the second possibility is that mass density and energy density are actually two different concepts —the first is part of a spacetime vector while the second is associated with a second-rank spacetime tensor—, so the indiscriminate use of Einstein's formula above cannot be justified (see [53–57] where this issue is analyzed in more details).

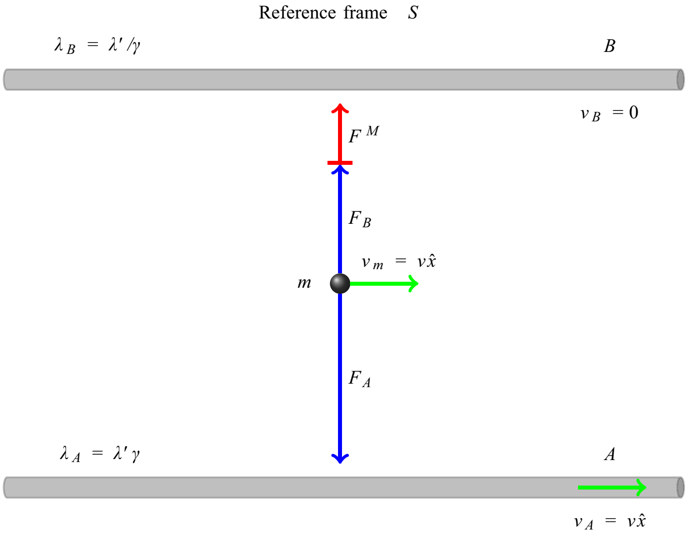

Fig. 3. The thought experiment in S: Here both the particle as line A move with the velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$, while line B is at rest. Thanks to the Lorentz contraction, the densities of the lines are no longer the same, we get that $\lambda_A = \lambda'\gamma$ and $\lambda_B = \lambda'/\gamma$, so that the resulting gravitational force exerted by the lines on the particle is not null anymore. The principle of relativity, however, ensures that the total force acting on the principle must also be null in S, which show us that there must exist a gravitational magnetic force in order to restore that symmetry.

In this way, we had shown from the principle of relativity that any moving body necessarily must create a gravitational magnetic field. This is illustrated in fig. 3.

Let us present another argument to show that the force (8) can really be thought as a magnetic force associated with the gravity. To this end, let us assume for the moment the existence *a priori* of a magnetic gravitational force $\boldsymbol{F}^M$ defined by

$$\boldsymbol{F}^M = m\boldsymbol{v} \times \boldsymbol{B}, \tag{9}$$

where $\boldsymbol{B}$ is the gravitational magnetic field, which is supposed to satisfy a law analogous to Ampère's law of electromagnetic theory, namely,

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu \iint \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{A}, \tag{10}$$

where $\boldsymbol{j} = \lambda\boldsymbol{v}/A$ is the mass current density and μ is a constant yet to be determined. In the reference frame S, the magnetic field generated by the system is only due to line A, since line B is at rest. Remembering that line A moves with velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ in this reference frame, and that its measured mass density should be $\lambda = \lambda'\gamma$, so that $\boldsymbol{j} = \lambda'\gamma\boldsymbol{v}/A$, we have from the Ampère's law above that

$$\boldsymbol{B} = \frac{\mu\lambda'\gamma v}{2\pi r}\hat{\boldsymbol{z}}. \tag{11}$$

Hence, the magnetic force acting on the particle (which moves with the same velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$) is

$$\boldsymbol{F}^M = m\boldsymbol{v} \times \boldsymbol{B} = -\frac{\mu\lambda' m\gamma v^2}{2\pi r}\hat{\boldsymbol{y}}. \tag{12}$$

Notice that this gravitational magnetic force is analogous to its electromagnetic counterpart, except that it is repulsive rather than attractive[2]. Moreover, we can see that, if we set

$$\mu = -\frac{4\pi g}{c^2} = -\frac{\kappa}{2}, \tag{13}$$

where κ is the Einstein gravitational constant, then both expressions (8) and (12) become the same, which show us that the force (8) can also be found without appeal to the relativity theory, in the same fashion as the magnetic force is calculated in the electromagnetic theory.

[2] In fact, if in our example we considered electric charges instead of gravitational mass, the same result would be obtained, except that the magnetic force would be attractive and more intense. This electromagnetic case is explored, for instance, in Feynman's *Lectures on Physics* [58].

3 The magnetism of gravity as a consequence of Lorentz transformations: The gravitomagnetic fields

In the previous section we gave only a glimpse of the gravitomagnetic effects provided by the special relativity theory. Now we shall begin to develop a more rigorous theory of the gravitomagnetic theory. Except when explicitly specified, hereafter we shall consider only the special theory of relativity. This means that any effect due to the curvature of spacetime implicated by the Einstein theory of gravitation will be despised. In a sufficient distant region of the gravitational sources, Newton's Law of gravitation gives a very good approximation for the gravitational interaction, provided that the speed of the gravitating bodies is small when compared with that of the light. Starting from this, we can impose covariance of the gravitational force w.r.t. the Lorentz transformations, from which exact formulæ for the gravitomagnetic fields can be deduced. In this way, Newton's theory can be extended to the case where the velocity of the orbiting bodies is comparable to the light speed. We highlight again that no use of electromagnetic theory (or any other analogy whatsoever) is employed in this approach —our results are derived purely from the transformation laws of the special theory of relativity.

Therefore, let us take a given distribution of matter which is at rest in a given inertial reference frame S'. This distribution of matter creates a static gravitational field $\boldsymbol{G}'(x',y',z')$, so that a particle of mass m situated at a point (x',y',z') of space will feel a gravitational force $\boldsymbol{F}' = \mathrm{d}\boldsymbol{p}'/\mathrm{d}t'$ given by

$$\boldsymbol{F}'(x',y',z') = m\boldsymbol{G}'(x',y',z'). \tag{14}$$

We shall suppose further that this particle is moving with a velocity $\boldsymbol{u}'$ w.r.t. S', although the total force does not depend on the particle velocity in S'.

Now we can ask what should be the (total) force $\boldsymbol{F}(x,y,z,t)$ acting on this particle w.r.t. another inertial reference frame S, which moves with the velocity $\boldsymbol{v} = v_x\hat{\boldsymbol{x}}$ w.r.t. the frame S'. The answer for this question can be found purely from the special theory of relativity —it is not necessary to know the nature of the force. In fact, the force $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ measured in S can be found directly through the respective transformation formulæ provided by the special relativity theory [59–61], which are,

$$F_x(x,y,z,t) = F'_x(x',y',z') + \frac{vu'_y}{c^2}\frac{F'_y(x',y',z')}{1+u'_xv/c^2} + \frac{vu'_z}{c^2}\frac{F'_z(x',y',z')}{1+u'_xv/c^2}\,,$$

$$F_y(x,y,z,t) = \frac{1}{\gamma}\frac{F'_y(x',y',z')}{1+u'_xv/c^2}\,, \tag{15}$$

$$F_z(x,y,z,t) = \frac{1}{\gamma}\frac{F'_z(x',y',z')}{1+u'_xv/c^2}\,. \tag{16}$$

Thus, from (14) we can find what is the total force $\boldsymbol{F}$ acting on the particle, as measured by the reference frame S. Notice, however, that this force in general will depend on the time t, even if the force $\boldsymbol{F}'$ does not depend on t'. This is because the primed coordinates should still be eliminated through the Lorentz transformations,

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \qquad x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z. \tag{17}$$

From (16) its not clear that the total force acting on the particle is composed by gravitational and magnetic forces. Nevertheless we can made this explicit if we write (16) actually in terms of the particle velocity $\boldsymbol{u}$, which is indeed the velocity measured by the reference frame S. From the transformation laws for the velocity components [59–61],

$$u_x = \frac{u'_x + v}{1+u'_xv/c^2}\,, \qquad u_y = \frac{1}{\gamma}\frac{u'_y}{1+u'_xv/c^2}\,, \qquad u_z = \frac{1}{\gamma}\frac{u'_z}{1+u'_xv/c^2}\,, \tag{18}$$

we can eliminate u'_x, u'_y and u'_z in (16) and, then, after simplify, the transformation laws for the force components become,

$$F_x(x,y,z,t) = F'_x(x',y',z') + \frac{\gamma v}{c^2}\left[u_yF'_y(x',y',z') + u_zF'_z(x',y',z')\right],$$

$$F_y(x,y,z,t) = \gamma\left(1 - \frac{u_xv}{c^2}\right)F'_y(x',y',z'),$$

$$F_z(x,y,z,t) = \gamma\left(1 - \frac{u_xv}{c^2}\right)F'_z(x',y',z'). \tag{19}$$

These formulæ are the key point of our approach. From this we can plainly see that the total force acting on the particle can be split into two parts: a part which does not depend on the particle velocity $\boldsymbol{u}$ and a part that does.

In this way, we define the (electric-like) *gravitational force* acting on the particle as that part of the *gravitomagnetic* force which does not depend on the velocity of the particle and, on the other hand, we define the (gravitational) *magnetic force* as that part of the gravitomagnetic force which does depend on the particle velocity. In terms of components, we get, for the gravitational force,

$$F_x^G(x,y,z,t) = F_x'(x',y',z'), \qquad F_y^G(x,y,z,t) = \gamma F_y'(x',y',z'), \qquad F_z^G(x,y,z,t) = \gamma F_z'(x',y',z'), \tag{20}$$

while for the components of the magnetic force, we get,

$$\begin{aligned} F_x^M(x,y,z,t) &= +\frac{\gamma v}{c^2}\left[u_y F_y'(x',y',z') + u_z F_z'(x',y',z')\right], \\ F_y^M(x,y,z,t) &= -\frac{\gamma v}{c^2}\left[u_x F_y'(x',y',z')\right], \\ F_z^M(x,y,z,t) &= -\frac{\gamma v}{c^2}\left[u_x F_z'(x',y',z')\right]. \end{aligned} \tag{21}$$

It should be kept in mind, although, that the Lorentz transformations should still be used in order to express those quantities in terms of x, y, z and t.

Now, it is straightforward to show that the total force $\boldsymbol{F} = \boldsymbol{F}^G + \boldsymbol{F}^M$ can be expressed in the Lorentz form,

$$\boldsymbol{F}(x,y,z,t) = m\left[\boldsymbol{G}(x,y,z,t) + \boldsymbol{u}\times\boldsymbol{B}(x,y,z,t)\right], \tag{22}$$

by introducing the gravitational field

$$\boldsymbol{G}(x,y,z,t) = \boldsymbol{F}^G(x,y,z,t)/m, \tag{23}$$

and defining the magnetic field as

$$\boldsymbol{B}(x,y,z,t) = \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{G}(x,y,z,t). \tag{24}$$

The formulæ deduced above are quite general: they hold to any distribution of mass which is at rest in a given reference frame S' and, consequently, moves with a velocity $\boldsymbol{v}$ in another reference frame S. Besides, since the total force acting on the particle given by the Lorentz formula (22) makes no direct mention to what reference frame it refers (the force depends only on the gravitomagnetic fields and the instantaneous velocity of the particle), this means that the Lorentz force should be valid in any inertial reference frame. This result will be confirmed in sect. 6.

