Eur. Phys. J. Plus (2018) **133**: 111
DOI 10.1140/epjp/i2018-11961-8

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Atangana-Batogna numerical scheme applied on a linear and non-linear fractional differential equation*

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Received: 4 December 2017 / Revised: 29 December 2017
Published online: 15 March 2018 – © Società Italiana di Fisica / Springer-Verlag 2018

Abstract. Recently, Atangana and Batogna suggested a new numerical scheme to solve linear and non-linear equations with classical and fractional differential operators. The method can be understood as a combination of forward (or backward) approximation and the Adams-Bashforth one. This paper further presents the application of the new method to a linear and non-linear partial differential equation with integer– and non-integer–order derivative. The stability and convergence analyses are presented in detail. Some simulations are done to verify the efficiency of the new numerical scheme for solving linear and non-linear equations.

1 Introduction

It is well known that some non-linear ordinary differential equations can be solved very efficiently using the well-known Adams-Bashforth method. The method is known to be efficient in solving even the most unstable natural occurrences, including chaotic models and biological models [1–3]. So far, one can say that it is the most used method for solving non-linear and unstable ordinary differential equations with classical and non-local differential operators [4–10]. Most recently, this model was extended to the realm of partial differential equations with integer– and non-integer–order derivatives by Atangana and Batogna in their paper [11]. Their method is thus a mixture of the forward/backward approximation on space and the Adams-Bashforth one in time. In this work, we will use this method to solve numerically linear and nonlinear partial differential equations with fractional differential operators with singular and non-singular kernel. We note that this model is a non-linear partial differential equation that describes a very important physical problem. We shall recall that, when the equator of the pole energy moves from one position to another through heat diffusion, the energy balance model is that of reaction diffusion, which can be described by the Chafee-Infante equation [12]. On the other hand, to check the accuracy of this model for solving linear equations, we study the perturbed heat model with classical and non-local derivatives. The structure of the work is as follows: The new numerical scheme is presented in sect. 2. In sect. 3, we present the application to the linear equation with detailed stability analysis. In sect. 4, the method is applied to a nonlinear differential equation. In sect. 5, we apply the method to the well-known Chafee-Infante equation with Caputo fractional derivative and, finally, numerical simulation in sect. 6.

2 Atangana-Batogna numerical scheme

To accommodate researchers that are not aware of this newly introduced method, we present, in this section, the derivation of the method for a general partial differential equation with local and non-local derivatives. We start with a general partial differential equation with local derivative. In their paper [11], Batogna and Atangana considered the following general partial differential equation:

$$\partial_t u(x,t) = Lu(x,t) + Nu(x,t), \tag{1}$$

* Focus Point on "Modelling Complex Real-World Problems with Fractal and New Trends of Fractional Differentiation" edited by A. Atangana, Z. Hammouch, G. Mophou, K.M. Owolabi.

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where L is a linear operator and N is a non-linear operator. Then, they applied the Laplace transform on both sides of eq. (1) to make it an ordinary differential equation, thus obtaining

$$\partial_t U(P,t) = LU(P,t) + NU(P,t), \tag{2}$$
$$\partial_t V(t) = LV(t) + NV(t), \tag{3}$$

with $U(P,t)$ the Laplace transform of $u(x,t)$ with respect to space and $V(t) = U(P,t)$. Thus eq. (3) can be converted to

$$\partial_t V(t) = F(t,V). \tag{4}$$

Then they applied the well-known Adams-Basforth scheme on eq. (4) and obtained

$$V_{n+1} = V_n + \frac{3h}{2}F_n - \frac{h}{2}F_{n-1}, \tag{5}$$
$$V_n = V(t_n), \tag{6}$$
$$F_n = F(t_n, V_n). \tag{7}$$

After that, the authors applied the inverse Laplace transform on both sides of eq. (5) to obtain

$$u(x,t_{n+1}) = u(x,t_n) + \frac{3h}{2}F(t_n, u(x,t_n)) - \frac{h}{2}F(t_{n-1}, u(x,t_{n-1})). \tag{8}$$

