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New exact solutions for a discrete electrical lattice using the analytical methods

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Abstract. This paper retrieves soliton solutions to an equation in nonlinear electrical transmission lines using the semi-inverse variational principle method (SIVPM), the $\exp(-\Omega(\xi))$-expansion method (EEM) and the improved $\tan(\phi/2)$-expansion method (ITEM), with the aid of the symbolic computation package Maple. As a result, the SIVPM, EEM and ITEM methods are successfully employed and some new exact solitary wave solutions are acquired in terms of kink-singular soliton solution, hyperbolic solution, trigonometric solution, dark and bright soliton solutions. All solutions have been verified back into their corresponding equations with the aid of the Maple package program. We depicted the physical explanation of the extracted solutions with the choice of different parameters by plotting some 2D and 3D illustrations. Finally, we show that the used methods are robust and more efficient than other methods. More importantly, the solutions found in this work can have significant applications in telecommunication systems where solitons are used to codify data.

1 Introduction

This study focuses on the nonlinear transmission line [1] described by the modified Zakharov-Kuznetsov (MZK) equation. Based on [1], one can acquire the discrete differential equation as follows:

$$\frac{\partial^2 Q_{n,m}}{\partial T^2} = \frac{1}{L}\left(V_{n+1,m} - V_{n,m} + V_{n-1,m}\right) + C_s \frac{\partial^2}{\partial T^2}\left(V_{n,m+1} - V_{n,m} + V_{n,m-1}\right) \tag{1}$$

and the nonlinear charge, in terms of $V_{n,m} = V_{n,m}(T)$, is given as

$$Q_{n,m} = C_0\left(V_{n,m} + \frac{\beta_1}{2}V_{n,m}^2 + \frac{\beta_2}{3}V_{n,m}^3\right), \tag{2}$$

where β_1 and β_2 are constants. Putting (2) into eq. (1), we achieve

$$C_0\frac{\partial^2}{\partial T^2}\left(V_{n,m} + \frac{\beta_1}{2}V_{n,m}^2 + \frac{\beta_2}{3}V_{n,m}^3\right) = \frac{1}{L}\left(V_{n+1,m} - V_{n,m} + V_{n-1,m}\right) + C_s \frac{\partial^2}{\partial T^2}\left(V_{n,m+1} - V_{n,m} + V_{n,m-1}\right). \tag{3}$$

Inserting $V_{n,m}(T) = V(n,m,T)$ into eq. (3), the following equation can be obtained:

$$C_0\frac{\partial^2}{\partial T^2}\left(V + \frac{\beta_1}{2}V^2 + \frac{\beta_2}{3}V^3\right) = \frac{1}{L}\frac{\partial^2}{\partial n^2}\left(V + \frac{1}{12}\frac{\partial^2 V}{\partial n^2}\right) + C_s\frac{\partial^2}{\partial T^2\partial m^2}\left(V + \frac{1}{12}\frac{\partial^2 V}{\partial m^2}\right). \tag{4}$$

Using the independent variable transformations

$$x = \chi^{1/2}(n - v_s T), \qquad y = \chi^{1/2}m, \qquad t = \chi^{1/2}T, \qquad V(n,m,T) = \chi u(x,y,t), \tag{5}$$

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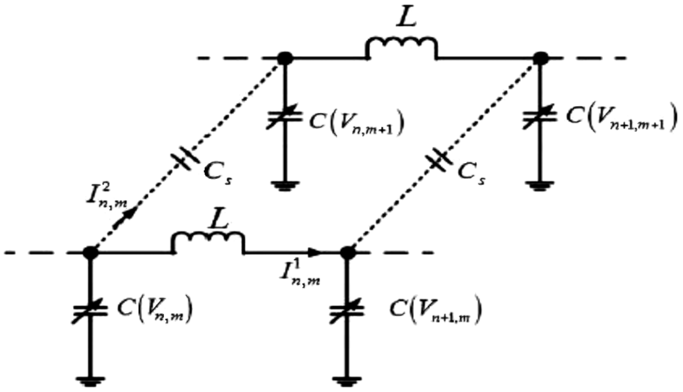

Fig. 1. Profile of the nonlinear electrical transmission line.

where χ is the formal parameter, $v_s = 1/(LC_0)$, and by utilizing the reductive perturbation method, afterwards eq. (4) can be transformed into the modified MZK equation [2,3] as follows:

$$u_t + Auu_x + Bu^2 u_x + Mu_{xxx} + Nu_{xyy} = 0, \tag{6}$$

where

$$A = -\beta_1 v_s, \qquad B = -\beta_2 v_s, \qquad M = \frac{1}{24\beta_1 L v_s}\,, \qquad N = \frac{\beta_1}{288 L^2 v_s C_0^2}\,. \tag{7}$$

The system model studied here is given by fig. 1. The mathematical model used in this work plays a prominent role in the theory of a nonlinear network and arises also in other physical applications such as the lines in the transverse direction [2]. During the last several years, exact solutions of the modified ZK equation and its related equations became important for scientists and engineers due to their application to the wave propagation problems and in the transverse direction modeling in the field of electrical engineering. Several researchers worked out some new solutions for the modified ZK equation using different analytical methods. We will review some of the literature about the modified ZK equation and analytical methods. In this respect, Yu and Feng [4] introduced and investigated the Darboux transformation to determine one-soliton solution as well as other soliton solutions for the modified ZK equation. Moreover, Zhen *et al.* [5] applied the Hirota bilinear method to find soliton solutions of the MZK equation. Recently, Sardar and coworkers adopted the especial analytical methods to seek soliton solutions of the MZK equation and obtained different kinds of solutions which are presented in [3]. This equation also appeared in many scientific applications such as discrete networks, wave propagating systems, laser [6–9]. Due to the rapid expansion of some powerful symbolic computations based mathematical packages such as Maple and Mathematica, the extraction process of exact solutions is now much easier than in the past. In this context, researchers have gained a platform to produce new exact solutions of well-known partial differential equations (PDEs) that arise in applied sciences with numerous robust methods such as the Exp-function method [10–12], the trial solution approach [13], the generalized Kudryashov method [14], the extended Jacobi elliptic function expansion method [15], the improved $\tan(\phi/2)$-expansion method [16–19], the G'/G-expansion method [20,21], the generalized G'/G-expansion method [22], the Bernoulli sub-equation function method [23,24], the Ricatti equation expansion [25,26], the rational function transformations [27], the multiple exp-function method [28,29], the invariant subspace method [30], the formal linearization method [31], the Lie symmetry [32], the Hirota bilinear method [33–36] , the Darboux transformation(DT) [37–39], the inverse scattering transformation (IST) [40,41], and so on. The main aim of this study is to introduce the modified ZK equation for converting the PDE into the ordinary differential equation with the help of the semi-inverse variational principle method [22,42–44], the improved $\tan(\phi/2)$-expansion method [45,46] and the $\exp(-\Omega(\xi))$-expansion method [47,48]. Besides, we will explore the new exact solutions for the modified ZK equation with the aid of the aforementioned methods. The obtained solutions are expressed by exponential, hyperbolic, trigonometric and rational function forms.

The remainder of the paper is organized as follows. A brief discussion about the semi-inverse variational principle method and its application to the modified ZK equation is presented in sect. 2. Section 3 and its sub-sections deal with the applications of the EEM to look for new closed forms of exact solutions of the modified ZK equation. Moreover, in sect. 4, we present the ITEM along with its application to MZK equation. Finally, we draw a conclusion about the executed methods and generated results in sect. 5.

2 The SIVPM

The main steps of the SIVPM are as follows:

Step 1. Consider a general form of a PDE, say in two independent variables x and t as

$$\mathcal{N}(u, u_x, u_y, u_t, u_{xx}, u_{tt}, \ldots) = 0. \tag{8}$$

In eq. (8) $u = u(x,t)$ is an unknown function, $\mathcal{N}$ is a polynomial in $u(x,t)$ and its various partial derivatives, in which the nonlinear terms and highest order derivatives are involved. The PDE can be converted to an ODE

$$\mathcal{Q}(U, r_1 U', r_2 U', -r_3 U', r_1^2 U'', r_3^2 U'', \ldots) = 0, \tag{9}$$

by the transformation $\xi = r_1 x + r_2 y - r_3 t$ in which the wave variable is. Also, r_1, r_2 and r_3 are arbitrary constants to be determined later.

Step 2. According to He's semi-inverse method, we construct the following trial-functional:

$$J(U) = \int L \, \mathrm{d}\xi, \tag{10}$$

where L is an unknown function of U and its derivatives.

Step 3. Utilizing the Ritz method, we can acquire various forms of solitary wave solutions, such as

$$U(\xi) = F \operatorname{sech}(G\xi), \tag{11}$$

$$U(\xi) = F \operatorname{sech}^2(G\xi), \tag{12}$$

$$u(\xi) = \frac{F}{D + \cosh(G\xi)}\,, \tag{13}$$

where F and G are constants to be further determined. Upon substituting (11)–(13) into (10) and making J stationary with respect to F and G results in

$$\frac{\partial J}{\partial F} = 0, \tag{14}$$

$$\frac{\partial J}{\partial G} = 0. \tag{15}$$

Solving eqs. (14) and (15), we obtain F and G. Therefore, the solitary wave solutions (11), (12) and (13) are well determined.