Let us apply now the arguments presented above[3] in the simplest case of a point-like particle of mass M, which we suppose to be at rest in the origin of the reference frame S' at $t' = 0$. The gravitational field generated by this particle is given, in the reference frame S', by Newton's law of gravitation:

$$\boldsymbol{G}'(x',y',z') = -gM\frac{x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}}}{(x'^2+y'^2+z'^2)^{3/2}}. \tag{25}$$

Equations (23) and (24) give, respectively, the gravitational and magnetic fields in the frame S, namely,

$$\boldsymbol{G}(x,y,z,t) = -\gamma\frac{gM}{R^3}\left[(x-vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right] \tag{26}$$

and

$$\boldsymbol{B}(x,y,z,t) = \frac{\gamma v}{c^2}\frac{gM}{R^3}\left(z\hat{\boldsymbol{y}} - y\hat{\boldsymbol{z}}\right) = -\frac{\gamma}{c^2}\frac{gM}{R^3}\left(\boldsymbol{v}\times\boldsymbol{R}\right), \tag{27}$$

where $\boldsymbol{R} = \gamma(x-vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ is the vector that measures the distance $R = |\boldsymbol{R}|$ from the present position of the particle (*i.e.*, at $t = 0$) to the space point where the fields are evaluated; we had also made use of the Lorentz transformations (17) to eliminate the coordinates x', y' and z'. Notice further that the formula for the magnetic field is analogous to the relativistic Biot-Savart law of electromagnetism.

Finally, from the Lorentz formula (22) we find the gravitational and the magnetic forces acting on the particle:

$$\boldsymbol{F}^G(x,y,z,t) = -\gamma\frac{gMm}{R^3}\left[(x-vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right] \tag{28}$$

and

$$\boldsymbol{F}^M(x,y,z,t) = -\frac{\gamma v}{c^2}\frac{gMm}{R^3}\left[\left(u_y y + u_z z\right)\hat{\boldsymbol{x}} - u_x y\hat{\boldsymbol{y}} - u_x z\hat{\boldsymbol{z}}\right], \tag{29}$$

so that the total force is given by

$$\boldsymbol{F}(x,y,z,t) = -\gamma\frac{gMm}{R^3}\left\{\left[(x-vt) + \frac{v}{c^2}\left(u_y y + u_z z\right)\right]\hat{\boldsymbol{x}} + \left(1 - \frac{vu_x}{c^2}\right)\left(y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right)\right\}. \tag{30}$$

[3] We highlight that this approach had been recently employed to explain the *Supplee submarine paradox* [62, 63] in the framework of special relativity in [64].

4 Comparison with the linearized theory of general relativity

In the previous section we derived the formulæ describing the gravitomagnetic fields according to the special theory of relativity. The standard way in which these phenomena are studied, however, is through the general theory of relativity, usually considering the linearized or post-Newtonian approximations of Einstein's gravitation theory. Our purpose in this section is to briefly comment on the differences regarding the description of the gravitomagnetic phenomena implicated by these two approaches.

In the simplest case of the linearized Einstein theory, we consider that spacetime is almost flat, so that the metric $\xi^{\mu\nu}$ of the curved spacetime can be split as $\xi^{\mu\nu} = \eta^{\mu\nu} + \zeta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowski metric of the flat spacetime and $\zeta^{\mu\nu}$ is a small perturbation. With these assumptions, the gravitomagnetic phenomena are deduced as follows: we first need to solve the Einstein field equations, taking into account the approximation employed (*i.e.*, disregarding all quantities which are of second, or higher, order in $\zeta^{\mu\nu}$). Then, the geodesic equations should be solved, so that we can obtain the acceleration of a particle that moves only under the influence of the gravitational fields. Finally, after we know the acceleration of the particle, then we can get the gravitomagnetic forces acting on it [65–68].

In this way, if we assume that the source of the gravitational field is at rest and the orbiting particle does not move so fast, then we are led to Newton's theory of gravitation [11]. On the other hand, in order to obtain gravitomagnetic effects, we should allow the source to move. In this case, the linearized Einstein field equations can be reduced, in certain circumstances, to a set of equations resembling the Maxwell equations of electromagnetism, although they are not the same. For instance, solving the geodesic equations we find that the gravitomagnetic force is given by a modified Lorentz formula [68], namely,

$$\boldsymbol{F} = m\left(\boldsymbol{G} + 4\boldsymbol{u} \times \boldsymbol{B}\right). \tag{31}$$

Therefore, we can see that the linearized theory of general relativity predicts different results if compared with our theory. This is expected, of course, since we did not take into account the effects of spacetime curvature in our approach. In fact, the apparition of this factor of 4 in the magnetic force above seems to be of the same nature as that factor of 2 which was obtained by Einstein in his calculation of the gravitational bending of light, after he took into account the curvature of spacetime [11].

Besides, it should be noticed that there are some implicit and subtle assumptions considered in the approximation above. First of all, when solving the geodesic equations we should actually go beyond the first-order approximation because, if we did not, the geodesics would become the same as that obtained in a flat spacetime [68]. Hence, terms of different orders should be carefully kept in order to compare the resulting equations with the electromagnetic theory. Secondly, these results are generally obtained only when specific gauge conditions are imposed to the field equations —the theory thus obtained is not gauge-invariant in the same sense as the electromagnetic theory is, neither it is covariant regarding the Lorentz transformations. Moreover, we should remember that Einstein's field equations actually contain terms which depend on the pressure and stresses as well and, if these terms were taken into account, then we would also get effects with no electromagnetic analogue —see, for instance, the Matte-Bel decomposition of Riemann tensor [69,70]. It should be mentioned further that attempts to remove the factor 4 in (31) were also considered [30–35]; these attempts however basically try to redefine the gravitomagnetic fields or potentials but, in general, the resulting gravitational Maxwell equations become different from that obtained in the electromagnetic theory (see also [71–73] for alternative perturbative methods that seems to successfully reproduce the Maxwell equations for gravity without the factor of 4 in the magnetic force above).

From what has been said, we can see that our approach is quite different from that obtained through weak field approximations of general relativity. The gravitational magnetic field obtained here is due only to the motion of matter (without restriction to the velocity of the gravitational source), while in the general relativity framework it is mainly due to small perturbations of the metric. Therefore the concepts involved in both approaches are quite different in nature and we cannot expected that the results from both theories exactly agree. In fact, the results that will be discussed in the present work should be regarded as a first approximation to the theory of gravitomagnetic phenomena, valid with good precision at great distances of the gravitational sources so that the gravitation interaction can be described by an effective force field; the gravitomagnetic effects predicted by the linearized theory of general relativity should be regarded as the next more accurate approximation, namely, that one on which the spacetime is supposed to be just a little bit curved.

5 Transformation laws for the mass and current densities and the principle of equivalence

An important issue that is very debated in discussions about the foundations of relativity theory, and that is very important in our theory as well, is the question of invariance of the gravitational mass w.r.t. the Lorentz transformations. It is unanimous accepted fact that electric charge is such an invariant, but no agreement at all is reached for the gravitational mass. Our intention in this section is to clarify this matter.

The beginning of this controversy can be dated to the first formulations of the relativity theory and it also relies on the definition of mass that is employed [53–57]. First of all, we have at least two different conceptions of mass: the *inertial mass* and the *gravitational mass*. In the old times of special relativity, several authors (including Einstein) have considered that the inertial mass depends on the velocity through the equation $m(u) = m_0\gamma_u$, where m_0 is the so-called *rest mass*. The development of relativity showed us, however, that it is more reasonable to regard the inertial mass as an invariant quantity, from which the theory of relativity can be formulated in a simpler and more elegant way[4]. In the case of the gravitational mass this issue becomes most dramatic because it seems, at first sight, that if the gravitational mass were dependent on the velocity, then the strength of the gravitational field of moving bodies would be greater than that of bodies at rest. This, however, is not the case, since we had shown that the total force acting on the moving particle is provided by the transformations (19) and the expression for the gravitational force in the proper frame of the source. Hence, no matter what definition of inertial mass we use: a different definition would just lead to a different definition of the gravitomagnetic fields —the forces cannot be changed by a mere redefinition of the mass.

Independently on what definition of mass is used, what is important to be noticed is that the source of the gravitational interaction must be regarded as an invariant quantity[5], namely, the mass (*i.e.*, the rest mass, if the older interpretation is adopted), so that the gravitational mass density must transform, from the reference frame S' to S as

$$\rho = \gamma\rho'. \tag{32}$$

As commented in the footnote[1], a naive argument would suggest that the gravitational mass density should transform as $\rho = \gamma^2\rho'$ instead, but we shall see in the sequel that this claim is wrong. In fact, the validity of (32) is ultimately a consequence of the equivalence principle. To see that, notice that the mass of a particle can be determined by the relativistic relation,

$$m^2c^4 = E^2 - c^2p^2, \tag{33}$$

where E is the particle energy and $\boldsymbol{p}$ its the momentum. Let us to consider, then, the following example: suppose an ensemble of particles distributed in a uniform way and at rest in the reference frame S'. Let $\sigma' = N'/V'$ be the concentration of particles (*i.e.*, the number of particles per unit volume), as measured by S'. Now we ask what will be this concentration $\sigma = N/V$ w.r.t. the reference frame S, where the ensemble moves with the velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. It is clear that this concentration is given by $\sigma = \gamma\sigma'$, since the number of particles per unit volume will be increased by a γ-factor, thanks to the Lorentz contraction. Now, if we consider that each particle of the ensemble has an inertial mass μ, so that the total mass contained in a giving volume is m, then it follows from the fact that the inertial mass is an invariant of Lorentz that the inertial mass density must transform according to (32). But since the inertial mass should equal the gravitational mass if the equivalence principle is true, we may conclude that the same transformation law (32) must hold for the gravitational mass density. Besides, since the Lorentz formula should be valid in any inertial reference frame (see sect. 6), the invariance of the gravitational mass also follows, and further, as we shall see in sect. 7, this is also necessary for the gravitational Maxwell equations be covariant w.r.t. the Lorentz transformations —in short, the invariance of the inertial and gravitational mass is mandatory for a covariant theory.

At this point, we would like to remark that the concepts of mass and current densities are not the same as the concepts of energy and momentum densities —this is the source of many confusions in the literature, which usually leads to mistakes as that one commented above. In fact, since in relativity theory energy and momentum are regarded as the components of a spacetime vector, the same cannot be true for their densities. Indeed, from the transformation law for the energy and momentum from S' to S,

$$E = \gamma E', \qquad \boldsymbol{p} = \frac{\gamma\boldsymbol{v}}{c^2}E', \tag{34}$$

it follows that the energy density ε and also the momentum density $\boldsymbol{\pi}$ must transform as

$$\varepsilon = \gamma^2\varepsilon', \qquad \boldsymbol{\pi} = \frac{\gamma^2\boldsymbol{v}}{c^2}\varepsilon', \tag{35}$$

[4] Notice that the dependence of the mass with the velocity can be seen either as a matter of convention or as a true experimental issue. The second interpretation, however, face difficulties because it is hard to conceive an independent measurement of the inertial mass, since it is generally attached to other physical quantities as the momentum or energy, for instance.

[5] This should be true both in the special as in the general theory of relativity. As it is known, the gravitational interaction in Einstein's theory is determined by the spacetime momentum-flux tensor $T^{\mu\nu}$. However, the *strength* of the gravitational interaction (*i.e.*, the strength of the spacetime curvature) must be determined only by the *proper* spacetime momentum-flux tensor, otherwise a single particle could even become a black hole in a reference frame where it has a great enough speed, which is clearly an absurd. Therefore, the effects implicated by the motion of matter should be of the same nature as the magnetic effects introduced here, although its mathematical description becomes more complex due to the non-linearity of Einstein's equations.

if we neglect the stresses contributions —so we can see where the confusion comes from. The densities of energy and momentum do not form a spacetime vector; on the contrary, they are (together with the stresses) the components of a two-rank tensor —the *spacetime momentum-flux tensor*. See sects. 10 and 11 for more details.