To conclude, they applied either the backward or the forward Euler approximation in space to obtain

$$u(x_i,t_{n+1}) = u(x_i,t_n) + \frac{3h}{2}F^i(t_n, u(x,t_n)) - \frac{h}{2}F^i(t_{n-1}, u(x,t_{n-1})), \tag{9}$$

or

$$u(x_{i+1},t_{n+1}) = u(x_{i+1},t_n) + \frac{3h}{2}F^{i+1}(t_n, u(x,t_n)) - \frac{h}{2}F^{i+1}(t_{n-1}, u(x,t_{n-1})). \tag{10}$$

They presented a detailed error analysis, which can be found in [12]. For fractional partial differential equations with Caputo derivative, using the same derivation, they obtained the following:

$$u(x,t_{n+1}) = u(x,t_n) + A_n^\alpha F(t_n, u(x,t_n)) - B_n^\alpha F(t_{n-1}, u(x,t_{n-1})), \tag{11}$$
$$A_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right), \tag{12}$$
$$B_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right). \tag{13}$$

Thus, they used forward or backward approximation on space obtaining

$$u(x_i,t_{n+1}) = u(x_i,t_n) + A_n^\alpha F^i(t_n, u(x,t_n)) - B_n^\alpha F^i(t_{n-1}, u(x,t_{n-1})), \tag{14}$$

or

$$u(x_{i-1},t_{n+1}) = u(x_{i-1},t_n) + A_n^\alpha F^{i-1}(t_n, u(x,t_n)) - B_n^\alpha F^{i-1}(t_{n-1}, u(x,t_{n-1})). \tag{15}$$

3 Application to the perturbed heat equation

In this section, we solve numerically the perturbed heat equation with classical and non-local differential operators. We shall start with the classical version.

3.1 Numerical solution of the perturbed heat equation with integer– and non-integer–order derivatives

The partial differential equation under analysis here is given as

$$\partial_t u(x,t) = \partial_{x,x}^2 u(x,t) + \epsilon u(x,t), \tag{16}$$
$$u(x,0) = g(x). \tag{17}$$

Equation (16) occurs in the heat flow system that occurs in real-world situations and, also, it finds application in chemical reactions. Thus applying the first step of the new numerical scheme on eq. (16) we obtain

$$\begin{aligned} u(x,t_{n+1}) &= u(x,t_n) + \frac{3h}{2}F(t_n,u(x,t_n)) - \frac{h}{2}F(t_{n-1},u(x,t_{n-1})), \quad (18)\\ F(t_n,u(x,t_n)) &= \partial^2_{x,x}u(x,t_n) + \epsilon u(x,t_n), \quad (19)\\ F(t_{n-1},u(x,t_{n-1})) &= \partial^2_{x,x}u(x,t_{n-1}) + \epsilon u(x,t_{n-1}). \quad (20)\end{aligned}$$

Now applying the second step on eq. (16), thus using the forward Euler approximation on space, we obtain

$$\begin{aligned} u(x_i,t_{n+1}) &= u(x_i,t_n) + \frac{3h}{2}F^i(t_n,u(x,t_n)) - \frac{h}{2}F^i(t_{n-1},u(x,t_{n-1})), \quad (21)\\ F(t_n,u(x_i,t_n)) &= \frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \epsilon u(x_i,t_n), \quad (22)\\ F(t_{n-1},u(x,t_{n-1})) &= \frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \epsilon u(x_i,t_{n-1}). \quad (23)\end{aligned}$$

Thus

$$\begin{aligned} u(x_i,t_{n+1}) = u(x_i,t_n) &+ \frac{3h}{2}\left(\frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \epsilon u(x_i,t_n)\right)\\ &- \frac{h}{2}\left(\frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \epsilon u(x_i,t_{n-1})\right). \quad (24)\end{aligned}$$

If we put, for simplicity, $u(x_i,t_n) = u_i^n$ the above equation is converted to

$$u_i^{n+1} = u_i^n + \frac{3h}{2}\left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{l^2} + \epsilon u_i^n\right) - \frac{h}{2}\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \epsilon u_i^{n-1}\right). \quad (25)$$

Rearranging, we obtain

$$u_i^{n+1} = u_i^n\left(\frac{3h}{2l^2} + 1 + \epsilon\right) + \frac{3h}{2}\left(\frac{-2u_{i-1}^n + u_{i-2}^n}{l^2}\right) - \frac{h}{2}\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \epsilon u_i^{n-1}\right). \quad (26)$$

