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On some results of third-grade non-Newtonian fluid flow between two parallel plates

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Abstract. In this paper, the main focus is to discuss the non-Newtonian third-grade fluid flow between two parallel plates. The mathematical model is consequent to the continuity and momentum equations. Three cases have been discussed such as the plane Couette flow, the plane Poiseuille flow and the last one is a plane Couette-Poiseuille flow. The modeled differential equations are converted into dimensionless form by suitable non-dimensional parameters, then they are solved by the variational iteration method (VIM). It is noticed that the variational iteration method is convenient to apply and is very helpful for finding solutions of a wide class of nonlinear problems.

1 Introduction

The study of fluid flow has become one important research area for engineers and scientists because of its wide applications in many engineering fields and industrial processes like multiphase mixtures, natural products, biological fluids, food products, agricultural and dairy wastes. Sir Isaac Newton is famous for establishing several scientific postulates. Newton recognized the uniform viscosity of the fluids and also reported the reaction of fluids. The flow behavior or viscosity of fluids alters only by changing the pressure or temperature. For example, water changes into solid at $0\,^{\circ}$C when it freezes and rolls into gas at $100\,^{\circ}$C. Liquid takes the shape of a vessel or container into which it is poured. Such type of liquids called Newtonian fluids (normal liquids). However those fluids that do not follow this formula are called non-Newtonian fluids (strong liquids) [1]. A fluid where there is no constant relation between shear stress and deformation rate is called non-Newtonian fluid. For such types of fluids consistency of Newtonian's law does not hold. Examples of non-Newtonian fluids are ketchup, custard, toothpaste, starch suspension paint, blood shampoo and biological fluids.

In later days, non-Newtonian fluids are more important than Newtonian fluids because of their several applications in industry and technology. When modeling non-Newtonian incompressible fluid flow, the differential equations that appear are highly nonlinear and tricky. As a result the nonlinear equation can be solved either analytically or numerically. Different methods were introduced to obtain solutions of nonlinear problems, the homotopy perturbation method (HPM) [2,3], the homotopy analysis method (HAM) [4,5], the Adomian decomposition method (ADM) [6–8], the differential transform method (DTM) [9–13] and the variation of parameter method (VPM) [14]. The variational iteration method (VIM) [15–17] is used to solve the higher orders of initial and boundary value problems.

Ji Huan He first introduced the VIM [18,19] in 1998. This method is valid and applicable for contracting the calculation size. The VIM [20,21] makes the procedure of solution simple without disturbing the accuracy of higher level. This method has been justified by many authors and is more useful and effective than current techniques like perturbation and Adomian techniques, etc. The method of VIM [22–30] is used to study the wave equations, partial differential equations, fractional differential equations etc.

At present the prominent and valuable progress is made in the field of physical sciences. The great achievement is the development of various techniques to hunt for exact solitary wave solutions of nonlinear differential equations. In nonlinear physical sciences, an essential contribution is that of exact solutions as we can therefore study physical behaviours and discuss more features of the problem which give direction to more applications. Consequently, a lot

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of work has been done in the formulation of several convincing and significant techniques. These techniques are applied by different researchers on mathematical and physical models. Such as, the solution of nonlinear coupled Higgs field equation by the (G′/G)-expansion method [31], the modified alternative (G′/G)-expansion method [32], the exact solutions to the $(2+1)$-dimensional Boussinesq equation by the $\exp(-\phi(\eta))$-expansion method [33], the solution of the unsteady Kortweg de-Vries equation [34], Exact traveling wave solutions of the KP-BBM equation by the (G′/G)-expansion method [35], some new exact traveling wave solutions to the simplified MCH and the $(11+1)$-dimensional combined KdV-mKdV equations [36], exact traveling wave solutions to the $(2+1)$-dimensional Boussinesq and Kadomtsev-Petviashvilli equation [37], exact solutions to the Benney-Luke equation and the phi-4 equations by using the modified simple equation method, the improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations [37], exact and solitary wave solutions for the Tzitzeica-Dodd-Bullough and the modified KdV-Zakharov-Kuznetsov equations using the modified simple equation method [38].