2.1 Application of semi-inverse variational principle

By using He's semi-inverse principle [42–44], one can get the variational formulation as follows

$$J = \int_0^\infty \left[\frac{r_3}{2} u(\xi)^2 + \frac{A r_1}{6} u(\xi)^3 + \frac{B r_1}{12} u(\xi)^4 + \frac{M r_1^3 + N r_1 r_2^2}{2} \left(\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi} \right)^2 \right] \mathrm{d}\xi. \tag{16}$$

Utilizing a Ritz-like method, a solitary wave solution will be as follows.

Case I:

$$u(\xi) = F \operatorname{sech}(G\xi), \tag{17}$$

where F and G are unknown constants to be further determined. Upon substituting (17) into (16) and carrying out the integration gives

$$J = \frac{1}{6} F^2 G M r_1^3 + \frac{1}{6} F^2 G N r_1 r_2^2 - \frac{1}{2} \frac{r_3 F^2}{G} + \frac{1}{24} \frac{A r_1 F^3 \pi}{G} + \frac{1}{18} \frac{B r_1 F^4}{G}\,. \tag{18}$$

Making J stationary with A and B supplies

$$\frac{\partial J(F,G)}{\partial F}=\frac{1}{3}FGMr_1^3+\frac{1}{3}FGNr_1r_2^2-\frac{r_3F}{G}+\frac{1}{8}\frac{Ar_1F^2\pi}{G}+\frac{2}{9}\frac{Br_1F^3}{G}=0, \tag{19}$$

$$\frac{\partial J(F,G)}{\partial G}=\frac{1}{6}F^2Mr_1^3+\frac{1}{6}F^2Nr_1r_2^2+\frac{1}{2}\frac{r_3F^2}{G^2}-\frac{1}{24}\frac{Ar_1F^3\pi}{G^2}-\frac{1}{18}\frac{Br_1F^4}{G^2}=0. \tag{20}$$

Solving eqs. (19) and (20), one can obtain

$$F=\frac{-5Ar_1\pi\pm\sqrt{25A^2r_1^2\pi^2+1536Br_1r_3}}{16Br_1},$$
$$G=\frac{\sqrt{-6r_1(Mr_1^2+r_2^2N)\left(\frac{A(5Ar_1\pi-\sqrt{25A^2r_1^2\pi^2+1536Br_1r_3})\pi}{16B}+24r_3\right)}}{12r_1(Mr_1^2+r_2^2N)}. \tag{21}$$

The domain of definition of the above relations is:

$$25A^2r_1^2\pi^2+1536Br_1r_3>0,\qquad 6r_1(Mr_1^2+r_2^2N)\left(\frac{A(5Ar_1\pi-\sqrt{25A^2r_1^2\pi^2+1536Br_1r_3})\pi}{16B}+24r_3\right)<0. \tag{22}$$

Hence, finally, the 1-soliton solution to the nonlinear electrical transmission lines is given by

$$u(x,y,t)=\frac{-5Ar_1\pi\pm\sqrt{25A^2r_1^2\pi^2+1536Br_1r_3}}{16Br_1}$$
$$\times\,\mathrm{sech}\left[\frac{\sqrt{-6r_1(Mr_1^2+r_2^2N)\left(\frac{A(5Ar_1\pi-\sqrt{25A^2r_1^2\pi^2+1536Br_1r_3})\pi}{16B}+24r_3\right)}}{12r_1(Mr_1^2+r_2^2N)}(r_1x+r_2y-r_3t)\right]. \tag{23}$$

Also, a bright soliton wave solution can be found as follows.

Case II:

$$u(\xi)=\frac{F}{\cosh^2(G\xi)}, \tag{24}$$

where F and G are unknown constants to be further determined. Upon inserting (24) into (16) and carrying out the integration gives

$$J=\frac{1}{315}\frac{F^2(84r_1^3G^2M+84r_1G^2Nr_2^2-105r_3+28Ar_1F+12Br_1F^2)}{G}. \tag{25}$$

Making J stationary with F and G one obtains

$$\frac{\partial J(F,G)}{\partial F}=\frac{2}{315}\frac{F(84r_1^3G^2M+84r_1G^2Nr_2^2-105r_3+28Ar_1F+12Br_1F^2)}{G}+\frac{1}{315}\frac{F^2(28Ar_1+24Br_1F)}{G}=0, \tag{26}$$

$$\frac{\partial J(F,G)}{\partial G}=\frac{168}{315}F^2(r_1^3M+r_1Nr_2^2)-\frac{1}{315}\frac{F^2(84r_1^3G^2M+84r_1G^2Nr_2^2-105r_3+28Ar_1F+12Br_1F^2)}{G^2}=0. \tag{27}$$

Solving eqs. (26) and (27), one can acquire

$$F=\frac{-35Ar_1\pm\sqrt{1225A^2r_1^2+7560Br_1r_3}}{36Br_1},$$
$$G=\pm\frac{1}{36}\frac{\sqrt{-2Br_1(Mr_1^2+Nr_2^2)(270Br_3+35A^2r_1-A\sqrt{35r_1(35A^2r_1+216Br_3)})}}{Br_1(Mr_1^2+Nr_2^2)}. \tag{28}$$

The domain of definition of the above relations is

$$1225A^2r_1^2+7560Br_1r_3>0,\qquad 2Br_1(Mr_1^2+Nr_2^2)(270Br_3+35A^2r_1-A\sqrt{35r_1(35A^2r_1+216Br_3)})<0. \tag{29}$$

Thus, finally, the 1-soliton solution to the nonlinear electrical transmission lines is given by

$$u(x,y,t) = \frac{-35Ar_1 \pm \sqrt{1225A^2r_1^2 + 7560Br_1r_3}}{36Br_1} \times \mathrm{sech}^2\left[\pm\frac{1}{36}\sqrt{\frac{-2Br_1(Mr_1^2+Nr_2^2)(270Br_3+35A^2r_1-A\sqrt{35r_1(35A^2r_1+216Br_3)})}{Br_1(Mr_1^2+Nr_2^2)}}(r_1x+r_2y-r_3t)\right]. \tag{30}$$

Finally, another singular wave solution can be considered as follows.

Case III:

$$u(\xi) = \frac{F}{D+\cosh(G\xi)}, \tag{31}$$

where F and G are unknown constants to be further determined. Upon putting (31) into (16) and carrying out the integration gives

$$\begin{aligned}
J &= \frac{F^2}{36G(D+1)^2(D-1)^5(D^2-1)^{\frac{3}{2}}}\left(3X(D-1)^3\Omega_1 + \sqrt{D^2-1}\Omega_2\right);\\
\Omega_1 &= -6G^2r_1D(D^2-1)(Mr_1^2+r_2^2N) + Br_1F^2D(2D^2+3) + 2Ar_1F(D^2-1)(2D^2+1) - 12r_3D(D^2-1),\\
\Omega_2 &= Br_1F^2(6D^2-D+3)(3D^2-D+2) + 6G^2r_1D(D^2-1)(9D^2+10D+13)(Mr_1^2+r_2^2N)\\
&\quad + 6Ar_1F(D+1)(4D^2-D+1)(D-1)^2 - 36r_3(D+1)^2(D-1)^4,\\
X &= \arctan\left(\frac{D-1}{\sqrt{D^2-1}}\right).
\end{aligned} \tag{32}$$

Making J stationary with F and G yields

$$\begin{aligned}
\frac{\partial J(F,G)}{\partial F} &= \frac{F}{18G(D+1)^2(D-1)^5(D^2-1)^{\frac{3}{2}}}\left(3X(D-1)^3\Omega_1 + \sqrt{D^2-1}\Omega_2\right)\\
&\quad + \frac{F^2}{36G(D+1)^2(D-1)^5(D^2-1)^{\frac{3}{2}}}\left(3X(D-1)^3\Omega_{1F} + \sqrt{D^2-1}\Omega_{2F}\right) = 0,\\
\Omega_{1F} &= 2Br_1FD(2D^2+3) + 2Ar_1(D^2-1)(2D^2+1),\\
\Omega_{2F} &= 2Br_1F(6D^2-D+3)(3D^2-D+2) + 6Ar_1(D+1)(4D^2-D+1)(D-1)^2,
\end{aligned} \tag{33}$$

$$\begin{aligned}
\frac{\partial J(F,G)}{\partial G} &= -\frac{F^2}{36G^2(D+1)^2(D-1)^5(D^2-1)^{\frac{3}{2}}}\left(3X(D-1)^3\Omega_1 + \sqrt{D^2-1}\Omega_2\right)\\
&\quad + \frac{F^2}{36G(D+1)^2(D-1)^5(D^2-1)^{\frac{3}{2}}}\left(3X(D-1)^3\Omega_{1G} + \sqrt{D^2-1}\Omega_{2G}\right) = 0,\\
\Omega_{1G} &= -12Gr_1D(D^2-1)(Mr_1^2+r_2^2N), \qquad \Omega_{2G} = 12Gr_1D(D^2-1)(9D^2+10D+13)(Mr_1^2+r_2^2N).
\end{aligned} \tag{34}$$