We next consider the fractional version of the heat equation. The fractional partial differential equation under analysis here is given as

$$\begin{aligned} {}_0^C D_t^\alpha u(x,t) &= \partial^2_{x,x}u(x,t) + \epsilon u(x,t), \quad (27)\\ u(x,0) &= g(x). \quad (28)\end{aligned}$$

Equation (27) is able to describe the diffusion of heat in elastic media, and the effect of memory is thus included into mathematical equations via the convolution of the power law and the rate of change; the parameter ϵ is a small perturbation term. Thus applying the first step of the new numerical scheme for the fractional case on eq. (27) we obtain

$$\begin{aligned} u(x,t_{n+1}) &= u(x,t_n) + A_n^\alpha F(t_n,u(x,t_n)) - B_n^\alpha F(t_{n-1},u(x,t_{n-1})), \quad (29)\\ F(t_n,u(x,t_n)) &= \partial^2_{x,x}u(x,t_n) + \epsilon u(x,t_n), \quad (30)\\ F(t_{n-1},u(x,t_{n-1})) &= \partial^2_{x,x}u(x,t_{n-1}) + \epsilon u(x,t_{n-1}), \quad (31)\\ A_n^\alpha &= \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right), \quad (32)\\ B_n^\alpha &= \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right). \quad (33)\end{aligned}$$

Application of the second step of the new numerical scheme for the fractional case yields

$$u(x_i,t_{n+1}) = u(x_i,t_n) + A_n^{\alpha} F^i(t_n,u(x,t_n)) - B_n^{\alpha} F^i(t_{n-1},u(x,t_{n-1})), \tag{34}$$

$$F(t_n,u(x_i,t_n)) = \frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \epsilon u(x_i,t_n), \tag{35}$$

$$F(t_{n-1},u(x,t_{n-1})) = \frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \epsilon u(x_i,t_{n-1}), \tag{36}$$

$$A_n^{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha)}\left(\frac{2(n+1)^{\alpha} - n^{\alpha}}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right), \tag{37}$$

$$B_n^{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha)}\left(\frac{(n+1)^{\alpha}}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right). \tag{38}$$

Nevertheless, the above eq. (34) can be converted to

$$\begin{aligned} u(x_i,t_{n+1}) = u(x_i,t_n) &+ A_n^{\alpha}\left(\frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \epsilon u(x_i,t_n)\right) \\ &- B_n^{\alpha}\left(\frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \epsilon u(x_i,t_{n-1})\right). \end{aligned} \tag{39}$$

Thus

$$u_i^{n+1} = u_i^n + A_n^{\alpha}\left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{l^2} + \epsilon u_i^n\right) - B_n^{\alpha}\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \epsilon u_i^{n-1}\right). \tag{40}$$

Rearranging, we obtain

$$u_i^{n+1} = u_i^n\left(\frac{A_n^{\alpha}}{2l^2} + 1 + A_n^{\alpha}\epsilon\right) + A_n^{\alpha}\left(\frac{-2u_{i-1}^n + u_{i-2}^n}{l^2}\right) - B_n^{\alpha}\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \epsilon u_i^{n-1}\right). \tag{41}$$

3.2 Stability analysis

In this section, we present the stability analysis using the method used for solving classical and the non-local derivatives. Here we give the following definition for the round-off error ϵ_i^n:

$$\epsilon_i^n = L_i^n - h_i^n, \tag{42}$$

with, of course, u_i^n as the solution of the discretized eq. (42). The difference equation error is linear, thus,

$$\epsilon_i^n = \exp[at]\exp[jk_m x]. \tag{43}$$

Replacing eq. (43) into eq. (41) yields

$$\begin{aligned} \exp[a(t+\Delta t)]\exp[jk_m x] = \exp[at]\exp[jk_m x]&\left(\frac{A_n^{\alpha}}{2l^2} + 1 + A_n^{\alpha}\epsilon\right) \\ &+ A_n^{\alpha}\left(\frac{-2\exp[at]\exp[jk_m(x-\Delta x)] + \exp[at]\exp[jk_m(x-2\Delta x)]}{l^2}\right) \\ &- B_n^{\alpha}\left(\frac{1}{2l^2}\left(\exp[a(t-\Delta t)]\exp[jk_m x]\right) - 2\exp[a(t-\Delta t)]\exp[jk_m(x-\Delta x)]\right) \\ &- B_n^{\alpha}\left(\frac{1}{2l^2}\left(\exp[a(t-\Delta t)]\exp[jk_m(x-\Delta x)]\right) + \epsilon\exp[a(t-\Delta t)]\exp[jk_m x]\right). \end{aligned} \tag{44}$$