This article is divided into different segments. In the next section the problem is formulated mathematically. In Section 3 analysis of method is given and used to attain analytical solutions. Section 4 is devoted to the application of the variational iteration method. Results and discussion are given in sect. 5. In the last section, conclusion has been drawn.

2 Mathematical formulation

Continuity equation and momentum equation for incompressible fluid are given by

$$\vec{\nabla}\cdot v=0, \tag{1}$$

$$\rho\frac{Dv}{D\tilde{t}}=\rho b+\operatorname{div}\dot{T}, \tag{2}$$

where

$$\frac{D}{D\tilde{t}}=\frac{\partial}{\partial\tilde{t}}+v\cdot\vec{\nabla}, \tag{3}$$

where $\dot{T}$ is the Cauchy stress tensor, given as

$$\dot{T}=-pI\dot{+}\underline{S}. \tag{4}$$

The constitutive equation for a third-grade fluid is

$$\underline{S}=\mu B_1+\lambda_1 B_2+\lambda_2 B_1^2+\Upsilon_1 B_3+\Upsilon_2(B_1B_2+B_2B_1)\Upsilon_3(\operatorname{tr}B_2)B_1, \tag{5}$$

where

$$\begin{aligned}B_1&=\vec{\nabla}\tilde{u}+(\vec{\nabla}\tilde{u})^t\\ B_{n+1}&=\frac{\mathrm{d}}{\mathrm{d}\hat{t}}B_n+B_n\vec{\nabla}\tilde{u}+\left(\vec{\nabla}\tilde{u}\right)^t B_n\quad n=1,2.\end{aligned} \tag{6}$$

2.1 Plane Couette flow

Let us consider the steady flow of a third-grade fluid between two equidistant plates [21]. For the given flow field the velocity is zero, when the external pressure gradient is constant. For maintaining the velocity field it is essential to adjust one of the plates in motion. In this case, it is assumed that the top plate is moving with velocity $\breve{U}$ and the bottom plate is at rest. The velocity and temperature fields are deduced in the form of

$$\tilde{u}=[\tilde{u},0,0],\qquad \tilde{u}=\tilde{u}(y). \tag{7}$$

Using (7), eq. (2) in the absence of body forces results in

$$\begin{aligned}0&=-\frac{\mathrm{d}\grave{p}}{\mathrm{d}\hat{x}}+\frac{\mathrm{d}}{\mathrm{d}\hat{y}}\left[\mu\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}+2(\Upsilon_2+\Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^3\right],\\ \frac{\mathrm{d}}{\mathrm{d}\hat{y}}&\left[\mu\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}+2(\Upsilon_2+\Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^3\right]=\frac{\mathrm{d}\grave{p}}{\mathrm{d}\hat{x}}.\end{aligned} \tag{8}$$

As the pressure gradient is zero, eq. (8) reduces to

$$\mu\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}+6(\Upsilon_2+\Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=0. \tag{9}$$

The associated boundary conditions are

$$\begin{aligned}\tilde{u}&=0, \qquad y=0,\\ \tilde{u}&=\check{U}, \qquad y=d.\end{aligned} \tag{10}$$

The resulting equation (9) and boundary conditions (10) are written in the form of dimensionless parameters. For this, the non-dimensional parameters are

$$\hat{y}^*=\frac{\hat{y}}{h}, \qquad \tilde{u}^*=\frac{\tilde{u}}{\tilde{w}}. \tag{11}$$