Solving eqs. (33) and (34), one can get

$$F=-\frac{1}{2}(D-1)^2(D+1)\left[5r_1XA(2D^2+1)(D-1)^2+5r_1\sqrt{D^2-1}A(4D^2-D+1)-\Omega_3^{\frac{1}{2}}\right];$$

$$\Omega_3=25A^2(D-1)r_1^2\Big[X^2(2D^2+1)^2(D-1)^3+2X\sqrt{D^2-1}(D-1)(2D^2+1)(4D^2-D+1)$$
$$+(D+1)(4D^2-D+1)^2\Big]+96B(D-1)r_1r_3\Big[3D^2X^2(2D^2+3)(D-1)^3$$
$$+DX\sqrt{D^2-1}(24D^4-21D^3+37D^2-23D+15)+(D+1)(6D^2-D+3)(3D^2-D+2)\Big], \tag{35}$$

$$G=-\frac{1}{3}(D-1)\frac{\sqrt{-3Dr_1\Omega_4\Omega_5}}{Dr_1\Omega_6},\qquad \Omega_4=-9DX^2(2D^2+3)(D-1)^5(Nr_2^2+r_1^2M)+6\sqrt{D^2-1}X(9D^5+D^4$$
$$+31D^3+4D^2+22D-3)(D-1)^2(Nr_2^2+r_1^2M)+(D+1)(6D^2-D+3)(3D^2-D+2)(9D^2+10D$$
$$+13)(Nr_2^2+r_1^2M),$$
$$\Omega_5=18r_3D^2X^2(D+1)(2D^2+3)(D-1)^4+Ar_1R\sqrt{D^2-1}DX(-49D^6+10+36D^5+18D^4+54D^7)$$
$$+6Dr_3(-12D^2+8D^3-1-5D^4-11D+18D^6+9D^5)-36r_3-27Ar_1R,$$
$$\Omega_6=-9DX^2(2D^2+3)(D-1)^5(Nr_2^2+r_1^2M)+6\sqrt{D^2-1}X(9D^5+D^4+31D^3+4D^2+22D-3)(D-1)^2(Nr_2^2$$
$$+r_1^2M)+(D+1)(6D^2-D+3)(3D^2-D+2)(9D^2+10D+13)(Nr_2^2+r_1^2M). \tag{36}$$

The domain of definition of above relations is

$$D^2-1>0,\qquad \Omega_3>0,\qquad Dr_1\Omega_4\Omega_5<0,\qquad Dr_1\Omega_6\neq 0. \tag{37}$$

Thus, one can state that the 1-soliton solution to the nonlinear electrical transmission lines is given by

$$u(x,y,t)=-\frac{\frac{1}{2}(D-1)^2(D+1)\left[5r_1XA(2D^2+1)(D-1)^2+5r_1\sqrt{D^2-1}A(4D^2-D+1)-\Omega_3^{\frac{1}{2}}\right]}{D+\cosh\left[\frac{1}{3}(D-1)\frac{\sqrt{-3Dr_1\Omega_4\Omega_5}}{Dr_1\Omega_6}(r_1x+r_2y-r_3t)\right]}. \tag{38}$$

3 The EEM

The EEM has been utilized to discover traveling wave solutions of nonlinear PDEs [47, 48]. Consider the following steps.

Step 1. Consider a general form of a PDE, say in two independent variables x and t as

$$\mathcal{N}(u,u_x,u_y,u_t,u_{xx},u_{tt},\ldots)=0. \tag{39}$$

In eq. (39) $u=u(x,t)$ is an unknown function, $\mathcal{N}$ is a polynomial in $u(x,t)$ and its various partial derivatives, in which the nonlinear terms and highest order derivatives are involved. The PDE can be converted to an ODE

$$\mathcal{Q}\left(U,r_1U',r_2U',-r_3U',r_1^2U'',r_3^2U'',\ldots\right)=0, \tag{40}$$

by the transformation $\xi=r_1x+r_2y-r_3t$ in which the wave variable is. Also, r_1, r_2 and r_3 are arbitrary constants to be determined later.

Step 2. Suppose the solution of the nonlinear equation (40) can be expressed by an exponential polynomial in $F(\xi)$ as

$$U(\xi)=\sum_{i=0}^{N}A_iF^i(\xi)+\sum_{i=1}^{N}B_iF^{-i}(\xi), \tag{41}$$

where $F(\xi)=\exp(-\Omega(\xi))$, $A_i(0\le i\le N)$, and $B_i(1\le i\le M)$ are constants to be determined, such that $A_N\neq 0$, $B_N\neq 0$, and, $\Omega=\Omega(\xi)$ gratifies the following ODE:

$$\Omega'=\mu F^{-1}(\xi)+F(\xi)+\lambda. \tag{42}$$

The exact solutions [49, 50] can be considered from eq. (42) as follows.

Solution 1: If $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$, then get to

$$\Omega(\xi) = \ln\left(-\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + E)\right) - \frac{\lambda}{2\mu}\right), \tag{43}$$

where E is integral constant.

Solution 2: If $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$, then achieve to

$$\Omega(\xi) = \ln\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\xi + E)\right) - \frac{\lambda}{2\mu}\right). \tag{44}$$

Solution 3: If $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$, the solution will be as

$$\Omega(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1}\right). \tag{45}$$

Solution 4: If $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$, the solution will be as

$$\Omega(\xi) = \ln\left(-\frac{2\lambda(\xi + E) + 4}{\lambda^2(\xi + E)}\right). \tag{46}$$

Solution 5: If $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$, the solution can be found at

$$\Omega(\xi) = \ln(\xi + E), \tag{47}$$

where A_i $(0 \leq i \leq N)$, B_i $(1 \leq i \leq M)$, λ and μ are constants to be determined. The N is a natural number which is determined by the homogeneous balance principle.

Step 3. Inserting a new solution from eq. (41) into eq. (40) along with eq. (42) and comparing the terms results in a set of nonlinear equations which by solving it using the Maple package, we will acquire new exact solutions of the fractional partial differential equation. Solving the algebraic equations including coefficients of $A_0, \ldots, A_N$, $B_1, \ldots, B_N$, r_1, r_2, r_3, λ, and μ into (41) one gets the exact solution of the considered problem.

3.1 Application of the EEM

This section is devoted to the application of the EEM to discover the exact solutions of eq. (6). By utilizing $u(x, y, t) = u(\xi)$ and $\xi = r_1 x + r_2 y - r_3 t$ eq. (6) can be reduced to the following ODE:

$$-r_3 u + A r_1 \frac{u^2}{2} + B r_1 \frac{u^3}{3} + (M r_1^3 + N r_1 r_2^2) u'' = 0. \tag{48}$$

Balancing u'' and u^3, we obtain $m = 1$; thus, (41) reduce to

$$u(\xi) = d_0 + d_1 \exp(-\Omega(\xi)) + e_1 \exp(\Omega(\xi)). \tag{49}$$

Putting (49) along with (42) into (48) and collecting all the coefficients of $Y^j = \exp(-j\Omega(\xi))$; $(j = 0, 1, \ldots, 6)$ and setting them to zero, the following results get:

Set I:

$$\mu = \mu, \qquad \lambda = \frac{1}{Be_1}\sqrt{A^2\mu^2 + 4\mu B^2 e_1^2}, \qquad \lambda^2 - 4\mu = \frac{A^2\mu^2}{B^2 e_1^2}, \qquad d_0 = \frac{Be_1\lambda - \mu A}{2\mu B}, \qquad d_1 = 0, \qquad e_1 = e_1,$$

$$r_1 = \frac{1}{\mu}\sqrt{-\frac{6Nr_2^2\mu^2 + Be_1^2}{6M}}, \qquad r_2 = r_2, \qquad r_3 = -\frac{A^2}{6\mu B}\sqrt{-\frac{6Nr_2^2\mu^2 + Be_1^2}{6M}}, \tag{50}$$