The above can be further simplified as

$$\begin{aligned} H_{n+1}\exp[a\Delta t] = &\left(\frac{A_n^{\alpha}}{2l^2} + 1 + A_n^{\alpha}\epsilon\right) + A_n^{\alpha}\left(\frac{-2H_n\exp[-jk_m\Delta x] + H_n\exp[-2jk_m\Delta x]}{l^2}\right) \\ &- B_n^{\alpha}\left(\frac{1}{2l^2}\left(H_{n-1} - 2H_{n-1}\exp[-jk_m\Delta x]\right)\right) - B_n^{\alpha}\left(\frac{1}{2l^2}\left(H_{n-1}\exp[-jk_m\Delta x]\right) + \epsilon H_{n-1}\right). \end{aligned} \tag{45}$$

We have, in addition to the above, that

$$\exp[a\Delta t] = \frac{\epsilon_i^{n+1}}{\epsilon_i^n}\,. \tag{46}$$

The numerical scheme will be stable if

$$\|\exp[a\Delta t]\|| = \left\|\frac{\epsilon_i^{n+1}}{\epsilon_i^n}\right\| < 1, \quad \forall n \geq 0. \tag{47}$$

In other words, we shall prove, via the induction formula, that

$$\|H_{n+1}\| < \|H_0\|, \quad \forall n \geq 1, \tag{48}$$

with

$$\begin{aligned} H_{n+1} = {} & H_n\left(\frac{A_n^\alpha}{2l^2} + 1 + A_n^\alpha \epsilon\right) \\ & + A_n^\alpha\left(\frac{-2H_n\exp[-jk_m\Delta x] + H_n\exp[-2jk_m\Delta x]}{l^2}\right) \\ & - B_n^\alpha\left(\frac{1}{2l^2}\left(H_{n-1} - 2H_{n-1}\exp[-jk_m\Delta x]\right)\right) - B_n^\alpha\left(\frac{1}{2l^2}\left(H_{n-1}\exp[-jk_m\Delta x]\right) + \epsilon H_{n-1}\right). \end{aligned} \tag{49}$$

Thus, taking $n = 0$, we have

$$H_1 = H_0\left(\frac{A_0^\alpha}{2l^2} + 1 + A_0^\alpha\epsilon\right) + A_0^\alpha\left(\frac{-2H_0\exp[-jk_m\Delta x] + H_0\exp[-2jk_m\Delta x]}{l^2}\right). \tag{50}$$

Now, taking the absolute value on both sides of the above yields

$$\left\|\frac{H_1}{H_0}\right\| < \left\|\frac{A_0^\alpha}{2l^2} + 1 + A_0^\alpha\epsilon - \frac{A_0^\alpha}{l^2}\right\| < 1. \tag{51}$$

The stability is achieved if

$$l^2 < \frac{1}{\epsilon}\,. \tag{52}$$

We now assume that $\forall n \geq 0$, so inequality (48) holds, and now we evaluate it for $n+1$:

$$\begin{aligned} \|H_{n+1}\| \leq {} & \|H_n\|\left\|\left(\frac{A_n^\alpha}{2l^2} + 1 + A_n^\alpha\epsilon\right)\right\| + \left\|A_n^\alpha\left(\frac{-2H_n\exp[-jk_m\Delta x] + H_n\exp[-2jk_m\Delta x]}{l^2}\right)\right\| \\ & + \left\|-B_n^\alpha\left(\frac{1}{2l^2}\left(H_{n-1} - 2H_{n-1}\exp[-jk_m\Delta x]\right)\right)\right\| + \left\|-B_n^\alpha\left(\frac{1}{2l^2}\left(H_{n-1}\exp[-jk_m\Delta x]\right) + \epsilon H_{n-1}\right)\right\|. \end{aligned} \tag{53}$$