Notice moreover that we can define in S the *current density* associated with the motion of the ensemble as

$$\boldsymbol{j} = \boldsymbol{v}\rho. \tag{36}$$

The conservation law for the number of particles leads us then to the conservation of matter. This means that the mass and current densities satisfy the *continuity equation*

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{37}$$

This equation, as a matter of fact, also results directly from covariance requirements: take a static distribution of matter in S', so that $\mathrm{d}\rho'/\mathrm{d}t' = 0$. Since $\rho' = \rho/\gamma$, we get that

$$\frac{\mathrm{d}\rho'}{\mathrm{d}t'} = \gamma\left(\partial_t \rho' + v\partial_x \rho'\right) = \partial_t \rho + v\partial_x \rho = 0, \tag{38}$$

where we had used the relation $\mathrm{d}/\mathrm{d}t' = \gamma(\partial/\partial t + v\partial/\partial x)$, deduced from the Lorentz transformations. Thus, since in the case considered here we have $\boldsymbol{j} = v\rho\hat{\boldsymbol{x}}$, this means that (38) is equivalent to the continuity equation (37).

General formulæ for the transformations of the mass and current densities can be found by considering the reference frames S and S''. In the example above, suppose that the ensemble moves with the velocity $\boldsymbol{u}$ w.r.t. the reference frame S and with the velocity $\boldsymbol{u}''$ w.r.t. S'', and also consider that the particles composing the ensemble are still uniformly distributed. Therefore, the mass and current densities will be ρ and $\boldsymbol{j}$ when measured by S and ρ'' and $\boldsymbol{j}''$ w.r.t. S''. Then it is easy to show from the Lorentz transformations for the velocity components (18) that the mass density and the components of the current densities transform as

$$\rho'' = \gamma\left(\rho - vj_x\right), \qquad j''_x = \gamma\left(j_x - \frac{v}{c^2}\rho\right), \qquad j''_y = j_y, \qquad j''_z = j_z, \tag{39}$$

that is, the mass density and the components of the current density form a spacetime vector —the *spacetime current density*.

These results can also be generalized to the case where the mass and current densities are not uniform nor constant in time. To this end, we may consider that the number of particles of the ensemble goes to infinity, while the distance between them goes to zero (so that we get a continuous distribution of matter) and also that those particles contained in a given volume element have a specific velocity. Then, all we need to do is repeat the arguments above to each one of these volume elements. Notice, however, that each volume element will be led to a different time when the Lorentz transformation is performed. It might appear, therefore, that these transformation laws for the mass density $\rho(x, y, z, t)$ and the current density $\boldsymbol{j}(x, y, z, t)$ cannot be defined to extended objects. The reason is that, if these quantities are measured in S in the same instant of time, say t_0, then the mass density $\rho''(x'', y'', z'', t'')$ and the current density $\boldsymbol{j}''(x'', y'', z'', t'')$ will be led to volume elements *in different instants of time*, but when a given observer in S'' measures these densities, he always do it *in the same instant of time* t''_0, therefore it is not clear at first that he would get the same result as given by (39). Fortunately, he indeed gets the same result, as was proven by Podolsky [74] for the analogous electromagnetic case.

6 Transformation laws for the fields

In sect. 3 we have shown that any massive moving body must present both a gravitational as a magnetic field. Now in this section we shall deduce the transformation laws for these gravitomagnetic fields from an inertial reference frame to another.

To this end, consider that in the reference frame S there exists a moving distribution of matter which creates both a gravitational field $\boldsymbol{G}(x, y, z, t)$ as a magnetic field $\boldsymbol{B}(x, y, z, t)$. If a particle of mass m moves in this gravitomagnetic field, say with an instantaneous velocity $\boldsymbol{u}$ measured at $t = 0$, then it will be subject to a gravitomagnetic force that is given by the Lorentz formula (22), as we showed in sect. 3. In terms of its components, the total force acting on the particle will be, therefore,

$$F_x = mG_x + m\left(u_y B_z - u_z B_y\right), \qquad F_y = mG_y + m\left(u_z B_x - u_x B_z\right), \qquad F_z = mG_z + m\left(u_x B_y - u_y B_x\right). \tag{40}$$

Now we may ask what will be the total force acting on this particle for another reference frame S'', where the (still undetermined) gravitomagnetic fields are $\boldsymbol{G}''(x'', y'', z'', t'')$ and $\boldsymbol{B}''(x'', y'', z'', t'')$, respectively, and where the particle moves with the instantaneous velocity $\boldsymbol{u}''$. The expressions for the force components in S'' can be found through the inverse of the force transformation laws (19), namely,

$$F''_x = F_x - \frac{\gamma v}{c^2}\left(u''_y F_y + u''_z F_z\right), \qquad F''_y = \gamma\left(1 + \frac{u''_x v}{c^2}\right) F_y, \qquad F''_z = \gamma\left(1 + \frac{u''_x v}{c^2}\right) F_z, \tag{41}$$

where we omitted dependence of the forces on x, y, z, t, etc., for short. In fact, replacing the components F_x, F_y and F_z in this equation by their respective expressions given by (40) and using (18) to eliminate the velocities u_x, u_y and u_z, we get, after simplification, the expressions

$$\begin{aligned} F''_x &= mG_x + \gamma m u''_y\left(B_z - \frac{v}{c^2}G_y\right) + \gamma m u''_z\left(B_y + \frac{v}{c^2}G_z\right),\\ F''_y &= \gamma m\left(G_y - vB_z\right) + m u''_z B_x - \gamma m u''_x\left(B_z - \frac{v}{c^2}G_y\right),\\ F''_z &= \gamma m\left(G_z + vB_y\right) + \gamma m u''_x\left(B_y + \frac{v}{c^2}G_z\right) - m u''_y B_x. \end{aligned} \tag{42}$$

Now, we may realize that this force can also be written as a gravitational force plus a magnetic force. In fact, the gravitational force is defined as the part of the total force that does not depend on the particle velocity $\boldsymbol{u}''$, and it is given by

$$F^{G''}_x = mG_x, \qquad F^{G''}_y = m\gamma\left(G_y - vB_z\right), \qquad F^{G''}_z = m\gamma\left(G_z + vB_y\right), \tag{43}$$

while the magnetic force is defined as the part that does depend on the particle velocity and, hence, it is given by

$$\begin{aligned} F^{B''}_x &= m\left[u''_y\gamma\left(B_z - \frac{v}{c^2}G_y\right) + u''_z\gamma\left(B_y + \frac{v}{c^2}G_z\right)\right],\\ F^{B''}_y &= m\left[u''_z B_x - u''_x\gamma\left(B_z - \frac{v}{c^2}G_y\right)\right],\\ F^{B''}_z &= m\left[u''_x\gamma\left(B_y + \frac{v}{c^2}G_z\right) - u''_y B_x\right]. \end{aligned} \tag{44}$$

Therefore, we conclude that in the reference frame S'' the total force can also be written in the Lorentz form as

$$\boldsymbol{F}'' = m\left(\boldsymbol{G}'' + \boldsymbol{u}'' \times \boldsymbol{B}''\right), \tag{45}$$

provided that the gravitomagnetic fields $\boldsymbol{G}''$ and $\boldsymbol{B}''$, as measured by S'', are related to the gravitomagnetic fields $\boldsymbol{G}$ and $\boldsymbol{B}$, as measured by S, by the formulæ,

$$G''_x = G_x, \qquad G''_y = \gamma\left(G_y - vB_z\right), \qquad G''_z = \gamma\left(G_z + vB_y\right) \tag{46}$$

and

$$B''_x = B_x, \qquad B''_y = \gamma\left(B_y + \frac{v}{c^2}G_z\right), \qquad B''_z = \gamma\left(B_z - \frac{v}{c^2}G_y\right). \tag{47}$$

In this way we have proved that the Lorentz force holds for any inertial reference frame, a result that was indeed anticipated at sect. 3. At the same time, we have found the transformation laws for the gravitomagnetic fields, expressed through (46) and (47).

7 The gravitomagnetic Maxwell equations

In this section we shall concern ourselves with the differential equations governing the gravitomagnetic fields. As we have seen, a static distribution of mass (for instance, w.r.t. the reference frame S') generates only a static gravitational field $\boldsymbol{G}'$. In the approximation considered here, where the gravitational force can be interpreted as an ordinary force field, the universal Newton's law of gravitation holds, from which follows that the gravitational field $\boldsymbol{G}'$ enjoys the following properties:

1) $\boldsymbol{G}'$ satisfies the Gauss law,

$$\boldsymbol{\nabla}' \cdot \boldsymbol{G}' = -4\pi g \rho'; \tag{48}$$

2) $\boldsymbol{G}'$ is an irrotational field,

$$\boldsymbol{\nabla}' \times \boldsymbol{G}' = 0; \tag{49}$$

3) $\boldsymbol{G}'$ is independent of time,

$$\partial'_t \boldsymbol{G}' = 0. \tag{50}$$

From these properties and the Lorentz transformations we can find what should be the differential equations describing the gravitomagnetic fields in the reference frame S. For this end, remember that in S there exist both a gravitational field $\boldsymbol{G}$ as a magnetic field $\boldsymbol{B}$ and that the magnetic field is related to the gravitational field by formula (24), that is, $\boldsymbol{B} = \boldsymbol{v}/c^2 \times \boldsymbol{G}$. Hence, to find the differential equations satisfied by the gravitomagnetic fields, we need to express the derivatives ∂'_x, ∂'_y, ∂'_z and ∂'_t in terms of ∂_x, ∂_y, ∂_z and ∂_t. We easily get the relations necessary:

$$\partial'_x = \gamma \left(\partial_x + \frac{v}{c^2}\partial_t\right), \qquad \partial'_y = \partial_y, \qquad \partial'_z = \partial_z, \qquad \partial'_t = \gamma\left(\partial_t + v\partial_x\right). \tag{51}$$

We shall need further the transformations formulæ for the gravitational field given at (46), which in this case reduce to the equations

$$G'_x = G_x, \qquad G'_y = G_y/\gamma, \qquad G'_z = G_z/\gamma, \tag{52}$$

since the magnetic field is null in S'. Now, inserting the relations (51) and (52) in (50), we get

$$\partial_x G_x + v\partial_t G_x = 0, \qquad \partial_x G_y + v\partial_t G_y = 0, \qquad \partial_x G_z + v\partial_t G_z = 0, \tag{53}$$

which will be of importance in the what follows.