Now in non-dimensional form, eq. (9) becomes

$$\begin{aligned}&\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}=\frac{\tilde{w}}{h}\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}^*},\\ &\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=\frac{\tilde{w}}{h^2}\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}},\\ &\mu\frac{\tilde{w}}{h^2}\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}+6(\Upsilon_2+\Upsilon_3)\left(\frac{\tilde{w}}{h}\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}^*}\right)^2\left(\frac{\tilde{w}}{h^2}\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}\right)=0,\\ &\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}+\frac{6(\Upsilon_2+\Upsilon_3)}{\mu}\frac{\tilde{w}^2}{h^2}\left(\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}^*}\right)^2\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}=0,\\ &\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}+6\nu\left(\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}^*}\right)^2\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}}=0,\end{aligned} \tag{12}$$

where

$$\nu=\frac{(\Upsilon_2+\Upsilon_3)}{\mu}\frac{\tilde{w}^2}{h^2}. \tag{13}$$

The boundary conditions are in eq. (10) becomes

$$\begin{aligned}\tilde{u}&=0, \qquad y=-1,\\ \tilde{u}&=1, \qquad y=1.\end{aligned} \tag{14}$$

2.2 Plane Poiseuille flow

In this case, a pressure gradient exists, so both top and bottom plates are fixed. For the velocity and temperature fields, other suppositions and modes remain constant. In this case the governing equation becomes

$$\mu\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}+6(\Upsilon_2+\Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}}, \tag{15}$$

$$\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{y}}=\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{z}}=0, \tag{16}$$

where modified pressure $\acute{\mathrm{p}}$ is defined as

$$\acute{\mathrm{p}}=\grave{p}-(2\lambda_1+\lambda_2)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2. \tag{17}$$

So, $\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}}=\text{const.}$

Thus, the following differential equation is solved to simplify the problem:

$$\mu\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}+6(\Upsilon_2+\Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}}. \tag{18}$$

On boundaries, the conditions are

$$\tilde{u}'(0) = 0, \qquad \tilde{u}(d) = 0. \tag{19}$$

Equation (18) in dimensionless form is

$$\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}} + 6\nu\left(\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}}\right)^2 \frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}} = \frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}}\,. \tag{20}$$

The boundary conditions (19) become

$$\tilde{u}(-1) = 0, \qquad \tilde{u}(1) = 0. \tag{21}$$

2.3 Plane Couette-Poiseuille flow

In this case, a pressure gradient also exists and the velocity distribution is dependent on both the motion of the top plate and the pressure gradient. The top plate is moving with velocity $\check{U}$. On the temperature and velocity field, all other conditions and suppositions remain unchanged. In this case the leading equation becomes in the absence of body forces

$$\begin{aligned}
&\mu\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2} + 6(\Upsilon_2 + \Upsilon_3)\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2 \frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2} = \frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}}\,,\\
&\frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}} = \text{const.},
\end{aligned} \tag{22}$$

with the given conditions

$$\tilde{u}(-d) = 0, \qquad \tilde{u}(d) = \check{U}. \tag{23}$$

The resulting boundary value problem in non-dimensional form becomes

$$\frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}} + 6\nu\left(\frac{\mathrm{d}\tilde{u}^*}{\mathrm{d}\hat{y}}\right)^2 \frac{\mathrm{d}^2\tilde{u}^*}{\mathrm{d}\hat{y}^{*2}} = \frac{\mathrm{d}\acute{\mathrm{p}}}{\mathrm{d}\hat{x}} \tag{24}$$

and the associated boundary conditions are

$$\tilde{u}(0) = 0, \qquad \tilde{u}(d) = \check{U}.$$

3 Analysis of variational iteration method

Consider the nonlinear differential equation

$$\mathcal{L}v(\xi) + \underline{N}v(\xi) = f(\xi), \tag{25}$$

where $\mathcal{L}$ is the linear operator. $\underline{N}$ is the nonlinear operator and $f(\xi)$ is the forcing term.