Using the inductive formula and also the inductive hypothesis, we have

$$\|H_{n+1}\| < \|H_0\|\left\|\left(\frac{A_n^\alpha}{2l^2} + 1 + A_n^\alpha\epsilon - \frac{A_n^\alpha}{2l^2} + \frac{B_n^\alpha}{2l^2} - B_n^\alpha\left(\epsilon + \frac{1}{2l^2}\right)\right)\right\|. \tag{54}$$

The inequality is achieved if

$$B_n^\alpha < A_n^\alpha, \tag{55}$$

and the above is true for $\forall n \geq 0$. Since the process is true for $n+1$ by the induction principle, the scheme is stable unconditionally because all inequalities are true for all natural numbers. We next present the stability for the classical derivative,

$$\begin{aligned} H_{n+1}\exp[\Delta xjk_m] = {} & H_n\exp[\Delta xjk_m]\left(\frac{3h}{2l^2} + 1 + \epsilon\right) + \frac{3h}{2}\left(\frac{-2H_n\exp[-jk_m\Delta x] + H_n\exp[-2jk_m\Delta x]}{l^2}\right) \\ & - \frac{h}{2}\left(\frac{H_{n-1}\exp[\Delta xjk_m] - 2H_{n-1}\exp[-\Delta xjk_m] + H_{n-1}\exp[-2\Delta xjk_m]}{l^2} + \epsilon H_{n-1}\exp[\Delta xjk_m]\right). \end{aligned} \tag{56}$$

Thus, after simplification, we obtain

$$H_{n+1} = H_n\left(\frac{3h}{2l^2}+1+\epsilon\right)+\frac{3h}{2}\left(\frac{-2H_n\exp[-2jk_m\Delta x]+H_n\exp[-3jk_m\Delta]}{l^2}\right)$$
$$-\frac{h}{2}\left(\frac{H_{n-1}-2H_{n-1}\exp[-2\Delta xjk_m]+H_{n-1}\exp[-3\Delta xjk_m]}{l^2}+\epsilon H_{n-1}\right). \quad (57)$$

Again, as done before, we establish the stability if the following condition is reached:

$$2l^2 < h. \quad (58)$$

4 Application to the non-linear equation

In this section, we solve numerically the perturbed heat equation with classical and non-local differential operators. We shall start with the classical version.

4.1 Numerical solution with classical differentiation

The partial differential equation under analysis here is given as

$$\partial_t u(x,t) = \partial^2_{x,x}u(x,t)u^2(x,t), \quad (59)$$
$$u(x,0) = g(x). \quad (60)$$

Equation (59) is a non-linear equation. Nevertheless a direct application of the first step of the new numerical scheme on eq. (59) leads to

$$u(x,t_{n+1}) = u(x,t_n)+\frac{3h}{2}F(t_n,u(x,t_n))-\frac{h}{2}F(t_{n-1},u(x,t_{n-1})), \quad (61)$$
$$F(t_n,u(x,t_n)) = \partial^2_{x,x}u(x,t_n)u^2(x,t_n), \quad (62)$$
$$F(t_{n-1},u(x,t_{n-1})) = \partial^2_{x,x}u(x,t_{n-1})u^2(x,t_{n-1}). \quad (63)$$

Thus applying the second step of the suggested method on (59), again making use of the forward Euler approximation on space, we obtain

$$u(x_i,t_{n+1}) = u(x_i,t_n)+\frac{3h}{2}F^i(t_n,u(x,t_n))-\frac{h}{2}F^i(t_{n-1},u(x,t_{n-1})), \quad (64)$$
$$F(t_n,u(x_i,t_n)) = \frac{u(x_i,t_n)-2u(x_{i-1},t_n)+u(x_{i-2},t_n)}{l^2}u^2(x_i,t_n), \quad (65)$$
$$F(t_{n-1},u(x,t_{n-1})) = \frac{u(x_i,t_{n-1})-2u(x_{i-1},t_{n-1})+u(x_{i-2},t_{n-1})}{l^2}+u^2(x_i,t_{n-1}). \quad (66)$$