Therefore, let us begin analyzing what should be the divergence of the gravitational field in the reference frame S. Replacing the coordinate derivatives present in (48) by the relations (51) we get, at once,

$$\begin{aligned}
\partial'_x G'_x + \partial'_y G'_y + \partial'_z G'_z &= -4\pi g\rho', \\
\gamma\left(\partial_x + \frac{v}{c^2}\partial_t\right) G_x + \partial_y\left(\frac{G_y}{\gamma}\right) + \partial_z\left(\frac{G_z}{\gamma}\right) &= -4\pi g\rho', \\
\gamma\left(1 - \frac{v^2}{c^2}\right)\partial_x G_x + \partial_y\left(\frac{G_y}{\gamma}\right) + \partial_z\left(\frac{G_z}{\gamma}\right) &= -4\pi g\rho', \\
\partial_x G_x + \partial_y G_y + \partial_z G_z &= -4\pi g\rho'\gamma, \\
\partial_x G_x + \partial_y G_y + \partial_z G_z &= -4\pi g\rho,
\end{aligned} \tag{54}$$

where we have used the first of relations (53) and the transformation law for the gravitational mass density (32) as well. We found therefore that the gravitational field also satisfies the Gauss law in S, namely,

$$\boldsymbol{\nabla} \cdot \boldsymbol{G} = -4\pi g\rho. \tag{55}$$

Now, let us verify what should be the curl of the gravitational field. Replacing the coordinate derivatives given in (51) in (49) and using (52) we get, for each component,

$$\begin{aligned}
\partial'_y G'_z - \partial'_z G'_y &= \partial_y\left(\frac{G_z}{\gamma}\right) - \partial_y\left(\frac{G_y}{\gamma}\right) = 0, \\
\partial'_z G'_x - \partial'_x G'_z &= \partial_z G_x - \gamma\left(\partial_x + \frac{v}{c^2}\partial_t\right)\left(\frac{G_z}{\gamma}\right) = 0, \\
\partial'_x G'_y - \partial'_y G'_x &= \gamma\left(\partial_x + \frac{v}{c^2}\partial_t\right)\left(\frac{G_y}{\gamma}\right) - \partial_y G_x = 0,
\end{aligned} \tag{56}$$

that is,

$$\partial_y G_z - \partial_z G_y = 0, \qquad \partial_z G_x - \partial_x G_z = \frac{v}{c^2}\partial_t G_z, \qquad \partial_x G_y - \partial_y G_x = -\frac{v}{c^2}\partial_t G_y. \tag{57}$$

Now, remembering that the magnetic field is given in S by (24), we get

$$B_x = 0, \qquad B_y = -\frac{v}{c^2}G_z, \qquad B_z = \frac{v}{c^2}G_y, \tag{58}$$

from which we conclude that

$$\boldsymbol{\nabla} \times \boldsymbol{G} = -\partial_t \boldsymbol{B}. \tag{59}$$

Notice that the gravitational field is no longer irrotational in S, which express the content of the gravitational analogue of Faraday's law. Since $\boldsymbol{\nabla} \times \boldsymbol{G} \neq 0$, it might seem that the gravitational field is not conservative anymore. This is only apparent, however, since we shall see, in sects. 9 and 10, that the gravitational fields have, besides an energy, also a momentum associated, so that the *spacetime momentum* of the fields is conserved in any inertial reference frame.

Together with (55), eq. (59) completes the set of differential equations for the gravitational field $\boldsymbol{G}$ as measured by S. Still remains, however, to find the divergence and the curl of the magnetic field $\boldsymbol{B}$. This is quite simple to be found. In fact, since the magnetic field components are given by (58), we get, at once, that

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = \partial_y \left(-\frac{v}{c^2} G_z\right) + \partial_z \left(\frac{v}{c^2} G_y\right) = -\frac{v}{c^2} \left(\partial_y G_z - \partial_z G_y\right) = 0, \tag{60}$$

where we made use of the first equation in (57). Therefore, we conclude that the divergence of the magnetic field is always zero[6],

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{61}$$

Finally, let us verify what should be the curl of the magnetic field. We have, at first, that

$$\begin{aligned}
\partial_y B_z - \partial_z B_y &= \partial_y \left(\frac{v}{c^2} G_y\right) - \partial_z \left(-\frac{v}{c^2} G_z\right) = \frac{v}{c^2} \left(\partial_y G_y + \partial_z G_z\right), \\
\partial_z B_x - \partial_x B_z &= -\partial_x \left(\frac{v}{c^2} G_y\right) = -\frac{v}{c^2} \partial_x G_y, \\
\partial_x B_y - \partial_y B_x &= \partial_x \left(-\frac{v}{c^2} G_z\right) = -\frac{v}{c^2} \partial_x G_z.
\end{aligned} \tag{62}$$

However, from the Gauss law we know that $\partial_y G_y + \partial_z G_z = -4\pi g\rho - \partial_x G_x$ and, hence, replacing the derivatives in x by the derivatives in t through (53), we get the equations,

$$\partial_y B_z - \partial_z B_y = -\frac{4\pi g}{c^2} \rho v + \partial_t G_x, \qquad \partial_z B_x - \partial_x B_z = -\frac{1}{c^2} \partial_t G_y, \qquad \partial_x B_y - \partial_y B_x = -\frac{1}{c^2} \partial_t G_z, \tag{63}$$

which can be written as well in a vector form as

$$\boldsymbol{\nabla} \times \boldsymbol{B} = -\frac{4\pi g}{c^2} \boldsymbol{j} + \frac{1}{c^2} \partial_t \boldsymbol{G}, \tag{64}$$

where $\boldsymbol{j} = \rho \boldsymbol{v} = \rho v \hat{\boldsymbol{x}}$ is the current density.

In conclusion, the differential equations satisfied by the gravitomagnetic fields in the reference frame S are

$$\boldsymbol{\nabla} \cdot \boldsymbol{G} = -4\pi g \rho, \tag{65}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{66}$$
$$\boldsymbol{\nabla} \times \boldsymbol{G} = -\partial_t \boldsymbol{B}, \tag{67}$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = -\frac{4\pi g}{c^2} \boldsymbol{j} + \frac{1}{c^2} \partial_t \boldsymbol{G}. \tag{68}$$

They have the same form of the Maxwell equations of electromagnetic theory, besides the signs in front of the sources ρ and $\boldsymbol{j}$. These negative signs, of course, is due to the fact that the gravitational interaction is always attractive.

In these equations, notice that the only things that matters are the fields $\boldsymbol{G}$ and $\boldsymbol{B}$ and the densities of mass and current, ρ and $\boldsymbol{j}$, all quantities being measured in S. However, since the velocity in which S moves w.r.t. S' is quite arbitrary, we may conclude that the Maxwell equations should maintain their form in any inertial frame of reference. This can be in fact proved in another way, by considering the reference frame S'', in the same fashion as was made in sect. 6. This calculation will be concealed.

This approach shows us that the Maxwell equations have nothing to do with the nature of the electromagnetic fields —on the contrary, we showed that they are just a consequence of the covariance enjoyed by the fields regarding the Lorentz transformations. Therefore, any other *covariant field* (*e.g.*, the electromagnetic field or any other else) must satisfy a similar set of equations. This point can also be evidenced in an interesting work of Heras [75], where is stated that the Maxwell equations can be deduced uniquely from the continuity equation[7]. Since we have not used the continuity equation in our derivation of the Maxwell equations, and since, on the other hand, the continuity equation is a direct consequence of the Maxwell equations, this means that both set of equations are, surprisingly, mathematically equivalent.

[6] Notice that this represents a strong evidence that classical magnetic monopoles should not exist, since our argument shows that the vanishing of the divergence of the magnetic field follows from a direct consequence of the special relativity theory. Besides, our definition of magnetic field (which agrees with the usual one) as the field that arises in order to balance the forces in all inertial reference frames makes non-sense when we talk about magnetic monopoles classically, since every field generated by such a particle would be simply interpreted as some kind of gravitational (electric) field generated by ordinary gravitational (electric) charges anyway.

[7] To be more specific, Heras stated that, if there are two localized functions $\phi(x,y,z,t)$ and $\boldsymbol{j}(x,y,z,t)$ that satisfy the continuity equation (37), then there exist two vector functions $\boldsymbol{F}(x,y,z,t)$ and $\boldsymbol{G}(x,y,z,t)$ that satisfy a set of equations similar to that of Maxwell. See [75] for details.

8 Gravitational waves

The gravitational Maxwell equations presented in the previous section consist of a coupled system of linear partial differential equations for the fields $\boldsymbol{G}$ and $\boldsymbol{B}$. This set of equations nevertheless can be written in a decoupled form as well. To see this, we can follow the standard methods employed in the electromagnetic theory. Thus, let us begin by taking the curl of (67):

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{G})=\boldsymbol{\nabla}\times(-\partial_t\boldsymbol{B}), \quad \text{that is,} \quad \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{G})-\boldsymbol{\nabla}^2\boldsymbol{G}-\partial_t(\boldsymbol{\nabla}\times\boldsymbol{B}), \tag{69}$$

where we had used the vector identities (B.3) and (B.4). Using now (65) and (68), we get, after simplification,

$$\boldsymbol{\nabla}^2\boldsymbol{G}-\frac{1}{c^2}\partial_t^2\boldsymbol{G}=-4\pi g\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\partial_t\boldsymbol{j}\right), \tag{70}$$

which is a decoupled differential equation for the field $\boldsymbol{G}$. Likewise, taking the curl of (68) and using the identities (B.3) and (B.4) again, we get

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=\boldsymbol{\nabla}\times\left(-\frac{4\pi g}{c^2}\boldsymbol{j}+\frac{1}{c^2}\partial_t\boldsymbol{G}\right), \quad \text{or} \quad \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\boldsymbol{\nabla}^2\boldsymbol{B}=-\frac{4\pi g}{c^2}(\boldsymbol{\nabla}\times\boldsymbol{j})+\frac{1}{c^2}\partial_t(\boldsymbol{\nabla}\times\boldsymbol{G}) \tag{71}$$

and, from the Maxwell equations (66) and (67), we obtain

$$\boldsymbol{\nabla}^2\boldsymbol{B}-\frac{1}{c^2}\partial_t^2\boldsymbol{B}=\frac{4\pi g}{c^2}(\boldsymbol{\nabla}\times\boldsymbol{j}), \tag{72}$$

which is the decoupled differential equation for the field $\boldsymbol{B}$.

Equations (70) and (72) are the gravitomagnetic *inhomogeneous wave equations*. In the absence of matter, that is, whenever $\rho=0$ and $\boldsymbol{j}=0$, (70) and (72) simplify to

$$\boldsymbol{\nabla}^2\boldsymbol{G}-\frac{1}{c^2}\partial_t^2\boldsymbol{G}=0, \qquad \boldsymbol{\nabla}^2\boldsymbol{B}-\frac{1}{c^2}\partial_t^2\boldsymbol{B}=0, \tag{73}$$

and we get the gravitomagnetic *homogeneous wave equations*.

We highlight that (73) shows us clearly that gravitational waves propagate with the speed of the light in vacuum. We remark that this result follows directly from the special theory of relativity and that this result cannot be derived by making a mere analogy between gravitation and electromagnetism, since, in this case, there is no way to guess what should be the gravitational permittivity and permeability of the vacuum. The recent detection of gravitational waves by the LIGO experiments [49–52] confirmed that gravitational waves indeed propagate with the speed of light, c, which represents a strong argument in favor of our present theory —and that also disproofs any other gravitomagnetic theory concluding in a different way.