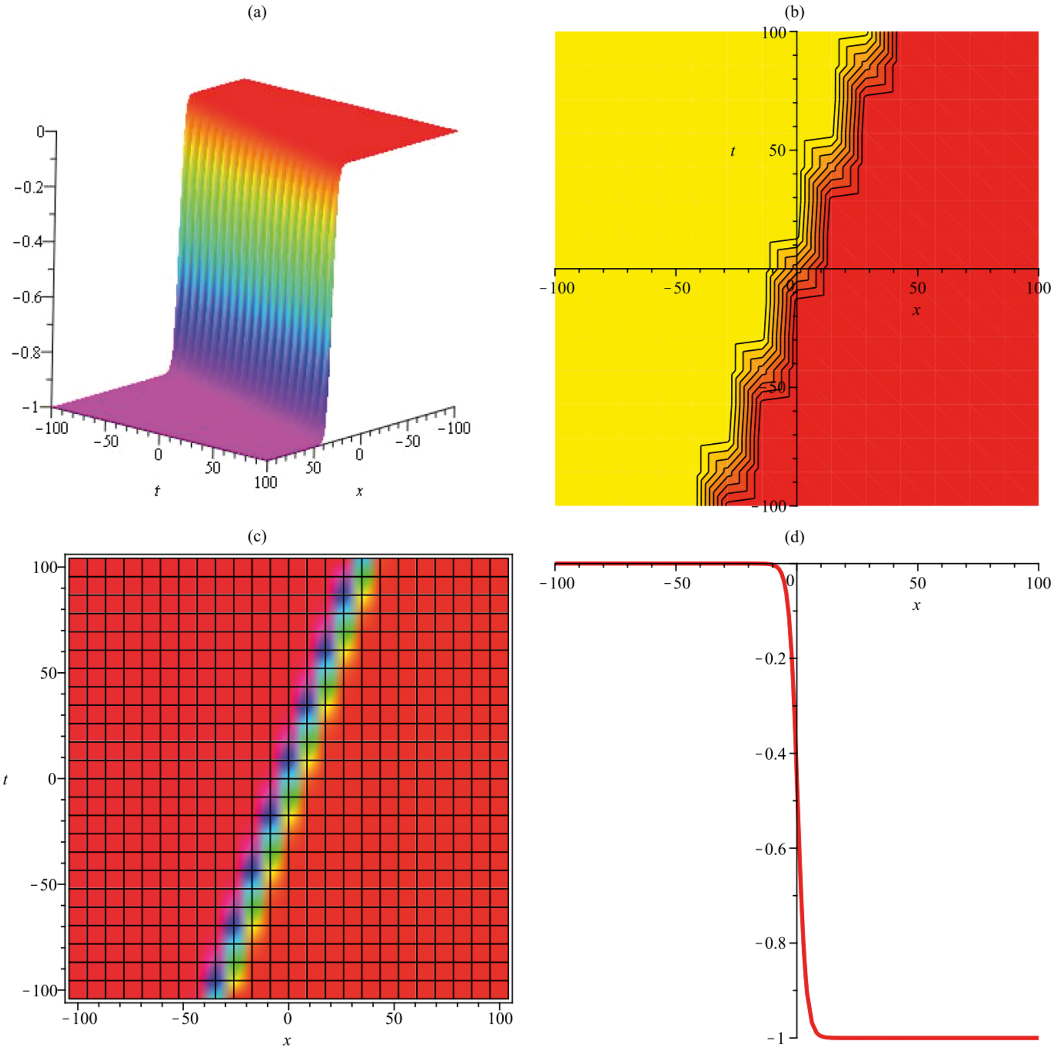

Fig. 2. Graph of eq. (51) by taking the parameters $A = B = 0.5$, $e_1 = 2$, $r_2 = 1$, $N = 1$, $M = -1$, $y = E = 0$, and (a) 3D plot, (b) contourplot, (c) density plot, and (d) 2D plot $t = 1$.

via (43), the exact solution can be found at

$$u_1(x,t) = -\frac{\mu A}{2\mu B} - \frac{A}{2B}\tanh\left[\frac{A\mu}{2Be_1}\left(\frac{1}{\mu}\sqrt{-\frac{6Nr_2^2\mu^2+Be_1^2}{6M}}x + r_2 y - \frac{A^2}{6\mu B}\sqrt{-\frac{6Nr_2^2\mu^2+Be_1^2}{6M}}t + E\right)\right] \qquad (51)$$

(see fig. 2).

Set II:

$$\mu = \frac{B^2 d_1^2\lambda^2 - A^2}{4B^2 d_1^2}, \qquad \lambda = \lambda, \qquad \lambda^2 - 4\mu = \frac{A^2}{B^2 d_1^2}, \qquad d_0 = -\frac{A}{2B} + \frac{d_1\lambda}{2}, \qquad d_1 = d_1, \qquad e_1 = 0,$$

$$r_1 = \sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}, \qquad r_2 = r_2, \qquad r_3 = -\frac{A^2}{6B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}. \qquad (52)$$

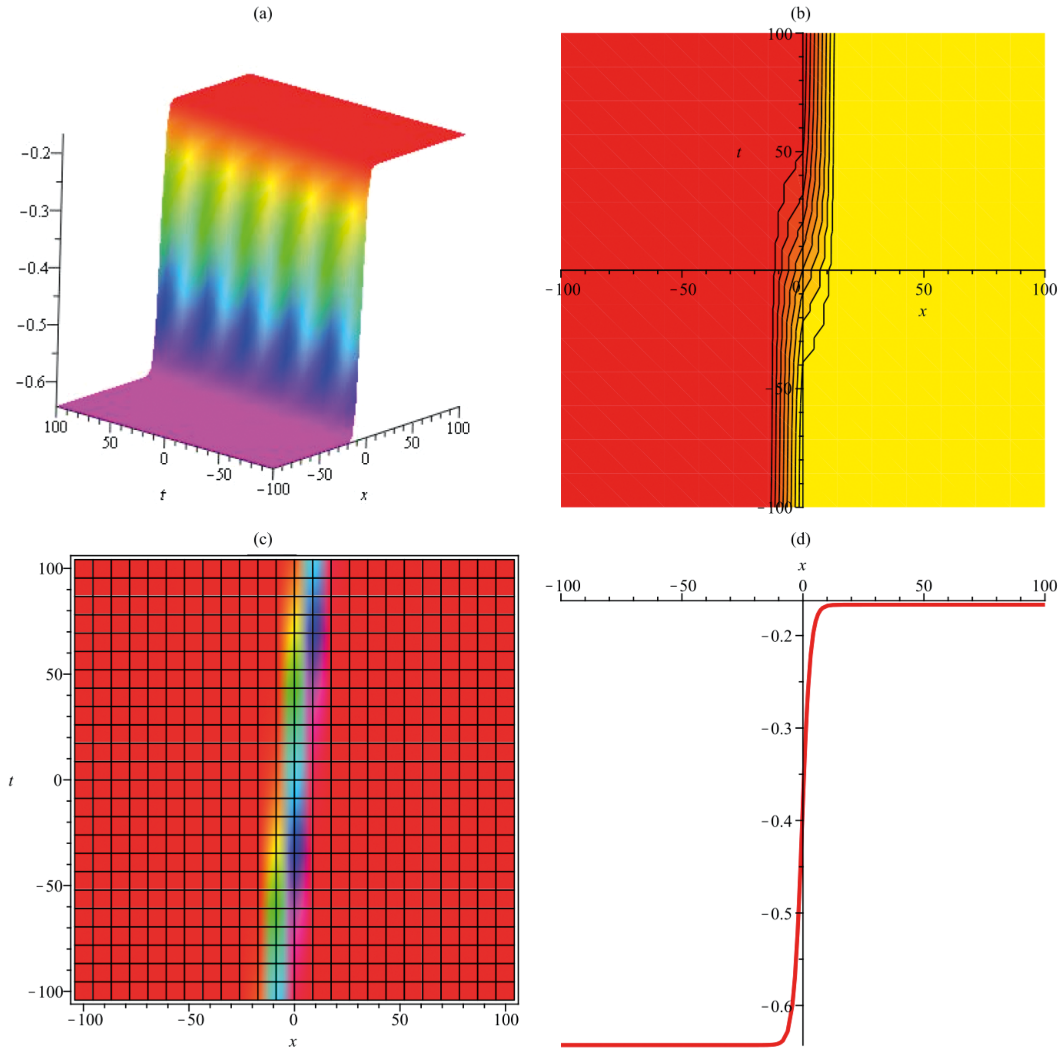

Fig. 3. Graph of eq. (53) by taking the parameters $A = B = 0.5$, $d_1 = 2$, $\lambda = 2$, $r_2 = 1$, $N = 1$, $M = -1$, $y = E = 0$, and (a) 3D plot, (b) contourplot, (c) density plot, and (d) 2D plot $t = 1$.

Via (43), the exact solution can be found at

$$u_2(x,t) = -\frac{A}{2B} + \frac{d_1\lambda}{2} + d_1 \left\{ -\frac{2A^2}{B^2 d_1^2 \lambda^2 - A^2} \tanh\left[\frac{A}{2Bd_1} \left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}\, x + r_2 y \right.\right.\right.$$
$$\left.\left.\left. - \frac{A^2}{6B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}\, t + E \right) \right] - \frac{2\lambda B^2 d_1^2}{B^2 d_1^2 \lambda^2 - A^2} \right\}^{-1} \tag{53}$$

(see fig. 3).

Set III:

$$\mu = \frac{A^2 - ABd_1\lambda}{8B^2 d_1^2}, \qquad \lambda = \lambda, \qquad \lambda^2 - 4\mu = \frac{d_1\lambda AB + 2B^2 d_1^2\lambda^2 - A^2}{2B^2 d_1^2},$$
$$d_0 = -\frac{A}{2B} + \frac{d_1\lambda}{2}, \qquad d_1 = d_1, \qquad e_1 = \frac{A^2 - ABd_1\lambda}{B^2 d_1},$$
$$r_1 = \sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}, \qquad r_2 = r_2, \qquad r_3 = -\frac{2A^2 + d_1AB\lambda - B^2 d_1^2\lambda^2}{12B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}. \tag{54}$$