Thus,

$$u(x_i,t_{n+1}) = u(x_i,t_n)+\frac{3h}{2}\left(\frac{u(x_i,t_n)-2u(x_{i-1},t_n)+u(x_{i-2},t_n)}{l^2}+u^2(x_i,t_n)\right)$$
$$-\frac{h}{2}\left(\frac{u(x_i,t_{n-1})-2u(x_{i-1},t_{n-1})+u(x_{i-2},t_{n-1})}{l^2}+u^2(x_i,t_{n-1})\right). \quad (67)$$

We put $u(x_i,t_n)=u_i^n$, such that eq. (67) becomes

$$u_i^{n+1} = u_i^n+\frac{3h}{2}\left(\frac{u_i^n-2u_{i-1}^n+u_{i-2}^n}{l^2}(u_i^n)^2\right)-\frac{h}{2}\left(\frac{u_i^{n-1}-2u_{i-1}^{n-1}+u_{i-2}^{n-1}}{l^2}(u_i^{n-1})^2\right). \quad (68)$$

4.2 Numerical solution with fractional differentiation

We next consider the fractional version of the non-linear equation:

$$ {}_0^C D_t^\alpha u(x,t) = \partial_{x,x}^2 u(x,t) + \epsilon u(x,t), \tag{69} $$
$$ u(x,0) = g(x). \tag{70} $$

Applying the first step of the new numerical scheme for the fractional case on eq. (69) yields

$$ u(x,t_{n+1}) = u(x,t_n) + A_n^\alpha F(t_n, u(x,t_n)) - B_n^\alpha F(t_{n-1}, u(x,t_{n-1})), \tag{71} $$
$$ F(t_n, u(x,t_n)) = \partial_{x,x}^2 u(x,t_n) u^2(x,t_n), \tag{72} $$
$$ F(t_{n-1}, u(x,t_{n-1})) = \partial_{x,x}^2 u(x,t_{n-1}) u^2(x,t_{n-1}), \tag{73} $$
$$ A_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)} \left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha} \right), \tag{74} $$
$$ B_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)} \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha} \right). \tag{75} $$

Application of the second step of the new numerical scheme for the fractional case yields

$$ u(x_i,t_{n+1}) = u(x_i,t_n) + A_n^\alpha F^i(t_n, u(x,t_n)) - B_n^\alpha F^i(t_{n-1}, u(x,t_{n-1})), \tag{76} $$
$$ F(t_n, u(x_i,t_n)) = \frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} u^2(x_i,t_n), \tag{77} $$
$$ F(t_{n-1}, u(x,t_{n-1})) = \frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} u^2(x_i,t_{n-1}), \tag{78} $$
$$ A_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)} \left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha} \right), \tag{79} $$
$$ B_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)} \left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha} \right). \tag{80} $$

Nevertheless, the above equation can be converted to

$$ \begin{aligned} u(x_i,t_{n+1}) = u(x_i,t_n) &+ A_n^\alpha \left(\frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} (u(x_i,t_n))^2 \right) \\ &- B_n^\alpha \left(\frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} (u(x_i,t_{n-1}))^2 \right). \end{aligned} \tag{81} $$

Thus

$$ u_i^{n+1} = u_i^n + A_n^\alpha \left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{l^2} (u_i^n)^2 \right) - B_n^\alpha \left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} (u_i^{n-1})^2 \right). \tag{82} $$

5 Application to the Chafee-Infante equation with the Caputo fractional derivative

We devote this part to the analysis of the Chafee-Infante equation, which takes into account the power law memory described by the Caputo fractional derivative. The mathematical equation under investigation here is given as

$$ {}_0^C D_t^\alpha u(x,t) = u_{x,x} + \lambda(u - u^3). \tag{83} $$

The above equation was introduced by Chafee and Infante in their work [12]. This equation is a scalar reaction diffusion model. It is also a deterministic parabolic equation. In this section, we apply the new numerical scheme to the extended model able to describe also the power memory in a reaction diffusion,

$$ {}_0^C D_t^\alpha u(x,t) = \partial_{x,x}^2 u(x,t) + \lambda u(x,t) - \lambda u^3(x,t), \tag{84} $$
$$ u(x,0) = g(x). \tag{85} $$