9 The gravitomagnetic potentials and their connection with the interaction energy and momentum

When a given distribution of matter is at rest, it is very known that the gravitational field so created is conservative. In fact, this is ensured by the third Maxwell equation (67), which states that the gravitational field is irrotational whenever there is no magnetic field and, thus, it is possible to write the gravitational field $\boldsymbol{G}'$ as the gradient of a *scalar potential* function ϕ', for instance, as

$$\boldsymbol{G}'(x',y',z')=-\boldsymbol{\nabla}'\phi'(x',y',z'). \tag{74}$$

The physical meaning of the function ϕ' is easily found to be the *interaction energy* by unit mass between the particle and the gravitational field. In other words, a particle of mass m moving in a static gravitational field has an interaction energy given by

$$U'(x',y',z')=m\phi'(x',y',z'). \tag{75}$$

This picture changes, however, when the source of the gravitational field is moving. In this case the gravitational field cannot be conservative in the sense above, since the presence of the magnetic fields prevents the field from being irrotational, as (67) states. Nevertheless, we shall see in the following that in the frame S there exists as well a *vector potential* function $\boldsymbol{A}$, which is associated with the *interaction momentum* between the gravitomagnetic fields and the particle and that solves the problem. The scalar potential ϕ and the vector potential $\boldsymbol{A}$ form, in fact, a spacetime vector (the *spacetime potential*) which is proportional to the spacetime momentum associated with the interaction between the fields and the particle. This *spacetime interaction momentum* is a conserved quantity and hence, in this more embracing sense, the gravitomagnetic field can be regarded as conservative.

We can easily show the existence of the gravitomagnetic potentials ϕ and $\boldsymbol{A}$ directly from the homogeneous Maxwell equations (66) and (67) —that is, the Maxwell equations which do not depend on the sources ρ and $\boldsymbol{j}$. In fact, the divergenceless property of the magnetic field stated in the second Maxwell equation (66) means, through identity (B.1), that there exists a C^2 class vector function $\boldsymbol{A}$ satisfying the condition

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{76}$$

Likewise, replacing (76) into the third Maxwell equation (67) we shall get, after using the previous formula (76) and the identity (B.3),

$$\boldsymbol{\nabla} \times \boldsymbol{G} + \partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{G} + \partial_t (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla} \times (\boldsymbol{G} + \partial_t \boldsymbol{A}) = 0. \tag{77}$$

Therefore we can realize, through the identity (B.2), that there exists a C^2 scalar function ϕ, such that

$$\boldsymbol{G} + \partial_t \boldsymbol{A} = -\boldsymbol{\nabla}\phi, \quad \text{that is,} \quad \boldsymbol{G} = -\boldsymbol{\nabla}\phi - \partial_t \boldsymbol{A}. \tag{78}$$

The existence of the gravitomagnetic potentials ϕ and $\boldsymbol{A}$ given by (76) and (78) provides the required relationship between the gravitomagnetic fields with the potentials.

The gravitomagnetic potentials, however, are not uniquely determined, since we can replace ϕ and $\boldsymbol{A}$, respectively, by $\phi' = \phi + \partial_t f$ and $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla} f$, where f is any C^2 scalar function, without any change in the gravitomagnetic fields. This feature is often called the *gauge freedom* of the potentials. Hence the covariant theory of gravitation here outlined has gauge symmetry as the electromagnetic theory. This feature, which is also a consequence of relativity (and therefore should be present in any relativistic force field), can be of importance in the formulation of a quantum version of this theory.

Let us see, now, why the potentials ϕ and $\boldsymbol{A}$ should be associated with the interaction energy and momentum between a particle and the gravitomagnetic fields. To this end, consider first a particle that is moving in a static gravitational field, say w.r.t. a reference frame S'. The energy of the particle is composed of two terms: the kinetic energy $K' = \gamma_{u'} mc^2$ and the interaction energy $U' = m\phi'$ as given by (75). Moreover, the particle has a kinetic momentum given by $\boldsymbol{p}' = \gamma_{u'} m \boldsymbol{u}'$. Now, take the reference frame S where the distribution of matter which generated the fields has a velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. Then we may ask what is the energy and the momentum of the particle as it moves in the gravitomagnetic fields. The answer is again provided by the special theory of relativity, since the total energy and momentum of the particle should be the components of a spacetime vector. This means that the total energy E and the total momentum $\boldsymbol{P}$ of the particle, as measured by S, should be related to its total energy E' and momentum $\boldsymbol{P}' = \boldsymbol{p}'$, as measured in S', by the formulæ

$$E = \gamma E', \qquad P_x = \frac{\gamma v}{c^2} E', \qquad P_y = 0, \qquad P_z = 0. \tag{79}$$

However, since the same transformation formulæ holds for the kinetic energy K and momentum $\boldsymbol{p}$ of the particle, this means that the same also holds for the interaction energy U and the interaction momentum $\boldsymbol{V}$. Therefore, the simple fact that the particle has an interaction energy in S' given by (75) implies that it has both an interaction energy U and an interaction momentum $\boldsymbol{V}$ in S, which are, respectively, given by

$$U(x,y,z,t) = m\phi(x,y,z,t), \qquad \boldsymbol{V}(x,y,z,t) = m\boldsymbol{A}(x,y,z,t), \tag{80}$$

where the potentials $\phi(x,y,z,t)$ and $\boldsymbol{A}(x,y,z,t)$ are

$$\phi(x,y,z,t) = \gamma\phi'(x',y',z') \quad \text{and} \quad \boldsymbol{A}(x,y,z,t) = \frac{\gamma\boldsymbol{v}}{c^2}\phi'(x',y',z') = \frac{\boldsymbol{v}}{c^2}\phi(x,y,z,t). \tag{81}$$

(The Lorentz transformations still must be used in order to eliminate the primed coordinates, of course.) From this we conclude that the gravitomagnetic potentials are the components of a spacetime vector, as stated above, since they are proportional to the interaction energy and momentum. Hence, they must transform, from a reference frame S to another frame S'', according to the formulæ

$$\phi'' = \gamma(\phi - vA_x), \qquad A''_x = \gamma\left(A_x - \frac{v}{c^2}\phi\right), \qquad A''_y = A_y, \qquad A''_z = A_z. \tag{82}$$

The results derived above can also be deduced from a variational perspective. To this end, let us take the relativistic Lagrangian for a particle moving in a static gravitational field, which is

$$\mathcal{L}' = -mc^2/\gamma_{u'} - m\phi'. \tag{83}$$

In the Lagrangian formulation, remember that the canonical energy and momentum of the particle are given, respectively, by

$$\boldsymbol{P}' = \dot{\boldsymbol{\nabla}}'\mathcal{L}' = \gamma_{u'} m\boldsymbol{u}', \qquad H' = \dot{\boldsymbol{\nabla}}'\mathcal{L}' \cdot \boldsymbol{u}' - \mathcal{L}' = \gamma_{u'} mc^2 + m\phi', \tag{84}$$

from where the relation between the interaction energy and the scalar potential becomes plain: $U' = m\phi'$. The action is, of course, given by

$$S = \int \mathcal{L}'\mathrm{d}t' = \int \boldsymbol{P}' \cdot \mathrm{d}\boldsymbol{r}' - \int H'\mathrm{d}t'. \tag{85}$$

Now we pass to the reference frame S, where the gravitational field is moving, and we ask what happens there. The key point here is to realize that the action is Lorentz invariant. This means that the canonical energy and the canonical momentum components must transform as a spacetime vector. Since the same holds for the kinetic energy and momentum, we found again that the same must be true for the interaction energy and momentum. In fact, it is not difficult to show that the canonical momentum and energy of the particle in the reference frame S, are given, respectively, by

$$\boldsymbol{P} = \gamma_u m\boldsymbol{u} + m\boldsymbol{A}, \qquad H = \gamma_u mc^2 + m\phi, \tag{86}$$

from where (80) and (81) directly follows. The relationship between the fields and the potentials can also be recovered from this variational approach [66]. Now we need the equation of motion of the particle in the reference frame S, which is provided by the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\boldsymbol{\nabla}}\mathcal{L} = \boldsymbol{\nabla}\mathcal{L}, \tag{87}$$

with the transformed Lagrangian

$$\mathcal{L} = -mc^2/\gamma_u - m\phi + m\left(\boldsymbol{A} \cdot \boldsymbol{u}\right). \tag{88}$$

Notice, however, that the force acting of the particle is given by the time derivative of its kinetic momentum $\boldsymbol{p}$, not by the time derivative of the canonical momentum $\boldsymbol{P}$. Thus, the Euler-Lagrange equations provide the force only indirectly. Nonetheless, it is easy to verify from (86) that

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} - m\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}. \tag{89}$$

Therefore, form the Lagrangian (88), we get that

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{\nabla}\mathcal{L} = -m\boldsymbol{\nabla}\phi + m\boldsymbol{\nabla}\left(\boldsymbol{A} \cdot \boldsymbol{u}\right) \tag{90}$$

and, remembering that the derivatives are carried out for constant $\boldsymbol{u}$, we get, after using the vector identity (B.8),

$$\boldsymbol{\nabla}\mathcal{L} = -m\boldsymbol{\nabla}\phi + m\left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\boldsymbol{A} + m\boldsymbol{u} \times \left(\boldsymbol{\nabla} \times \boldsymbol{A}\right). \tag{91}$$

On the other hand, the derivative $\mathrm{d}\boldsymbol{A}/\mathrm{d}t$ is

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \partial_t\boldsymbol{A} + \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\boldsymbol{A} \tag{92}$$

and, hence, inserting (91) and (92) on (89), we find that the force acting on the particle is

$$\boldsymbol{F} = m\left[-\left(\boldsymbol{\nabla}\phi + \partial_t\boldsymbol{A}\right) + \boldsymbol{u} \times \left(\boldsymbol{\nabla} \times \boldsymbol{A}\right)\right]. \tag{93}$$

Now, since, by definition, the gravitational force is the part of the gravitomagnetic force which does not depend on the particle velocity, while the magnetic force is that part which does depend, we conclude that the total force acting on the particle in the reference frame S can be written in the Lorentz form as

$$\boldsymbol{F} = m\left(\boldsymbol{G} + \boldsymbol{u} \times \boldsymbol{B}\right), \tag{94}$$

where the gravitational and magnetic fields, $\boldsymbol{G}$ and $\boldsymbol{B}$, are related to the scalar and vector potentials, ϕ and $\boldsymbol{A}$, respectively, by

$$\boldsymbol{G} = -\boldsymbol{\nabla}\phi - \partial_t\boldsymbol{A}, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{95}$$

This variational approach provides another deduction of (76) and (78) and of the results obtained in sect. 3.

Before closing this section, we would like to make some remarks about the association of the vector potential $\boldsymbol{A}$ with the interaction momentum. In the case of the electromagnetic theory, this association is by no means new and the first connections remount back to the original works of Maxwell [2], although this connection was getting lost in the course of time [76, 77]. Even today, this subject is controversial and many physicists believe that no physical meaning can be associated with the vector potential —their criticisms are commonly based on the fact that the vector potential has a gauge freedom, while it is hard to explain this feature in terms of the momentum. This argument, however, is flawed. There is no doubt that the kinetic momentum of a particle does not allow a gauge freedom, since it is only a function of its velocity, however, the same could be said about the kinetic energy. For the interaction energy, on the other hand, we can set the ground state as we please and since the interaction energy is proportional to the scalar potential, we have a gauge freedom as well. Therefore, the same arguments can be applied to interaction momentum, and there is no reason to prevent the vector potential of being associated with the interaction momentum. Moreover, the theory of relativity sets momentum and energy on an equal foot, whence this association is mandatory instead of optional. Finally, we should remember that in quantum mechanics this association is important to explain, for instance, the Aharonov-Bohm effect [78, 79].