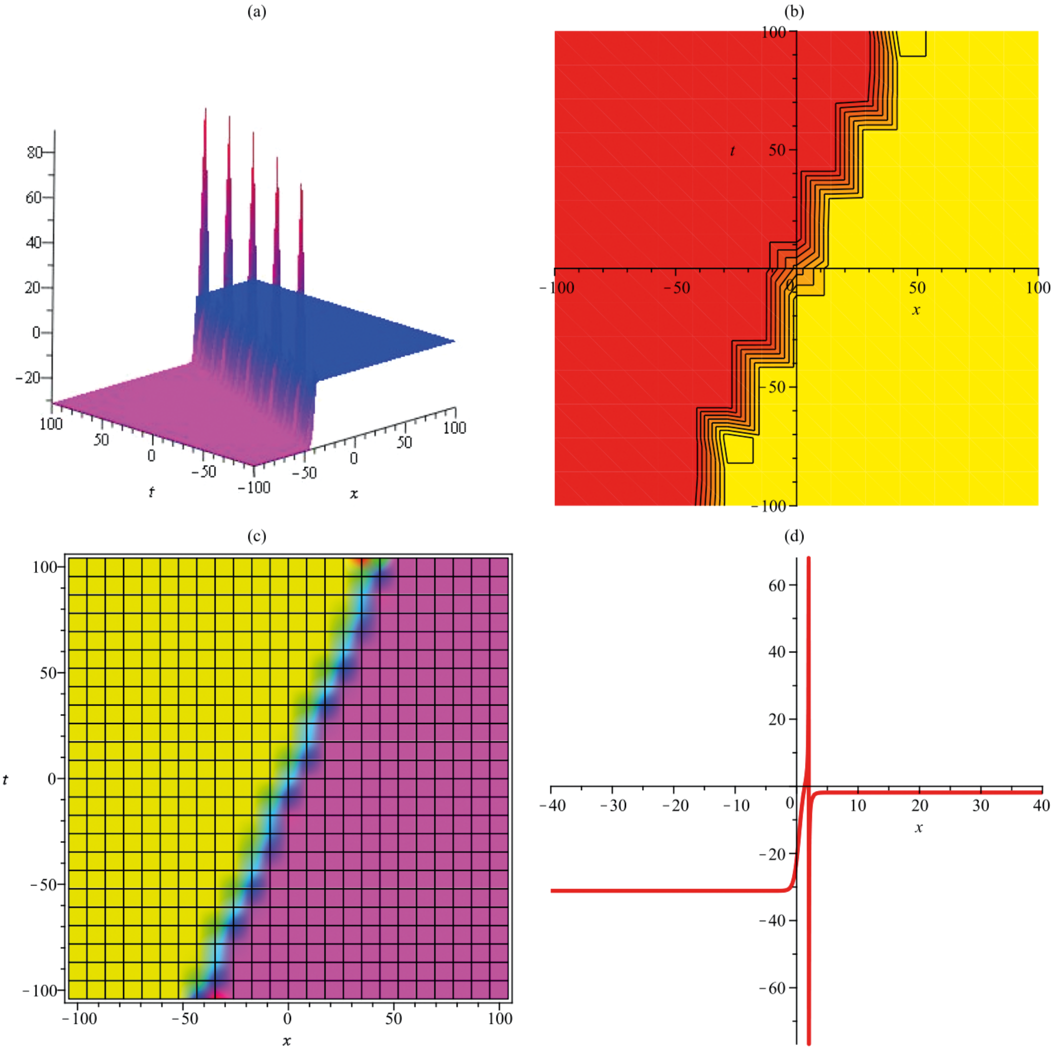

Fig. 4. Graph of eq. (55) by taking the parameters $A = B = 0.5$, $d_1 = 2$, $\lambda = 2$, $r_2 = 1$, $N = 1$, $M = -1$, $y = E = 0$, and (a) 3D plot, (b) contourplot, (c) density plot, and (d) 2D plot $t = 1$.

Via (43), the exact solution can be determined at

$$
\begin{aligned}
u_3(x,t) &= -\frac{A}{2B} + \frac{d_1\lambda}{2} + d_1 \Bigg\{ -\frac{2\sqrt{2}Bd_1\sqrt{d_1\lambda AB + 2B^2d_1^2\lambda^2 - A^2}}{A^2 - ABd_1\lambda} \\
&\times \tanh\left[\frac{\sqrt{d_1\lambda AB + 2B^2d_1^2\lambda^2 - A^2}}{2\sqrt{2}Bd_1}\left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}x + r_2 y - \frac{2A^2 + d_1AB\lambda - B^2d_1^2\lambda^2}{12B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}t + E\right)\right] \\
&- \frac{4\lambda B^2d_1^2}{A^2 - ABd_1\lambda}\Bigg\}^{-1} + \frac{A^2 - ABd_1\lambda}{B^2d_1}\Bigg\{ -\frac{2\sqrt{2}Bd_1\sqrt{d_1\lambda AB + 2B^2d_1^2\lambda^2 - A^2}}{A^2 - ABd_1\lambda} \\
&\times \tanh\left[\frac{\sqrt{d_1\lambda AB + 2B^2d_1^2\lambda^2 - A^2}}{2\sqrt{2}Bd_1}\left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}x + r_2 y - \frac{2A^2 + d_1AB\lambda - B^2d_1^2\lambda^2}{12B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}t + E\right)\right] \\
&- \frac{4\lambda B^2d_1^2}{A^2 - ABd_1\lambda}\Bigg\} \qquad (55)
\end{aligned}
$$

(see fig. 4).

Also, via (44), the exact solution can be determined at

$$u_4(x,t) = -\frac{A}{2B} + \frac{d_1\lambda}{2} + d_1 \left\{ \frac{2\sqrt{2}Bd_1\sqrt{A^2 - d_1\lambda AB - 2B^2d_1^2\lambda^2}}{A^2 - ABd_1\lambda} \right.$$

$$\times \tan\left[\frac{\sqrt{A^2 - d_1\lambda AB - 2B^2d_1^2\lambda^2}}{2\sqrt{2}Bd_1} \left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}x + r_2 y \right.\right.$$

$$\left.\left.\left. - \frac{2A^2 + d_1AB\lambda - B^2d_1^2\lambda^2}{12B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}t + E \right)\right] - \frac{4\lambda B^2 d_1^2}{A^2 - ABd_1\lambda} \right\}^{-1}$$

$$+ \frac{A^2 - ABd_1\lambda}{B^2 d_1} \left\{ \frac{2\sqrt{2}Bd_1\sqrt{A^2 - d_1\lambda AB - 2B^2d_1^2\lambda^2}}{A^2 - ABd_1\lambda} \right.$$

$$\times \tan\left[\frac{\sqrt{A^2 - d_1\lambda AB - 2B^2d_1^2\lambda^2}}{2\sqrt{2}Bd_1} \left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}x + r_2 y \right.\right.$$

$$\left.\left.\left. - \frac{2A^2 + d_1AB\lambda - B^2d_1^2\lambda^2}{12B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}t + E \right)\right] - \frac{4\lambda B^2 d_1^2}{A^2 - ABd_1\lambda} \right\}. \tag{56}$$

Moreover, via (45), the exact solution can be determined at

$$u_5(x,t) = d_1 \left(\frac{\frac{A}{Bd_1}}{\exp\left[\frac{A}{Bd_1} \left(\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}x + r_2 y - \frac{A^2}{6B}\sqrt{-\frac{6Nr_2^2 + Bd_1^2}{6M}}t + E \right) \right] - 1} \right). \tag{57}$$

Set IV:

$$\mu = \mu, \qquad \lambda = \frac{A}{6(Nr_2^2 + Mr_1^2)}\sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}}, \qquad \lambda^2 - 4\mu = -\frac{A^2 + 24B\mu(Nr_2^2 + Mr_1^2)}{6B(Nr_2^2 + Mr_1^2)}, \qquad d_0 = -\frac{A}{B},$$

$$d_1 = \sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}}, \qquad e_1 = \mu\sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}},$$

$$r_1 = r_1, \qquad r_2 = r_2, \qquad r_3 = -\frac{r_1(A^2 + 24B\mu(Nr_2^2 + Mr_1^2))}{6B}. \tag{58}$$

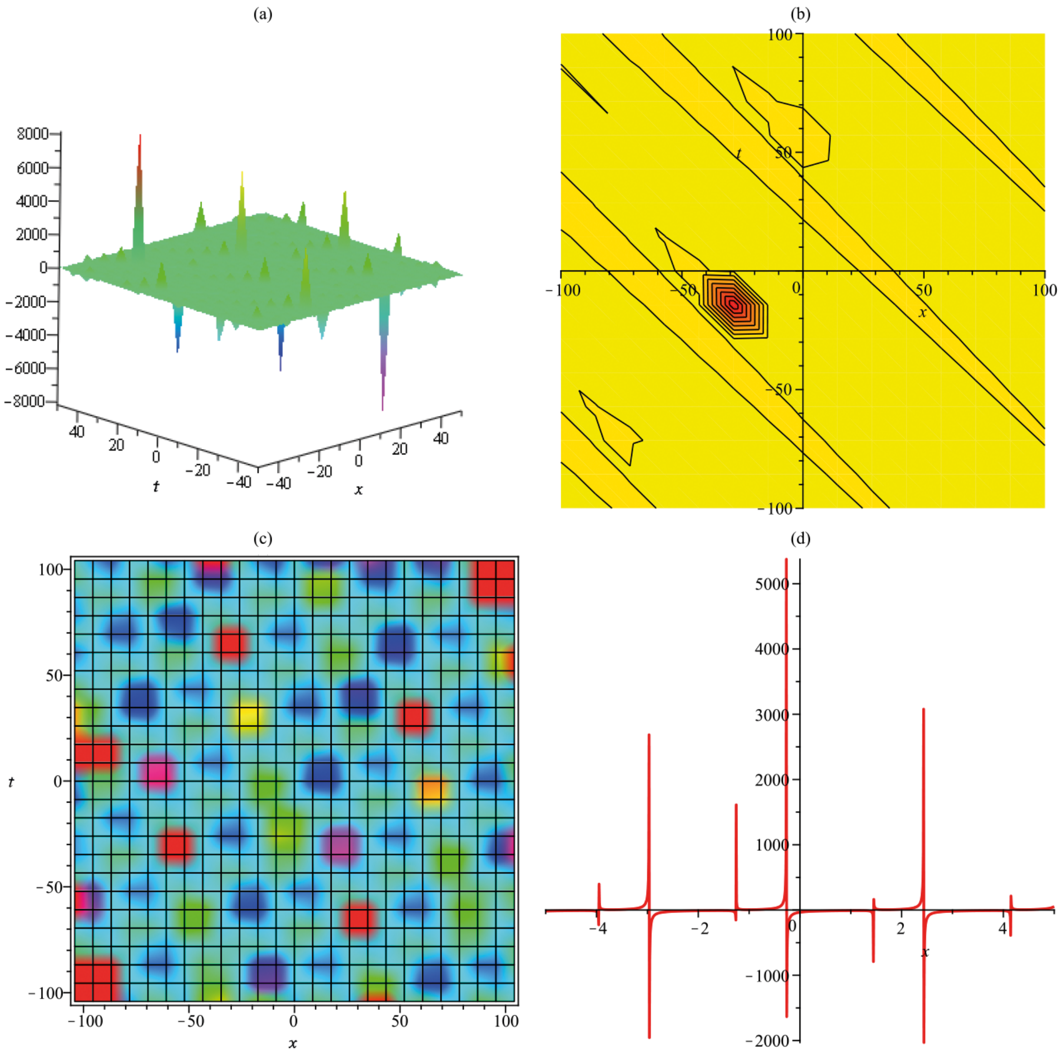

Fig. 5. Graph of eq. (59) by taking the parameters $A = 2$, $B = 0.5$, $d_1 = \lambda = 1$, $r_2 = 1$, $N = 1$, $M = -1$, $y = E = 0$, and (a) 3D plot, (b) contourplot, (c) density plot, and (d) 2D plot $t = 10$.