As done in the previous section, we can apply the first step of the new numerical scheme, which produces

$$u(x,t_{n+1}) = u(x,t_n) + A_n^\alpha F(t_n, u(x,t_n)) - B_n^\alpha F(t_{n-1}, u(x,t_{n-1})), \tag{86}$$

$$F(t_n, u(x,t_n)) = \partial_{x,x}^2 u(x,t_n) + \lambda(u(x,t_n) - u^3(x,t_n)), \tag{87}$$

$$F(t_{n-1}, u(x,t_{n-1})) = \partial_{x,x}^2 u(x,t_{n-1}) + \lambda(u(x,t_{n-1} - u^3(x,t_{n-1}))), \tag{88}$$

$$A_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right), \tag{89}$$

$$B_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right). \tag{90}$$

Application of the second step of the new numerical scheme for the fractional case yields

$$u(x_i,t_{n+1}) = u(x_i,t_n) + A_n^\alpha F^i(t_n, u(x,t_n)) - B_n^\alpha F^i(t_{n-1}, u(x,t_{n-1})), \tag{91}$$

$$F(t_n, u(x_i,t_n)) = \frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \lambda(u(x_i,t_n) - u^3(x_i,t_n)), \tag{92}$$

$$F(t_{n-1}, u(x,t_{n-1})) = \frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \lambda(u(x_i,t_{n-1}) - u^3(x_i,t_{n-1})), \tag{93}$$

$$A_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{2(n+1)^\alpha - n^\alpha}{\alpha} - \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right), \tag{94}$$

$$B_n^\alpha = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{(n+1)^\alpha}{\alpha} + \frac{n^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha}\right). \tag{95}$$

We reformulate the above equation as follows:

$$\begin{aligned} u(x_i,t_{n+1}) = u(x_i,t_n) &+ A_n^\alpha\left(\frac{u(x_i,t_n) - 2u(x_{i-1},t_n) + u(x_{i-2},t_n)}{l^2} + \lambda(u(x_i,t_n) - u^3(x_i,t_n))\right) \\ &- B_n^\alpha\left(\frac{u(x_i,t_{n-1}) - 2u(x_{i-1},t_{n-1}) + u(x_{i-2},t_{n-1})}{l^2} + \lambda(u(x_i,t_{n-1}) - u^3(x_i,t_{n-1}))\right). \end{aligned} \tag{96}$$

Thus

$$\begin{aligned} u_i^{n+1} = u_i^n + A_n^\alpha &\left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{l^2} + \lambda(u(x_i,t_n) - u^3(x_i,t_n))\right) \\ - B_n^\alpha &\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \lambda(u(x_i,t_n) - u^3(x_i,t_n))\right). \end{aligned} \tag{97}$$

Rearranging, we obtain

$$\begin{aligned} u_i^{n+1} = u_i^n &\left(\frac{A_n^\alpha}{2l^2} + 1 + A_n^\alpha\lambda(1 - (u_i^n)^2)\right) + A_n^\alpha\left(\frac{-2u_{i-1}^n + u_{i-2}^n}{l^2}\right) \\ - B_n^\alpha &\left(\frac{u_i^{n-1} - 2u_{i-1}^{n-1} + u_{i-2}^{n-1}}{l^2} + \lambda(u(x_i,t_{n-1}) - u^3(x_i,t_{n-1}))\right). \end{aligned} \tag{98}$$

The above numerical scheme can be used to generate numerical simulations and this shall be presented in the next section.

6 Numerical simulations

In this section, we will present the numerical simulations for the solved equations. We will present these simulations for $\alpha = 0.8$. These are shown in figs. 1 and 2 for $\alpha = 0.8$, we also present the contour plot of the exact and numerical solutions in figs. 3 and 4, respectively. In eq. (82) we consider the parameter "$\alpha = 0$". We present the numerical simulations of the exact and approximate solutions as depicted in figs. 1 and 2, for $\alpha = 0.8$, the contour plots of the exact and approximate solutions are shown in figs. 3 and 4 for $\alpha = 0.8$. The comparison of the exact and numerical solutions leads to the conclusion that the used numerical method is highly accurate and a powerful mathematical tool for solving fractional partial differential equations.