10 The energy and the momentum stored in the gravitomagnetic fields

We have seen in the previous section that a particle moving in a non-static gravitational field has both an interaction energy as well as an interaction momentum —these quantities are directly related to the gravitomagnetic potentials ϕ and A through (80) and (81). In this section, we shall show that the fields themselves store energy and momentum. The energy and momentum contained in the gravitomagnetic fields can be regarded as distributed throughout the space, thus, we can talk about the energy and momentum densities as well as the energy and momentum fluxes. In the one hand, we shall see that, if there are no mass or current in a given region of space, then the energy and the momentum of the fields will be conserved there, so that they will satisfy a kind of continuity equation. On the other hand, if matter is present in the space, then we shall see that the fields must transmit energy and momentum to the matter, so that the continuity equation will be replaced by an appropriated balance equation.

Let us begin our analysis by deducing what should be the energy stored in the gravitomagnetic fields. This can be made in the following mathematical way: first we dot the third Maxwell equation (67) with $\boldsymbol{G}$ and then the fourth Maxwell equation (68) with $\boldsymbol{B}$,

$$\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G}) = -\boldsymbol{B}\cdot\partial_t\boldsymbol{B}, \qquad \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = -\frac{4\pi g}{c^2}\boldsymbol{G}\cdot\boldsymbol{j} + \frac{1}{c^2}\boldsymbol{G}\cdot\partial_t\boldsymbol{G}. \tag{96}$$

Taking the difference of these two equations and using the vector identity (B.5), we obtain

$$\boldsymbol{\nabla}\cdot(\boldsymbol{G}\times\boldsymbol{B}) = \frac{4\pi g}{c^2}\boldsymbol{G}\cdot\boldsymbol{j} - \frac{1}{c^2}\boldsymbol{G}\cdot\partial_t\boldsymbol{G} - \boldsymbol{B}\cdot\partial_t\boldsymbol{B} \tag{97}$$

and, dividing it by $4\pi g/c^2$, using the identity (B.6), and rearranging, we get

$$\boldsymbol{\nabla}\cdot\left[\frac{c^2}{4\pi g}(\boldsymbol{G}\times\boldsymbol{B})\right] + \partial_t\left[\frac{1}{8\pi g}\left(\boldsymbol{G}\cdot\boldsymbol{G} + c^2\boldsymbol{B}\cdot\boldsymbol{B}\right)\right] = \boldsymbol{G}\cdot\boldsymbol{j}. \tag{98}$$

This equation represents the time rate per unit volume in which the gravitomagnetic fields deliver energy to the matter. In fact, by introducing the quantities

$$\boldsymbol{S} = -\frac{c^2}{4\pi g}(\boldsymbol{G}\times\boldsymbol{B}) \quad \text{and} \quad \mathcal{E} = -\frac{1}{8\pi g}\left(\boldsymbol{G}\cdot\boldsymbol{G} + c^2\boldsymbol{B}\cdot\boldsymbol{B}\right), \tag{99}$$

we may realize that (98) can be rewritten as

$$\boldsymbol{\nabla}\cdot\boldsymbol{S} + \partial_t\mathcal{E} = -\boldsymbol{G}\cdot\boldsymbol{j}. \tag{100}$$

The quantity $-\boldsymbol{G}\cdot\boldsymbol{j}$, however, is nothing but the power (per unit volume) exerted by the fields on the matter —the minus sign means that the fields lost energy in this process. Consequently, the quantity $\mathcal{E}$ represents the energy density of the gravitational field and the vector $\boldsymbol{S}$ consists in the flux of this energy (this vector is the gravitational analogue of the Poynting vector found in the electromagnetic theory). Equation (98) —or its equivalent, eq. (100)— states that in the presence of matter the energy of the gravitational fields, alone, is not conserved anymore. Only the total energy, that is, the energy of the fields plus the energy of the matter, is conserved.

Furthermore, we should remark that the energy of the gravitational fields is always negative —contrasting with the energy of the electromagnetic fields, which is always positive. This feature is sometimes subject to some criticism, since it is not clear at the first sight what a field with negative energy means. However, we argue here that this feature is not a problem at all and, on the contrary, that the negativeness of the gravitational energy is ultimately a consequence of the gravity being always attractive. To see why, take the simple example of a test particle initially at rest on the gravitational field of a spherical body. As the particle begins to move due to the gravitational interaction, it draws energy from the gravitational field, which is converted into kinetic energy (an always positive quantity). Hence, since the total energy of the particle should be conserved, its potential energy must decrease in this process. But for the potential energy to decrease while there is an attraction between the two bodies, it is necessary that this potential energy be negative, if we agree to set the zero of energy at infinity. Therefore, the negativeness of the gravitational field is only a manifestation of the attractive character of gravity. This feature seems to be also important to explain why a quantum gravity theory can be formulated, in the approximation of weak fields, by the exchange of spin-1 particles —see sect. 13 for more details.

Let us now concern ourselves with the momentum stored in the gravitomagnetic fields. Here we can proceed by crossing (from the left) the third Maxwell equation (67) with $\boldsymbol{G}$ and then crossing (from the right) the fourth Maxwell equation (68) with $\boldsymbol{B}$ in order to get

$$\boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) = -\boldsymbol{G}\times\partial_t\boldsymbol{B}, \qquad (\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{4\pi g}{c^2}\,(\boldsymbol{j}\times\boldsymbol{B}) + \frac{1}{c^2}\,(\partial_t\boldsymbol{G}\times\boldsymbol{B})\,. \tag{101}$$

Thus, multiplying the second equation above by $-c^2$ and summing with the first, we get, after we use the anti-commuting property of the vector product and the identity (B.6)

$$\boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + c^2\,[\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B})] = 4\pi g\,(\boldsymbol{j}\times\boldsymbol{B}) - \partial_t\,(\boldsymbol{G}\times\boldsymbol{B})\,. \tag{102}$$

Besides, from the vector identity (B.7), we can rewrite this as

$$\partial_t\,(\boldsymbol{G}\times\boldsymbol{B}) + \frac{1}{2}\boldsymbol{\nabla}\left[\boldsymbol{G}\cdot\boldsymbol{G} + c^2\boldsymbol{B}\cdot\boldsymbol{B}\right] - \left[(\boldsymbol{G}\cdot\boldsymbol{\nabla})\,\boldsymbol{G} + c^2\,(\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{B}\right] = 4\pi g\,(\boldsymbol{j}\times\boldsymbol{B}) \tag{103}$$

and, finally, from the tensor identity (B.9), we get

$$\partial_t\,(\boldsymbol{G}\times\boldsymbol{B}) + \frac{1}{2}\boldsymbol{\nabla}\left[\boldsymbol{G}\cdot\boldsymbol{G} + c^2\boldsymbol{B}\cdot\boldsymbol{B}\right] - \left[\boldsymbol{\nabla}\cdot(\boldsymbol{G}\otimes\boldsymbol{G}) + c^2\boldsymbol{\nabla}\cdot(\boldsymbol{B}\otimes\boldsymbol{B})\right] = 4\pi g\,(\rho\boldsymbol{G} + \boldsymbol{j}\times\boldsymbol{B})\,, \tag{104}$$

where we used the first and second Maxwell equations (65) and (66) as well. Equation (104) can also be written in a compact form as

$$\partial_t\left(\boldsymbol{S}/c^2\right) + \boldsymbol{\nabla}\cdot\mathsf{M} = -\,(\rho\boldsymbol{G} + \boldsymbol{j}\times\boldsymbol{B})\,, \tag{105}$$

by invoking the vector $\boldsymbol{S}$, defined by (99) and introducing the *gravitational Maxwell stress-tensor*,

$$\mathsf{M} = -\frac{1}{8\pi g}\left[\left(\boldsymbol{G}\cdot\boldsymbol{G} + c^2\boldsymbol{B}\cdot\boldsymbol{B}\right)\mathsf{I} - 2\left(\boldsymbol{G}\otimes\boldsymbol{G} + c^2\boldsymbol{B}\otimes\boldsymbol{B}\right)\right], \tag{106}$$

where I is the identity tensor. Equation (104) —or its equivalent, eq. (105)— represents the time rate per unit volume in which the fields delivers momentum to the matter. In fact, the quantity $-(\rho\boldsymbol{G}+\boldsymbol{j}\times\boldsymbol{B})$ is nothing but the opposite of the Lorentz force per unit volume, $\boldsymbol{F} = \rho\boldsymbol{G} + \boldsymbol{j}\times\boldsymbol{B}$. Therefore, in the presence of matter, the momentum of the fields is no longer conserved —only the total momentum, that is, the momentum of the fields plus the momentum of the matter, is conserved.

Notice further that the vector $\boldsymbol{S}$ plays a dual role in those formulæ: in (100) it represents the flux of the gravitational energy, while $\boldsymbol{S}/c^2$ in the formula (105) represents the momentum density of the fields. This symmetry is not a matter of coincidence: is a consequence of special relativity that the flux of energy and the momentum density are related in this way.

11 A manifestly covariant approach

In this section we shall present another derivation of the previous results, but following now a spacetime perspective. This means that all the results and equations will be written in a tensor form, that is, in a manifestly covariant fashion.

Let us begin by showing the existence of a magnetic gravitational force directly from this spacetime approach. The key point here is to replace the concept of ordinary force $\boldsymbol{F}$ by that one of spacetime force Γ^μ. Before doing so we should highlight, however, that in a pure spacetime approach there is no clear separation between gravitational and magnetic forces. In fact, this separation is ultimately a matter of convention —it was introduced first in the electromagnetic theory mainly due to historical reasons, namely, due to the chronological order on which the electric and magnetic phenomena were discovered. By this reason, it will be necessary at the end of our calculations to return to the concept of ordinary force in order to make the separation of the gravitomagnetic forces into gravitational and magnetic ones.

In the spacetime framework of special relativity, the spacetime force Γ^μ is defined as the derivative of the spacetime momentum p^μ w.r.t. the proper time τ,

$$\Gamma^\mu = \mathrm{d}p^\mu/\mathrm{d}\tau, \tag{107}$$

as well as in terms of the spacetime acceleration [58,60,61,66,67],

$$\Gamma^\mu = ma^\mu. \tag{108}$$

Once fixed a reference frame S, the spacetime force components are related to the power and ordinary force through the relations

$$\Gamma^0 = \gamma_u W/c, \qquad \Gamma^1 = \gamma_u F_x, \qquad \Gamma^2 = \gamma_u F_y, \qquad \Gamma^3 = \gamma_u F_z. \tag{109}$$

Now, let us consider a static gravitational force acting on a given particle, as measured in the proper frame S' of the gravitational source. Then the components of the spacetime force are given, in this reference frame, by

$$\Gamma'^0 = \gamma_{u'} W'/c, \qquad \Gamma'^1 = \gamma_{u'} F'_x, \qquad \Gamma'^2 = \gamma_{u'} F'_y, \qquad \Gamma'^3 = \gamma_{u'} F'_z, \tag{110}$$

where W', F'_x, F'_y and F'_z do not depend on the particle velocity $\boldsymbol{u}'$, since the force is purely gravitational in S'.