The VIM presents a correctional functional for eq. (25),

$$V_{n+1}(\xi) = V_n(\xi) + \int_0^{\xi} \lambda(k)[LV_n(k) + N\tilde{V}_n(k) - f(k)]\mathrm{d}k, \tag{26}$$

where $\lambda(k)$ is the Lagrange multiplier, and can be obtained by using the following expression:

$$\lambda(k) = (-1)^m \frac{(k-t)^{m-1}}{(m-1)!}\,.$$

So we can get the exact solution by using

$$v(\xi) = \lim_{n\to\infty} v_n(\xi). \tag{27}$$

4 Solution procedure

Case I: Plane Couette flow

$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}+6\nu\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=0, \tag{28}$$

with boundary conditions

$$\tilde{u}(-1)=0, \qquad \tilde{u}(1)=1.$$

Then the correctional functional for eq. (28) is

$$\tilde{u}_{n+1}(y)=\tilde{u}_n(y)+\int_0^y\lambda(k)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)+6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_nk\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)\right]\mathrm{d}k. \tag{29}$$

By using optimality conditions, we get

$$\lambda(k)=k-y. \tag{30}$$

Putting this value of Lagrange multiplier $\lambda(k)=k-y$ into eq. (29)

$$\tilde{u}_{n+1}(y)=\tilde{u}_n(y)+\int_0^y(k-y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)+6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_nk\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)\right]\mathrm{d}k. \tag{31}$$

We select $u_0(y)=A+By$, then using into eq. (31) we have the following approximants:

$$\tilde{u}_1(y)=(A+By)+\int_0^y(k-y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A+Bk)+6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}(A+Bk)\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A+Bk)\right]\mathrm{d}k,$$

$$\tilde{u}_1(y)=A+By,$$

$$\tilde{u}(y)=\tilde{u}_0(y)+\tilde{u}_1(y)+\tilde{u}_2(y)+\ldots, \tag{32}$$

$$\tilde{u}(y)=A+By+A+By+\ldots. \tag{33}$$

Now using boundary conditions

$$B=\frac{1}{4}, \qquad A=\frac{1}{4}\,.$$

Now eq. (32) becomes

$$\tilde{u}(y)=\frac{1}{2}+\frac{1}{2}y+\ldots.$$

Case II: Plane Poiseuille flow

$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}+6\nu\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2}=\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\,, \tag{34}$$

with boundary conditions

$$\tilde{u}(-1)=0, \qquad \tilde{u}(1)=0.$$

The correctional functional is

$$\tilde{u}_{n+1}(y)=\tilde{u}_n(y)+\int_0^y\lambda(k)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)+6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_nk\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)-\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k. \tag{35}$$

By using optimality conditions, we get

$$\lambda(k)=k-y. \tag{36}$$

Putting this value of Lagrange multiplier $\lambda(k)=k-y$ into eq. (35)

$$\tilde{u}_{n+1}(y)=\tilde{u}_n(y)+\int_0^y(k-y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)+6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_nk\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k)-\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k. \tag{37}$$

We select $u_0(y) = A + By$, then using into eq. (37) we have the following approximants:

$$\tilde{u}_1(y) = (A + By) + \int_0^y (k - y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A + Bk) + 6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}(A + Bk)\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A + Bk) - \frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k,$$

$$\tilde{u}_1(y) = A + By + \frac{y^2}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x},$$

$$\tilde{u}(y) = \tilde{u}_0(y) + \tilde{u}_1(y) + \tilde{u}_2(y) + \ldots, \tag{38}$$

$$\tilde{u}(y) = A + By + A + By + \frac{y^2}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\ldots. \tag{39}$$

Now using boundary conditions we get

$$B = 0, \qquad A = -\frac{1}{4}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x},$$

eq. (38) becomes

$$\tilde{u}(y) = -\frac{1}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}(1 - y^2) + \ldots.$$

Case III: Plane Couette-Poiseuille flow

$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2} + 6\nu\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\hat{y}}\right)^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\hat{y}^2} = \frac{\mathrm{d}\grave{p}}{\mathrm{d}x}, \tag{40}$$

with boundary conditions

$$\tilde{u}(0) = 0, \qquad \tilde{u}(d) = \check{U}.$$

The correctional functional is

$$\tilde{u}_{n+1}(y) = \tilde{u}_n(y) + \int_0^y \lambda(k)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k) + 6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_n k\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k) - \frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k. \tag{41}$$