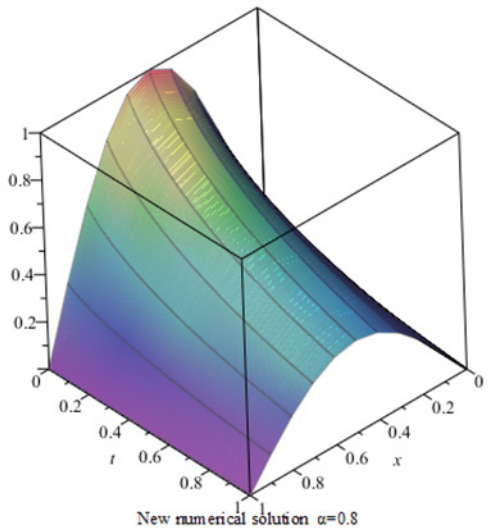

Fig. 1. Numerical solution with the Atangana-Batogna scheme for $\alpha = 0.8$.

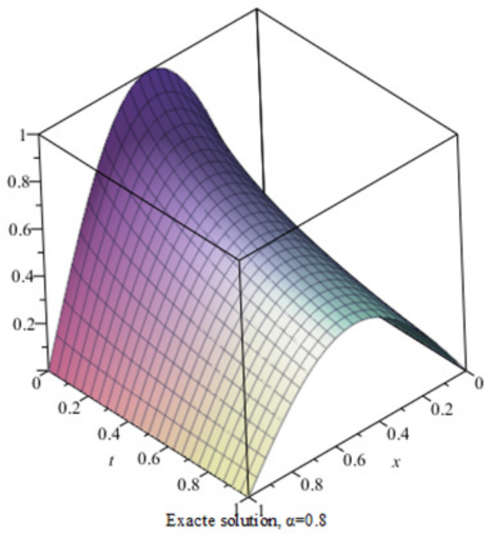

Fig. 2. Exact solution for $\alpha = 0.8$.

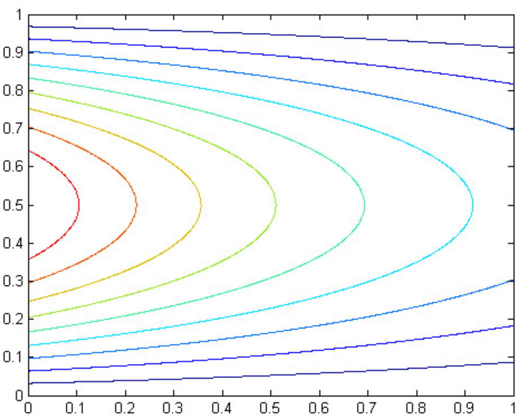

Fig. 3. Contour plot of the numerical solution with the Atangana-Batogna scheme for $\alpha = 0.8$.

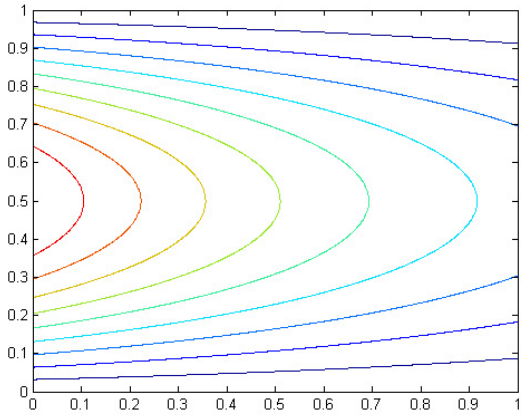

Fig. 4. Contour plot of the exact solution for $\alpha = 0.8$.

7 Conclusion

In this paper, we employed the new numerical scheme suggested by Atangana and Batogna to study the numerical solutions of linear and non-linear equations with local and non-local operators of differentiation. Their numerical scheme is as stable as the well-known Adams-Bashforth method for ordinary differential equations. The method is the combination of finite difference method, integral transforms, fundamental theorem of calculus and Lagrange polynomial interpolation. We presented in detail, for the non-local and local linear equations, the stability of the method using the inductive method. Some examples and simulations are presented. This method is especially powerful for fractional differential equations as it can be represented without the summation that always occurs while approximating a fractional derivative with the commonly used finite difference method.

The authors express their sincere gratitude to the Deanship of Science Research at King Saud University for funding this prolific research group PRG-1437-35.

Conflict of interest

The author confirms that there is no conflict of interest for this paper.

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