The spacetime force, in contrast to the usual force, is a spacetime vector, whence their components transform, from S' to S, according to

$$\Gamma^0 = \gamma_v\left(\Gamma'^0 + \frac{v}{c}\Gamma'^1\right), \qquad \Gamma^1 = \gamma_v\left(\Gamma'^1 + \frac{v}{c}\Gamma'^0\right), \qquad \Gamma^2 = \Gamma'^2, \qquad \Gamma^3 = \Gamma'^3. \tag{111}$$

From this we can verify that the power and the usual force components, as measured by S, are related to the power and the respective components of the force in the reference frame S' by

$$W = \frac{\gamma_v\gamma_{u'}}{\gamma_u}\left(W' + vF'_x\right), \qquad F_x = \frac{\gamma_v\gamma_{u'}}{\gamma_u}\left(F'_x + \frac{vW'}{c^2}\right), \qquad F_y = \frac{\gamma_{u'}}{\gamma_u}F'_y, \qquad F_z = \frac{\gamma_{u'}}{\gamma_u}F'_z. \tag{112}$$

However, it can be deduced from (18) that the γ-factors are related by the formula

$$\frac{\gamma_{u'}}{\gamma_u} = \gamma_v\left(1 - \frac{u_x v}{c^2}\right), \tag{113}$$

from which we get, after simplification, the expressions

$$\begin{aligned}
W &= \gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)\left(W' + vF'_x\right),\\
F_x &= \gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)\left(F'_x + \frac{vW'}{c^2}\right) = F'_x + \frac{\gamma_v v}{c^2}\left(F'_y u_y + F'_z u_z\right),\\
F_y &= \gamma_v\left(1 - \frac{u_x v}{c^2}\right)F'_y,\\
F_z &= \gamma_v\left(1 - \frac{u_x v}{c^2}\right)F'_z,
\end{aligned} \tag{114}$$

where we used, in the second equation, the identity $W' = \boldsymbol{F}'\cdot\boldsymbol{u}'$ and also the transformation formulæ for the velocity components. Thus we recovered the force transformations (19) presented in sect. 3, which enable us finally to separate the gravitomagnetic force into the gravitational and magnetic parts, leading us again to the formulæ (20) and (21).

All other results can be expressed in a manifestly covariant way as well. The mass and current densities, in this spacetime description, compose into a *spacetime current density*,

$$J^\mu = \rho_0 w^\mu, \tag{115}$$

where ρ_0 is the proper mass density and $w^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the spacetime velocity. The time component of this spacetime vector is proportional to the mass density: $J^0 = \rho_0\gamma c = \rho c$, while its space components are related to the current density: $J^a = \rho_0\gamma u^a = \rho u^a = j^a$. We can also verify that the continuity equation (37) is written just as a spacetime divergence,

$$\partial_\mu J^\mu = 0. \tag{116}$$

In sect. 9, we have seen that a particle moving in a gravitomagnetic field has both an interaction energy as an interaction momentum. These two quantities can also be unified into a spacetime vector, namely, the *spacetime interaction momentum*,

$$V^\mu = mA^\mu. \tag{117}$$

This means that the scalar potential ϕ and the vector potential $\boldsymbol{A}$ also form a spacetime vector: the *spacetime potential* A^μ, whose components are

$$A^0 = \phi/c, \qquad A^1 = A_x, \qquad A^2 = A_y, \qquad A^3 = A_z. \tag{118}$$

From this we can plainly see that the vector potential is associated with the interaction momentum as the scalar potential is associated with the interaction energy.

Moreover, it follows that the gravitomagnetic fields can be found through the derivatives of the spacetime potential A^μ w.r.t. the spacetime coordinates,

$$G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{119}$$

Therefore, the gravitomagnetic fields components form a two-rank anti-symmetric spacetime tensor, the *spacetime gravitomagnetic field* $G^{\mu\nu}$. In a matrix representation this tensor can be written as

$$G^{\mu\nu} = \begin{pmatrix} 0 & -G_x/c & -G_y/c & -G_z/c \\ G_x/c & 0 & -B_z & B_y \\ G_y/c & B_z & 0 & -B_x \\ G_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{120}$$

The *spacetime Lorentz force* can also be expressed in terms of the gravitomagnetic tensor as follows:

$$\Gamma^\mu = m w_\nu G^{\mu\nu}. \tag{121}$$

Notice that the time component of the spacetime Lorentz force is proportional to the power delivered by the gravitomagnetic fields to the matter, while its space components are proportional to the usual Lorentz force. Notice further that, since $\Gamma^\mu = m a^\mu$, the motion of a particle in a gravitomagnetic field does not depend on its mass —this property is the content of Einstein's equivalence principle and it is one of the most important features of the gravitomagnetic interaction.

Continuing with the manifestly covariant description, we can also verify that the four gravitational Maxwell equations become given by just two tensor equations: the inhomogeneous Maxwell equations (65) and (68) can be written as

$$\partial_\mu G^{\mu\nu} = -\frac{4\pi}{c^2} J^\nu, \tag{122}$$

while the homogeneous Maxwell equations (66) and (67), become given by

$$\partial_\alpha G_{\beta\gamma} + \partial_\beta G_{\gamma\alpha} + \partial_\gamma G_{\alpha\beta} = 0. \tag{123}$$

The balance equations for the energy and momentum stored in the fields, expressed by (100) and (105), can also be written as in a simple tensor form as

$$\partial_\nu T^{\mu\nu} = -J_\nu G^{\mu\nu} = -f^\mu, \tag{124}$$

where $f^\mu = \rho w_\nu G^{\mu\nu}$ is the density of the spacetime Lorentz force acting on the matter and

$$T^{\mu\nu} = \frac{c^2}{4\pi g}\left(\eta_{\alpha\beta} G^{\mu\alpha} G^{\nu\beta} - \frac{1}{4}\eta^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta}\right), \tag{125}$$

is the *spacetime momentum-flux* of the field, so that (124) represents the transfer of the spacetime momentum from the fields to the matter. When expressed in a matrix form, $T^{\mu\nu}$ becomes

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & S^1/c & S^2/c & S^3/c \\ S^1/c & M^{11} & M^{12} & M^{13} \\ S^2/c & M^{21} & M^{22} & M^{23} \\ S^3/c & M^{31} & M^{32} & M^{33} \end{pmatrix}, \tag{126}$$

from which we plainly see that $T^{\mu\nu}$ is a two-rank symmetric tensor. Notice further that we can introduce a spacetime momentum flux associated with the matter through

$$t^{\mu\nu} = \rho w^\mu w^\nu. \tag{127}$$

From this we see that the spacetime force acting on the matter can be obtained through derivatives:

$$\partial_\nu t^{\mu\nu} = \partial_\nu \left(\rho w^\mu w^\nu\right) = w^\nu \partial_\nu \left(\rho w^\mu\right) = \frac{\mathrm{d}\pi^\mu}{\mathrm{d}\tau} = f^\mu, \tag{128}$$

where π^μ are the components of the spacetime momentum density and we used the identity $\mathrm{d}/\mathrm{d}\tau = w^\mu \partial_\mu$. Substituting this into (124), we get promptly that

$$\partial_\mu \left(T^{\mu\nu} + t^{\mu\nu}\right) = 0, \tag{129}$$

which plainly shows the conservation law for the total energy and momentum of the system —fields plus matter.

Finally, we would like to remark that the analysis above allow us to make a point about the meaning of the spacetime momentum-flux tensor $\mathcal{T}^{\mu\nu}$ that is present in the Einstein field equations,

$$\mathcal{G}^{\mu\nu} = \kappa \mathcal{T}^{\mu\nu}. \tag{130}$$

This quantity $\mathcal{T}^{\mu\nu}$ should not be regarded as the spacetime momentum-flux tensor of the *gravitational field* itself —as one might think at first sight— but, on the contrary, $\mathcal{T}^{\mu\nu}$ should be associated with the spacetime momentum flux of the *matter* that generates the field. The reason is that in general relativity gravity is not regarded as a force anymore, but it is only a consequence of the curvature of the spacetime. Therefore, there can be no real transfer of momentum and energy from the gravitational fields to the matter and the momentum-flux tensor of the gravitomagnetic field alone must be trivially null (this is also in accordance with the fact that in general relativity $T^{\mu\nu}$ is considered null in the regions where there is no matter). This interpretation means that, in the last instance, the source of the gravitational fields —in both the special as in the general theory of relativity— should be regarded as the matter and that the gravitational magnetic effects are due to the motion of the matter[8] —see also the footnote[5]. Now, we can raise the following question: Why is there a non-null spacetime momentum-flux tensor in the present theory (which is an approximation of general relativity) if it is identically null in general relativity? The answer to this question is a conceptual matter. In general relativity the gravitational momentum flux $T^{\mu\nu}$ is null but the spacetime is curved; when we pass to a description where the spacetime is regarded as flat, the effects of the curvature of the spacetime become described by actions of force fields and, in the same way, the gravitational field acquire a proper energy and momentum in this interpretation. In other words, we might say that, for matter curve the spacetime some energy should be spent: all the energy stored in the gravitational field (according to a flat spacetime perspective), is spent on the bending of the spacetime, so that we end with a curved spacetime description where, now, gravity is no longer a force field and there is no gravitational proper energy associated with it anymore.

12 Differences and similarities between gravity and electricity

Finally, we shall comment some differences between gravity and electricity which evince their different nature. We have seen that the special theory of relativity requires that gravity and electricity should share many properties in common —thus, the existence of magnetic fields, the conservation of mass (electric charge), the continuity equation, the Maxwell equations, the existence of gravitomagnetic (electromagnetic) waves that propagates in the empty space with light's speed, among others, are all results that depend uniquely on the covariance of these fields regarding the Lorentz transformations. Other similarities, between these interactions, however, are not derived from covariance requirements, for instance, that both forces satisfy an inverse square law and that they also depend on the product of the interacting masses (charges).

Gravity and electricity have three main differences, although. The first and most obvious difference is that the electric force is incredibly greater than the gravitational one. The second difference, and indeed the most dramatic one, is the connection that exists between gravity and geometry, as Einstein's theory states —it seems that no similar relationship exists regarding the electromagnetic theory. The third difference and perhaps the most inexplicable one is that the gravitational interaction is always attractive, while the electric forces can be either attractive or repulsive. In other words, while the electric charges can be either positive or negative, the gravitational charge —that is, the gravitational mass— is found to be only of one type. The existence of only one kind of gravitational mass that always attract themselves implies huge differences between gravity and electricity. In the following we shall cite some of them.

First of all, the existence of an always attractive gravitational force is the true reason why the gravitational force overcomes the electric force at large distances: while the electric fields are all neutralized by the existence of positive and negative charges, the gravitational force only enhances itself. Besides, this also prevents us of talking about gravitational polarization charges, since there is no way to form such things with one kind of charge only

[8] This also suggests that gravitational waves should not generate themselves gravitational fields, since they are only massless perturbations of the spacetime itself. The energy of gravitational waves only acquires a meaning when they are interpreted in terms of gravitomagnetic fields propagating in a background flat spacetime.

(hence, there can be no distinction between the gravitational Maxwell equations in the matter and in the vacuum). Another difference can be seen if we consider some gravitational analogues of electric capacitors —for instance, a binary system consisting of a black hole plus a swallowing star. The fact that the gravitational interaction is always attractive implies that this gravitational system behave differently than its electromagnetic analogue. In fact, while in a discharging capacitor the electrons of the plate move in order to decrease the potential, so that the system reaches at some point to the equilibrium state, in the gravitational case the black hole (capacitor) continuously sucks the material of the star (plate), so that its mass increases while the mass of the star is depreciated and, hence, no state of equilibrium can be reached —the star is doomed to be completely swallowed by the black hole.