By using optimality conditions, we get

$$\lambda(k) = k - y. \tag{42}$$

Putting this value of Lagrange multiplier $\lambda(k) = k - y$ into eq. (41)

$$\tilde{u}_{n+1}(y) = \tilde{u}_n(y) + \int_0^y (k - y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k) + 6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}\tilde{u}_n k\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}\tilde{u}_n(k) - \frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k. \tag{43}$$

We select $u_0(y) = A + By$, then using into eq. (43) we have the following approximants:

$$\tilde{u}_1(y) = (A + By) + \int_0^y (k - y)\left[\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A + Bk) + 6\nu\left(\frac{\mathrm{d}}{\mathrm{d}k}(A + Bk)\right)^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}(A + Bk) - \frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right]\mathrm{d}k,$$

$$\tilde{u}_1(y) = A + By + \frac{y^2}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x},$$

$$\tilde{u}(y) = \tilde{u}_0(y) + \tilde{u}_1(y) + \tilde{u}_2(y) + \ldots, \tag{44}$$

$$\tilde{u}(y) = A + By + A + By + \frac{y^2}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\ldots. \tag{45}$$

Now using boundary conditions, we have

$$A = 0, \qquad B = \frac{\check{U}}{2d} - \frac{d}{4}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}.$$

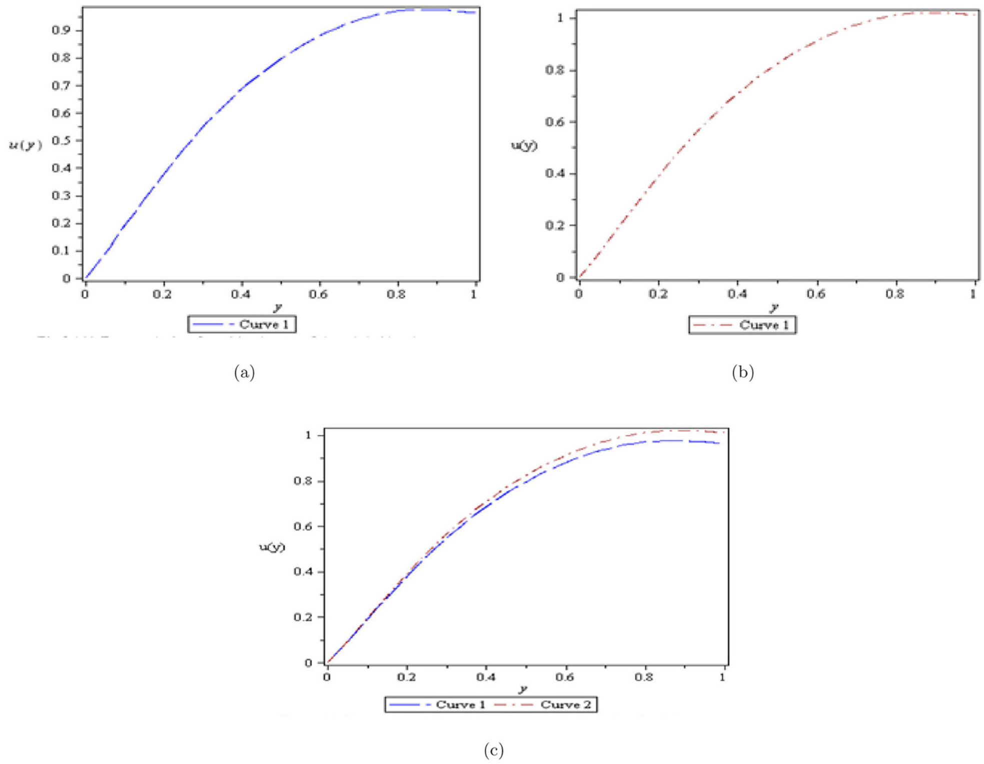


Fig. 1. (a) Exact solution for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -1$. (b) Results obtained by VIM for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -1$. (c) Comparison between VIM and exact solution for fully developed plane Couette-Poiseuille flow.