Via (43) and (44), the exact solutions, respectively, will be as

$$u_6(x,t) = -\frac{A}{B} + \sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}} \left\{ -\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E) \right) - \frac{\lambda}{2\mu} \right\}^{-1} + \mu\sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}} \left\{ -\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + E) \right) - \frac{\lambda}{2\mu} \right\}, \tag{59}$$

$$u_7(x,t) = -\frac{A}{B} + \sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}} \left\{ \frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}(\xi + E) \right) - \frac{\lambda}{2\mu} \right\}^{-1} + \mu\sqrt{-\frac{6Nr_2^2 + 6Mr_1^2}{B}} \left\{ \frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}(\xi + E) \right) - \frac{\lambda}{2\mu} \right\}, \tag{60}$$

in which $\xi = r_1 x + r_2 y - \frac{r_1(A^2 + 24B\mu(Nr_2^2 + Mr_1^2))}{6B} t$ and E is a free constant (see figs. 5 and 6).

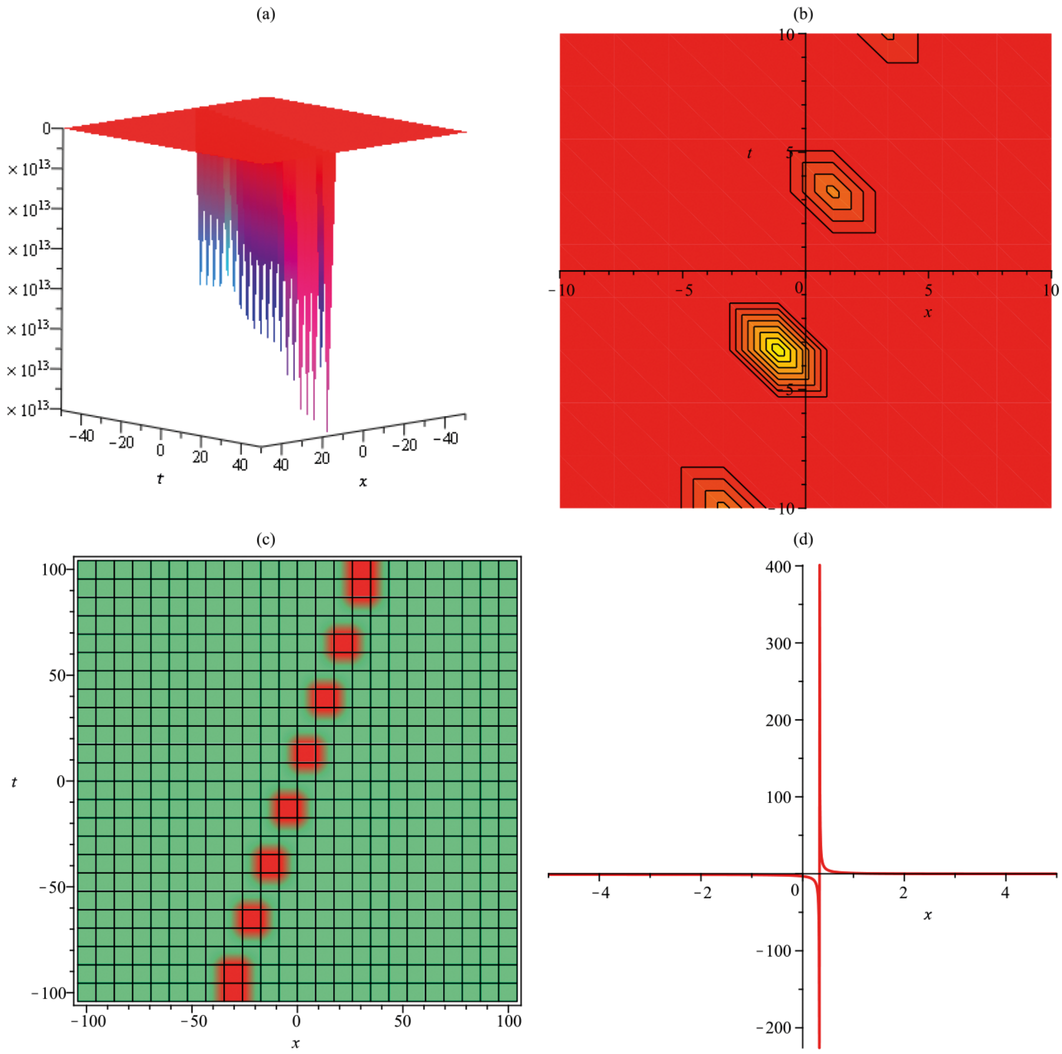

Fig. 6. Graph of eq. (60) by taking the parameters $A = B = 2$, $d_1 = \lambda = 1$, $r_2 = 1$, $N = 1$, $M = -1$, $y = E = 0$, and (a) 3D plot, (b) contourplot, (c) density plot, and (d) 2D plot $t = 1$.

3.2 Discussion and remarks

With the aid of the EEM, we obtain solutions including exponential, hyperbolic and trigonometric functions forms of the modified ZK equation. Whereas in [37], Yu and Feng used Darboux transformation and obtained one-soliton solution and other soliton solutions. Sardar and coworkers acquired various types of solutions which are solitary, shock, singular, periodic, rational and kink-shaped solitons obtained by using the special analytical methods given in [3]. Meanwhile, in this paper there are results, including the kink-singular soliton solution, the hyperbolic solution, the trigonometric solution, dark and bright soliton solutions which agree with the results of [3]. Moreover, in the current paper the better results are obtained when comparing these travelling wave solutions with the solutions achieved by Krishnan and Biswas [6]. These travelling wave solutions are shown to obey the modified ZK equation with the aid of Maple 13. To the best of our knowledge, the application of the EEM to the eq. (6) has not been submitted to literature so far.

4 Description of the ITEM

In this section, all procedures of the ITEM for solving the nonlinear PDEs are described. The essential steps of this method are as follows.

Step 1. Consider a general form of a PDE, say in two independent variables x and t as

$$\mathcal{N}(u, u_x, u_y, u_t, u_{xx}, u_{tt}, \ldots) = 0. \tag{61}$$

In eq. (61) $u = u(x,t)$ is an unknown function, $\mathcal{N}$ is a polynomial in $u(x,t)$ and its various partial derivatives, in which the nonlinear terms and highest-order derivatives are involved. The PDE can be converted to an ODE

$$\mathcal{Q}\left(U, r_1U', r_2U', -r_3U', r_1^2U'', r_3^2U'', \ldots\right) = 0, \tag{62}$$

by the transformation $\xi = r_1x + r_2y - r_3t$ in which the wave variable is. Also, r_1, r_2 and r_3 are arbitrary constants to be determined later.

Step 2. Suppose the traveling wave solution of eq. (62) can be expressed as follows:

$$u(\xi) = S(\phi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\phi/2)\right]^k, \tag{63}$$

where $A_k(0 \le k \le m)$, $B_k(1 \le k \le m)$ and p are constants to be determined, such that $A_m \neq 0$, $B_m \neq 0$ and $\phi = \phi(\xi)$ gratifies the following ODE:

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c, \tag{64}$$

where m is a natural number which is determined by the homogeneous balance principle.

Step 3. Inserting a new solution from eq. (63) into eq. (62) along with eq. (64) and comparing the terms results in a set of nonlinear equations which by solving it using the Maple package, we will acquire new exact solutions of the fractional partial differential equation. Solving the algebraic equations including coefficients of $A_0, \ldots, A_N$, $B_1, \ldots, B_N$, p, a, b, c, and r_1, r_2, r_3 into (41) one gets the exact solution of the considered problem.

Consider the following special solutions of eqs. (64):

Family 1: When $\Delta = a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\overline{\xi}\right)\right]$.

Family 2: When $\Delta = a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\overline{\xi}\right)\right]$.

Family 3: When $\Delta = a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ and $a = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\overline{\xi}\right)\right]$.

Family 4: When $a = 0$ and $c = 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{e^{2b\overline{\xi}}-1}{e^{2b\overline{\xi}}+1}, \frac{2e^{b\overline{\xi}}}{e^{2b\overline{\xi}}+1}\right]$.

Family 5: When $c = a$, then $\phi(\xi) = -2\tan^{-1}\left[\frac{(a+b)e^{b\overline{\xi}}-1}{(a-b)e^{b\overline{\xi}}-1}\right]$.

Family 6: When $b = -c$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{ae^{a\overline{\xi}}}{1-ce^{a\overline{\xi}}}\right]$.

Family 7: When $b = 0$ and $a = c$, then $\phi(\xi) = -2\tan^{-1}\left[\frac{c\overline{\xi}+2}{c\overline{\xi}}\right]$.