Another difference arises from the irradiating fields. In fact, while an electromagnetic dipole can contribute to the irradiated fields, the same cannot happen with the gravitomagnetic dipoles. The reason is that, in the first case, the gravitational dipole moment,

$$\boldsymbol{d} = \int \rho(\boldsymbol{r})\boldsymbol{r}\mathrm{d}V, \tag{131}$$

always vanishes in the center-of-mass frame and, in the second case, the gravitational magnetic dipole,

$$\boldsymbol{m} = \int \rho(\boldsymbol{r})\boldsymbol{u} \times \boldsymbol{r}\mathrm{d}V, \tag{132}$$

is just the angular momentum of the system w.r.t. the center-of-mass frame and hence it is also conserved. The conclusion is that gravitational dipoles cannot irradiate: the first contribution should come from the quadrupoles terms onward (notice that the same property is valid in Einstein's theory of gravitation [60, 61, 66, 68, 80, 81]).

Finally, we want to highlight that although the gravitational interaction between two particles is always attractive, there is still an open question regarding the gravitational interaction between a particle and an antiparticle (some arguments defending that particles and antiparticles should repel each other are presented in [82,83]). We remark that although the concept of antiparticle is usually introduced in the scope of quantum mechanics, their existence can be evidenced as well from classical point of view, for instance, through the so-called *Extended Theory of Relativity* [84–89] —an extension of the special relativity theory that describes antiparticles and tachyons[9]. Let us present briefly the lines of reasoning that support the possibility of a gravitational repulsion between matter and antimatter. Remember that in the *Stueckelberg-Feynman* interpretation [94, 95], antiparticles are thought as particles that travel back in time. In fact, in a spacetime description the energy of a retrograde particle should be negative, since, in this case, we have the relation

$$E = -\sqrt{m^2c^4 + p^2c^2}. \tag{133}$$

A particle with negative energy that travels back in time should be, although, actually observed as an ordinary particle with positive energy, since any observer measures the time from the past to the future In this process of measurement, however, some properties of the particle must be reversed, in particular its electric charge —this is in fact the content of the Stueckelberg-Feynman principle. The reversion of the electric charge can be proved through the CPT theorem or in terms of the extended Lorentz transformations (from which the CPT theorem can be classically proved [86, 88]). Now, the same argument that implies the reversal of the electric charge also must apply to the gravitational mass, since we had shown that both concepts are quite analogues. Hence, this would mean that the "gravitational charge" of antiparticles are negative, which suggests a repulsive force between a particle and an antiparticle. This conclusion, if confirmed, could be of importance in cosmology, since it might explain the observed asymmetry regarding matter and antimatter in the universe and it could also provide some insight on the problem of dark matter. We believe that this issue merits further analysis.

13 Conclusions and perspectives

In this work we discussed the magnetic effects of gravity according to the special theory of relativity. A fully covariant theory of gravitation with respect to the Lorentz transformations was constructed from first principles: all results were derived from the special theory of relativity only, making no use of any analogy with the electromagnetism theory or additional assumptions. This covariant theory of gravitation, nevertheless, share many of the properties of the electromagnetism theory, which is due to requirements imposed by the special theory of relativity for any relativistic

[9] Proposals to extend the special theory of relativity to describe tachyons and antiparticles were presented by several authors since the sixties, especially by Bilaniuk, Deshpande and Sudarshan [84], Antippa and Everett [85], Recami [86], Sutherland and Shepanski [87], among others. In recent years, some interest in this field was renewed after the works of Vieira [88] and Hill and Cox [89], which independently proposed the same extended Lorentz transformations. It seems now that there is some consensus about the correctness of these extended Lorentz transformations, at least in two dimensions [90–93]. A consistent extension of the theory of relativity in higher dimensions is, however, still an open problem.

force field. The present theory should be regarded as an approximation of the general theory of relativity that holds for bodies moving at large enough distances of the gravitational sources, but not necessarily with small velocities. The Newtonian limit corresponds to the case where the velocities are also very small when compared with the speed of light.

The main conclusions derived in this work are the following:

1) We showed through a simple thought experiment that any massive body in motion should create a gravitational magnetic-like field.
2) Imposing covariance of the gravitational force with respect to the Lorentz transformations, we derived the exact expressions for these gravitomagnetic fields. We showed as well that the total gravitational force can always be written (for any inertial frame) in the same form as the Lorentz force of the electromagnetism theory.
3) A comparison between the expression for the gravitational Lorentz force derived here and those obtained from other approximations of the general relativity was presented. We showed that linear approximations of general relativity usually leads to a modified Lorentz force (*e.g.*, introducing a factor of 4 in the magnetic gravitational force) and that this is due to the account of high-order terms used in those approximations.
4) The differences between the concepts of mass (current) density and energy (momentum) density were clarified. In particular, we showed that the gravitational mass and current densities transform under the Lorentz transformations as the components of a spacetime vector, while the energy and momentum densities transform as two-rank tensor components.
5) This implies that the gravitational mass of a body must be an invariant, in the same footing as the electric charge.
6) The differential equations satisfied by the gravitomagnetic fields were also derived through covariance requirements only. They are similar to the Maxwell equations of electromagnetism, except for the presence of negative signs, which are due to the attractive character of the gravitational interaction.
7) From these gravitational Maxwell equations, the existence of gravitational waves was demonstrated. Our approach leads to a unique value for the speed of gravitational waves, namely, the velocity of the light in vacuum, c. We highlight that this result is in agreement with the most recent experiments concerning the detection of gravitational waves, which provides a strong argument in favor of the validity of our theory in the approximation considered.
8) We introduced the gravitomagnetic potentials and we argued that a physical meaning can be attached to them (as well as to their electromagnetic analogues). Whenever a scalar potential is associated with an interaction energy of the bodies, the corresponding vector potential should be associated to a corresponding interaction momentum. The gauge freedom of the potentials does not prohibit their reality as physical quantities, since the ground state of the interaction energy and momentum can also be chosen at will.
9) Then, the energy and momentum stored on the gravitational fields were discussed and a four-dimensional formulation of the theory was also presented.
10) Finally, some differences between gravity and electricity were discussed. The most important one comes from the always attractive behavior of the gravitational interaction for usual massive bodies. We had speculated, however, that this could not be so if a gravitational interaction between particles and antiparticles is considered.

The present theory opens the door to the formulation of a quantum theory of gravity that is expected to hold at large distances of the gravitational sources, or in any situation where special relativity can be used with safety. Such a theory could be relevant in discussing some gravitational effects where quantum mechanics plays a significant role, for instance, in the Hawking radiation, Landau levels, gravitational entanglement, the gravitational analogues of the Aranov-Bohm, Casimir and Unruh effects, among others. We expect to not find any mathematical difficulty in formulating such a "quantum gravitodynamics" in a flat spacetime background, since in this case the gravitomagnetism is quite similar to the electromagnetism (being even gauge-invariant also) and there is already a well-established formulation of quantum electrodynamics (in fact, some work on this direction was presented recently in [96]). The conceptual issues, however, might be challenging. For instance, it is usually believed that gravity is mediated by the exchange of spin-2 particles —the so-called *gravitons*. This is mainly because the gravitational potentials in Einstein theory are described by a two-rank tensor [97, 98]. However, we have seen here that the gravitational potentials can be described as well by a spacetime vector in our approximation, in the same way as the electromagnetic potentials. This means, therefore, that we can also associate a spin-1 exchanging particle with the gravitational interaction in the approximation considered. The connection between this exchanging particle and the usual graviton, however, still need to be clarified further. A related issue concerns with the conclusion that spin-1 quantum field theory leads to a repulsive force between likewise particles. This is usually taken as an argument to disprove any attempt to construct a spin-1 theory of quantum gravity [97, 98]. This argument, however, does not take into account that the energy of the gravitational fields is negative, rather than positive as in the electromagnetic case. This fact, however, makes likewise particles to attract each other as they exchange spin-1 particles —*i.e.*, the apparent counter-intuitive negativeness of the energy stored in the gravitomagnetic fields is, notwithstanding, the ultimate reason for why gravity is attractive, even on a quantum level. A detailed exposition of these topics will be the subject of a future work.

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Appendix A. Conventions and notations

In this paper, three-dimensional vectors are written in bold-italics as in $\boldsymbol{A}$ or in the tensor notation as in A^a, in which case Latin indices run from 1 to 3. Tensor quantities defined on Minkowski spacetime are written in the usual tensor notation as A^μ, where Greek indices run from 0 to 3. We named four-dimensional quantities by their spatial component name only, preceded by the word "spacetime" (for example, we refer to the *spacetime position* x^μ, the *spacetime momentum* p^μ, the spacetime *momentum-flux* $T^{\mu\nu}$ and so on).

We consider here the metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and the spacetime coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. Moreover, the Lorentz factor,

$$\gamma(\boldsymbol{u}) = \frac{1}{\sqrt{1 - \boldsymbol{u}\cdot\boldsymbol{u}/c^2}}\,, \tag{A.1}$$

is usually denoted as γ_u, although $\gamma(v)$ is written as γ_v or just as γ whenever it is clear from the context. We write the usual partial derivatives as $\partial_t \equiv \partial/\partial t$, $\partial_x \equiv \partial/\partial x$, $\partial_y \equiv \partial/\partial y$ and $\partial_z \equiv \partial/\partial z$, while the spacetime derivatives are written as $\partial_\mu \equiv \partial/\partial x^\mu$ and $\partial^\mu \equiv \partial/\partial x_\mu$. Finally, the *dotted nabla symbol*,

$$\dot{\boldsymbol{\nabla}} = \hat{\boldsymbol{x}}\frac{\partial}{\partial \dot{x}} + \hat{\boldsymbol{y}}\frac{\partial}{\partial \dot{y}} + \hat{\boldsymbol{z}}\frac{\partial}{\partial \dot{z}}\,, \tag{A.2}$$

means the vector differential operator which takes derivatives w.r.t. the velocity components.

Here we consider mainly three frames of reference: the frame S' is a reference frame where a given distribution of mass (namely, that one which generates the gravitational field), is always at rest. On the other hand, the reference frame S is regarded as a reference frame where that distribution of mass moves with a constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ (in other words, S' moves w.r.t. S with the velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$). To avoid misunderstanding, in some situations we consider a third reference frame S'', which is, in all aspects, the same as the reference frame S' except that no fixed distribution of mass is attached to it. In fact, we assume that the velocity of that distribution of mass is $\boldsymbol{u}$ w.r.t. S and $\boldsymbol{u}''$ w.r.t. S''. As usual, we assume further that all axes of these reference frames are equally oriented and superposed at $t = t' = t'' = 0$.

Appendix B. Vector identities

The following vector identities are used in the paper:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a}) = 0, \tag{B.1}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0, \tag{B.2}$$

$$\partial_t(\boldsymbol{\nabla}\times\boldsymbol{a}) = \boldsymbol{\nabla}\times(\partial_t\boldsymbol{a})\,, \tag{B.3}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{a}) - \boldsymbol{\nabla}^2\boldsymbol{a}, \tag{B.4}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\,, \tag{B.5}$$

$$\partial_t(\boldsymbol{a}\times\boldsymbol{b}) = (\partial_t\boldsymbol{a})\times\boldsymbol{b} + \boldsymbol{a}\times(\partial_t\boldsymbol{b})\,, \tag{B.6}$$

$$\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{a}) - (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{a}, \tag{B.7}$$

$$\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} + \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a})\,, \tag{B.8}$$

where f is an at least C^2 class scalar function and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are at least C^2 vector functions. We also make use of the following tensor identity:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}\otimes\boldsymbol{a}) = (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{a}. \tag{B.9}$$

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