Equation (44) becomes

$$\tilde{u}(y) = \left(\frac{\check{U}}{2d} - \frac{d}{4}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right) y + \left(\frac{\check{U}}{2d} - \frac{d}{4}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x}\right) y + \frac{y^2}{2}\frac{\mathrm{d}\grave{p}}{\mathrm{d}x} + \dots.$$

The exact solution of the flow problem is calculated by symbolic computation using Maple 18 software. The attained analytical solution is compared with the exact solution and is described by graphical results.

5 Results and discussion

The physical behaviour of the velocity profile is studied in this section. In figs. 1(a) and (b), the exact solution and results obtained by VIM are shown respectively when the value of $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -1$. These figures illustrate that the velocity $u(y)$ increases with the increasing value of y. But figs. 2(a) and (b) show that the velocity function $u'(y)$ is decreasing with increase in the value of y for fully plane Couette-Poiseuille flow when $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -1$. The exact solution and results obtained by VIM are seen from figs. 3(a) and (b) for the velocity profile when the value of $\nu = 0.3$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -1$. It is observed from the figures that the velocity $u(y)$ is increasing with increase in the value of y. But from the figs. 4(a) and (b) it is clear that the velocity function $u'(y)$ is decreasing with increase in the value of y for the plane Couette-Poiseuille flow. Figure 5(a) illustrates that the exact solution for the velocity profile when the value of $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -2$ for fully plane Couette-Poiseuille flow. It is seen that velocity $u(y)$ is increasing with increase in the value of y. Figure 5(b) shows the results obtained by VIM for the velocity profile when the value of $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -2$. It is clear from the figure that velocity $u(y)$ is increasing with the increasing value of y. In figs. 6(a) and (b), the exact solution and results obtained by VIM for the plane Couette-Poiseuille flow are shown when the value of $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -2$. It is noted that the velocity function $u'(y)$ is decreased with the increasing value of y. Figures 7(a) and (b) show that the velocity $u(y)$ is increasing with the increasing value of y when $\nu = 0.1$ and $\frac{\mathrm{d}p}{\mathrm{d}x} = -3$ for the plane Couette-Poiseuille flow.

In order to justify the capability of the understudy method in comparison with the exact solution, the attained results are expressed in figs. 1(c) to 7(c) for the plane Couette-Poiseuille flow. Graphical representations in figs. 2(c), 4(c) and 6(c) depict that the solution obtained by VIM clearly matches the exact solution. Figures 1(c), 3(c), 5(c) and 7(c) show that the solution obtained by VIM matches the exact solution for a small domain. Overall it is observed that the solutions attained by VIM are closer to the exact solution. It is concluded that VIM is an efficient and reliable technique for solving nonlinear differential equations.