Family 8: When $a = 0$ and $b = c$, then $\phi(\xi) = 2\tan^{-1}\left[c\overline{\xi}\right]$.

Family 9: When $a = 0$ and $b = -c$, then $\phi(\xi) = -2\tan^{-1}\left[\frac{1}{c\overline{\xi}}\right]$.

Here, $\overline{\xi} = \xi + C$, p, A_0, A_k, B_k $(k = 1, 2, \ldots, m)$, a, b and c are constants to be determined later.

4.1 Application of the ITEM

In this subsection, the ITEM will be performed to handle the MZK equation for acquiring new soliton solutions. To this end, we use the transformation $u(x,y,t) = u(\xi)$ and $\xi = r_1x + r_2y - r_3t$ to reduce eq. (6) to the following nonlinear ODE:

$$-r_3u + Ar_1\frac{u^2}{2} + Br_1\frac{u^3}{3} + (Mr_1^3 + Nr_1r_2^2)u'' = 0. \tag{65}$$

Balancing u'' and u^3, we obtain $m = 1$; thus, (63) reduce to

$$u(\xi) = A_0 + A_1\tan(\phi/2) + B_1\cot(\phi/2). \tag{66}$$

Substituting (66) along with (64) into (65) and collecting all the coefficients of $Y^j = \tan^j(\phi/2)$; $(j = 0, 1, \ldots, 6)$ and inserting them to zero, the algebraic equations conclude:

$$\begin{cases} Y^0\colon & r_1B_1(2BB_1^2 + 3r_1^2b^2M + 6r_1^2bcM + 3r_1^2c^2M + 6bcr_2^2N + 3c^2r_2^2N + 3b^2r_2^2N) = 0, \\ Y^1\colon & 3r_1B_1(2B_1A_0B + B_1A + 3r_1^2baM + 3r_1^2caM + 3bar_2^2N + 3acr_2^2N) = 0, \\ Y^2\colon & 3B_1(2Br_1A_1B_1 + 2Nr_1r_2^2a^2 + r_1Nr_2^2c^2 - r_1Nr_2^2b^2 - 2r_3 - Mr_1^3b^2 \\ & +2Br_1A_0^2 + 2Ar_1A_0 + Mr_1^3c^2 + 2Mr_1^3a^2) = 0, \\ Y^3\colon & 3r_1B_1(-r_1^2baM + r_1^2caM - bar_2^2N + acr_2^2N + 4A_1A_0B) + A_0(2Br_1A_0^2 + 3Ar_1A_0 - 6r_3) \\ & +3r_1A_1(r_1^2baM + r_1^2caM + bar_2^2N + acr_2^2N + 2B_1A) = 0, \\ Y^4\colon & 3A_1(2Br_1A_1B_1 + N(2r_1r_2^2a^2 + r_1r_2^2c^2 - r_1r_2^2b^2) - 2r_3 + M(r_1^3c^2 + 2r_1^3a^2 - r_1^3b^2) + 2Br_1A_0^2 + 2Ar_1A_0) = 0, \\ Y^5\colon & 3r_1A_1(A_1A + 2A_1A_0B - 3r_1^2baM + 3r_1^2caM + 3acr_2^2N - 3bar_2^2N) = 0, \\ Y^6\colon & r_1A_1(2BA_1^2 + 3r_1^2c^2M - 6r_1^2bcM + 3r_1^2b^2M - 6bcr_2^2N + 3c^2r_2^2N + 3b^2r_2^2N) = 0. \end{cases} \tag{67}$$

Solving the above algebraic (67), the following sets of coefficients can be written as

Set I:

$$\begin{aligned} &a = \pm\sqrt{A^2(b+c)^2 - 4B^2B_1^2(b^2-c^2)}, \qquad b = b, \qquad c = c, \qquad \Delta = a^2 + b^2 - c^2 = \frac{A^2(b+c)^2}{4B^2B_1^2}, \\ &A_0 = -\frac{Ab - 2aBB_1 + Ac}{2B(b+c)}, \\ &A_1 = 0, \qquad B_1 = B_1, \qquad r_1 = \pm\frac{\sqrt{-3M(3Nr_2^2(b+c)^2 + 2BB_1^2)}}{3M(b+c)}, \qquad r_1 = r_2, \qquad r_3 = -\frac{A^2r_1}{6B}. \end{aligned} \tag{68}$$

Set II:

$$\begin{aligned} &a = \pm\sqrt{A^2(b-c)^2 - 4B^2A_1^2(b^2-c^2)}, \qquad b = b, \qquad c = c, \qquad \Delta = a^2 + b^2 - c^2 = \frac{A^2(b-c)^2}{4B^2A_1^2}, \\ &A_0 = -\frac{Ab + 2aBA_1 - Ac}{2B(b-c)}, \\ &A_1 = A_1, \qquad B_1 = 0, \qquad r_1 = \pm\frac{\sqrt{-3M(3Nr_2^2(b-c)^2 + 2BA_1^2)}}{3M(b-c)}, \qquad r_1 = r_2, \qquad r_3 = -\frac{A^2r_1}{6B}. \end{aligned} \tag{69}$$

Set III:

$$\begin{aligned} &a = 0, \qquad b = b, \qquad c = c, \qquad A_0 = -\frac{A}{2B}, \qquad A_1 = \frac{A}{2B}\sqrt{-\frac{b-c}{2(b+c)}}, \qquad B_1 = \frac{A}{4B}\sqrt{-\frac{2(b+c)}{b-c}}, \\ &r_1 = \pm\frac{\sqrt{3MB(b^2-c^2)(A^2 - 12NBr_2^2(b^2-c^2))}}{6MB(b^2-c^2)}, \qquad r_1 = r_2, \qquad r_3 = -\frac{A^2r_1}{6B}. \end{aligned} \tag{70}$$

Set IV:

$$\begin{aligned} &a = 0, \qquad b = b, \qquad c = c, \qquad A_0 = -\frac{A}{2B}, \qquad A_1 = \frac{A}{4B}\sqrt{\frac{b-c}{b+c}}, \qquad B_1 = \frac{A}{4B}\sqrt{\frac{b+c}{b-c}}, \\ &r_1 = \pm\frac{\sqrt{-6MB(b^2-c^2)(A^2 - 24NBr_2^2(b^2-c^2))}}{12MB(b^2-c^2)}, \qquad r_1 = r_2, \qquad r_3 = -\frac{A^2r_1}{6B}. \end{aligned} \tag{71}$$

Set V:

$$a=3\sqrt{\frac{c^2-b^2}{5}}\,,\qquad b=b,\qquad c=c,\qquad A_0=-\frac{5A}{4B}\,,\qquad A_1=-\frac{5Aa}{12B(b+c)}\,,\qquad B_1=\frac{5Aa}{12B(b-c)}\,,$$
$$r_1=\pm\frac{\sqrt{6MB(b^2-c^2)(5A^2-24NBr_2^2(b^2-c^2))}}{12MB(b^2-c^2)}\,,\qquad r_1=r_2,\qquad r_3=\frac{7A^2r_1}{48B}\,. \tag{72}$$

Set VI:

$$a=\frac{A(b-c)}{2BA_1}\,,\qquad b=b,\qquad c=c,\qquad A_0=-\frac{A}{B}\,,\qquad A_1=A_1,\qquad B_1=-\frac{(b+c)A_1}{b-c}\,,$$
$$r_1=\pm\frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M}\,,\qquad r_1=r_2,\qquad r_3=-\frac{(A^2(b-c)+4BA_1^2(b+c))r_1}{6B(b-c)}\,. \tag{73}$$

Set VII:

$$a=-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1}\,,\qquad b=b,\qquad c=c,\qquad A_0=\frac{4(b+c)BA_1^2}{A(b-c)}\,,\qquad A_1=A_1,$$
$$B_1=-\frac{(b+c)A_1}{b-c},\qquad r_1=r_2,$$
$$r_1=\pm\frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M},\qquad r_3=-\frac{(A^2(b-c)-8B^2A_1^2(b+c))(A^2(b-c)+4B^2A_1^2(b+c))r_1}{6B(b-c)^2A^2}\,. \tag{74}$$

To Set I, the following solution results:

$$u(\xi)=-\frac{Ab-2aBB_1+Ac}{2B(b+c)}+B_1\cot(\phi/2). \tag{75}$$

To Set II, the following solution results:

$$u(\xi)=-\frac{Ab+2aBA_1-Ac}{2B(b-c)}+A_1\tan(\phi/2). \tag{76}$$

To Set III, the following solution concludes:

$$u(\xi)=-\frac{A}{2B}+\frac{A}{2B}\sqrt{-\frac{b-c}{2(b+c)}}\tan(\phi/2)+\frac{A}{4B}\sqrt{-\frac{2(b+c)}{b-c}}\cot(\phi/2). \tag{77}$$

To Set IV, the following solution results:

$$u(\xi)=-\frac{A}{2B}+\frac{A}{4B}\sqrt{\frac{b-c}{b+c}}\tan(\phi/2)+\frac{A}{4B}\sqrt{\frac{b+c}{b-c}}\cot(\phi/2). \tag{78}$$

To Set V, the following solution results:

$$u(\xi)=-\frac{5A}{4B}-\frac{5A}{4B}\sqrt{\frac{c-b}{5(b+c)}}\tan(\phi/2)+\frac{5A}{4B}\sqrt{-\frac{b+c}{5(b-c)}}\cot(\phi/2). \tag{79}$$

To Set VI, the following solution concludes:

$$u(\xi)=-\frac{A}{B}+A_1\tan(\phi/2)-\frac{(b+c)A_1}{b-c}\cot(\phi/2). \tag{80}$$

To Set VII, the following solution results:

$$u(\xi)=\frac{4(b+c)BA_1^2}{A(b-c)}+A_1\tan(\phi/2)-\frac{(b+c)A_1}{b-c}\cot(\phi/2). \tag{81}$$

For different values of a, b and c, we have the following results:

Result 1: If we take Family 2, then we have the following (75) solution as

$$u(\xi) = -\frac{Ab - 2BB_1\sqrt{A^2(b+c)^2 - 4B^2B_1^2(b^2-c^2)} + Ac}{2B(b+c)} + B_1\left[\frac{\sqrt{A^2(b+c)^2 - 4B^2B_1^2(b^2-c^2)}}{b-c} + \frac{A(b+c)}{2BB_1(b-c)}\tanh\left(\frac{A(b+c)}{4BB_1(b-c)}\widetilde{\xi}\right)\right]^{-1}, \quad (82)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{-3M(3Nr_2^2(b+c)^2+2BB_1^2)}}{3M(b+c)}x + r_2 y - \frac{A^2\sqrt{-3M(3Nr_2^2(b+c)^2+2BB_1^2)}}{18MB(b+c)}t + C.$

Result 2: If we take Family 2, then we have the following (76) solution as

$$u(\xi) = -\frac{Ab + 2BA_1\sqrt{A^2(b-c)^2 - 4B^2A_1^2(b^2-c^2)} - Ac}{2B(b-c)} + A_1\left[\frac{\sqrt{A^2(b-c)^2 - 4B^2A_1^2(b^2-c^2)}}{b-c} + \frac{A}{2BA_1}\tanh\left(\frac{A}{4BA_1}\widetilde{\xi}\right)\right], \quad (83)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{-3M(3Nr_2^2(b-c)^2+2BA_1^2)}}{3M(b-c)}x + r_2 y - \frac{A^2\sqrt{-3M(3Nr_2^2(b-c)^2+2BA_1^2)}}{18MB(b+c)}t + C.$

Result 3: If we take Family 3 and 4, then we have the following (77) solutions, respectively, as

$$u(\xi) = -\frac{A}{2B} + \frac{A}{4B}\sqrt{-2}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\xi}\right) + \frac{A}{4B}\sqrt{-2}\coth\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\xi}\right), \quad (84)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{3MB(b^2-c^2)(A^2-12NBr_2^2(b^2-c^2))}}{6MB(b^2-c^2)}x + r_2 y - \frac{\sqrt{3MB(b^2-c^2)(A^2-12NBr_2^2(b^2-c^2))}A^2}{36MB^2(b^2-c^2)}t + C,$

$$u(\xi) = -\frac{A}{2B} + \frac{A}{4B}\sqrt{-2}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\widetilde{\xi}}-1}{e^{2b\widetilde{\xi}}+1}, \frac{2e^{b\widetilde{\xi}}}{e^{2b\widetilde{\xi}}+1}\right]\right) + \frac{A}{4B}\sqrt{-2}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\widetilde{\xi}}-1}{e^{2b\widetilde{\xi}}+1}, \frac{2e^{b\widetilde{\xi}}}{e^{2b\widetilde{\xi}}+1}\right]\right), \quad (85)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{3MB(A^2-12NBr_2^2b^2)}}{6MBb}x + r_2 y - \frac{\sqrt{3MB(A^2-12NBr_2^2b^2)}A^2}{36MB^2b}t + C.$

Result 4: If we take Family 3 and 4, then we have the following (78) solutions, respectively, as

$$u(\xi) = -\frac{A}{2B} + \frac{A}{4B}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\xi}\right) + \frac{A}{4B}\coth\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\xi}\right), \quad (86)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{-6MB(b^2-c^2)(A^2-24NBr_2^2(b^2-c^2))}}{12MB(b^2-c^2)}x + r_2 y - \frac{\sqrt{-6MB(b^2-c^2)(A^2-24NBr_2^2(b^2-c^2))}A^2}{36MB^2(b^2-c^2)}t + C,$

$$u(\xi) = -\frac{A}{2B} + \frac{A}{4B}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\widetilde{\xi}}-1}{e^{2b\widetilde{\xi}}+1}, \frac{2e^{b\widetilde{\xi}}}{e^{2b\widetilde{\xi}}+1}\right]\right) + \frac{A}{4B}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\widetilde{\xi}}-1}{e^{2b\widetilde{\xi}}+1}, \frac{2e^{b\widetilde{\xi}}}{e^{2b\widetilde{\xi}}+1}\right]\right), \quad (87)$$

where $\widetilde{\xi} = \pm\frac{\sqrt{-6MB(A^2-24NBr_2^2b^2)}}{12MBb}x + r_2 y - \frac{\sqrt{-6MB(A^2-24NBr_2^2b^2)}A^2}{36MB^2b}t + C.$

Result 5: If we take Family 1 and 2, then we have the following (79) solutions, respectively, as

$$u(\xi) = -\frac{5A}{4B} - \frac{\sqrt{5}A}{4B}\left[3 - \sqrt{-\frac{4}{5}}\tan\left(\sqrt{\frac{b^2-c^2}{5}}\widetilde{\xi}\right)\right] + \frac{\sqrt{5}A}{4B}\left[3 - \sqrt{-\frac{4}{5}}\tan\left(\sqrt{\frac{b^2-c^2}{5}}\widetilde{\xi}\right)\right]^{-1}, \quad (88)$$

$$u(\xi) = -\frac{5A}{4B} - \frac{\sqrt{5}A}{4B}\left[3 + \sqrt{\frac{4}{5}}\tanh\left(\sqrt{\frac{c^2-b^2}{5}}\widetilde{\xi}\right)\right] + \frac{\sqrt{5}A}{4B}\left[3 + \sqrt{\frac{4}{5}}\tanh\left(\sqrt{\frac{c^2-b^2}{5}}\widetilde{\xi}\right)\right]^{-1}, \quad (89)$$

where $\widetilde{\xi} = \pm \frac{\sqrt{-6MB(b^2-c^2)(A^2-24NBr_2^2(b^2-c^2))}}{12MB(b^2-c^2)} x + r_2 y + \frac{7\sqrt{-6MB(b^2-c^2)(A^2-24NBr_2^2(b^2-c^2))}A^2}{576MB^2(b^2-c^2)} t + C$.

Result 6: If we take Family 6, then we have the following (80) solutions, as

$$u(\xi) = -\frac{A}{B} + \left[\frac{\frac{A(b-c)}{2B} e^{\frac{A(b-c)}{2BA_1}\bar{\xi}}}{1 - ce^{\frac{A(b-c)}{2BA_1}\bar{\xi}}}\right] - \frac{(b+c)A_1}{b-c}\left[\frac{\frac{A(b-c)}{2BA_1} e^{\frac{A(b-c)}{2BA_1}\bar{\xi}}}{1 - ce^{\frac{A(b-c)}{2BA_1}\bar{\xi}}}\right]^{-1}, \tag{90}$$

where $\widetilde{\xi} = \pm \frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M} x + r_2 y - \frac{(A^2(b-c)+4BA_1^2(b+c))}{6B(b-c)} \frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M} t + C$,

$$u(\xi) = -\frac{A}{B} - \frac{A}{2B}\left[\frac{c\bar{\xi}+2}{c\bar{\xi}} + \frac{c\bar{\xi}}{c\bar{\xi}+2}\right], \tag{91}$$

where $\widetilde{\xi} = \pm \frac{\sqrt{-3M(2BA^2+3Nr_2^2c^2)}}{3M} x + r_2 y - \frac{(A^2-4BA_1^2)}{6B} \frac{\sqrt{-3M(2BA^2+3Nr_2^2c^2)}}{3M} t + C$.

Result 7: If we take Family 6, then we have the following (81) solutions, as

$$\begin{aligned} u(\xi) = {} & \frac{4(b+c)BA_1^2}{A(b-c)} + \left[\frac{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2AB} e^{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1}\bar{\xi}}}{1 - ce^{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1}\bar{\xi}}}\right] \\ & - \frac{(b+c)A_1}{b-c}\left[\frac{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1} e^{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1}\bar{\xi}}}{1 - ce^{-\frac{A^2(b-c)+8B^2A_1^2(b+c)}{2ABA_1}\bar{\xi}}}\right]^{-1}, \end{aligned} \tag{92}$$

where $\widetilde{\xi} = \pm \frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M} x + r_2 y - \frac{(A^2(b-c)-8B^2A_1^2(b+c))(A^2(b-c)+4B^2A_1^2(b+c))}{6B(b-c)^2A^2} \frac{\sqrt{-3M(2BA^2+3Nr_2^2(b-c)^2)}}{3M} t + C$.

5 Conclusion

The basic goal of this work was to execute the SIVPM, the EEM and the ITEM methods for exactly solving the equation of nonlinear electrical transmission lines described by a MZK equation. As a result, we received many new exact soliton solutions for the equation of nonlinear electrical transmission lines which are expressed by rational, hyperbolic, trigonometric and exponential functions forms.

The rational, hyperbolic, trigonometric and exponential functions forms are based on arbitrary parameters which can be zero or nonzero. To the best of our knowledge, the received results have not been reported in other studies on the MZK equation. Therefore, the obtained results show that the implemented method along with the symbolic computation package suggests a promising, robust, and the well-built mathematical tool for handling any nonlinear PDEs arising in mathematical physics and other applied fields.

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Conflict of interest

The authors also declare that there is no conflict of interest.

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