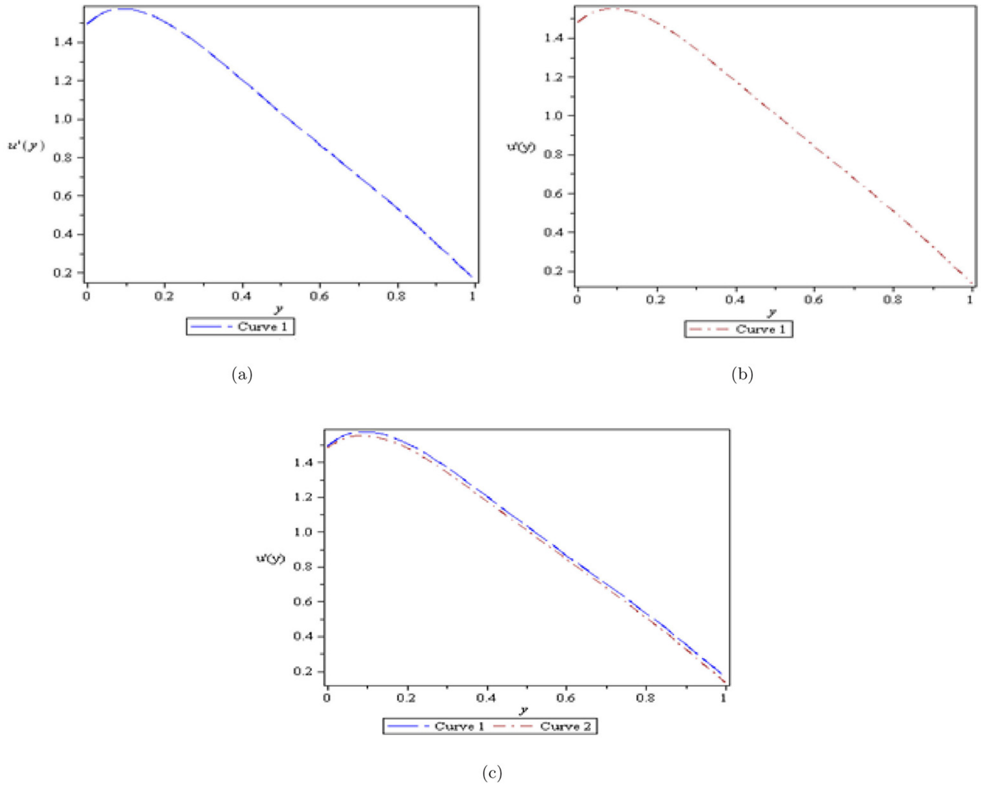


Fig. 2. (a) Exact solution for $u'(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -1$. (b) Results obtained by VIM for $u'(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -1$. (c) Comparison between VIM and exact solution for fully plane Couette-Poiseuille flow.

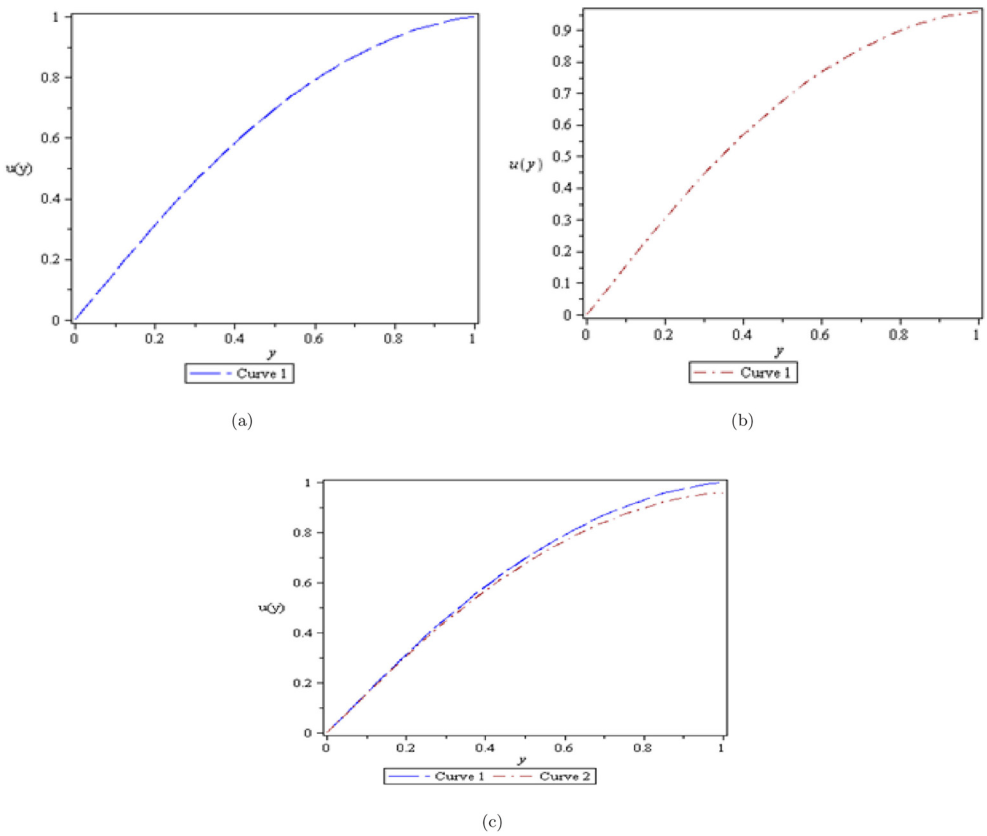
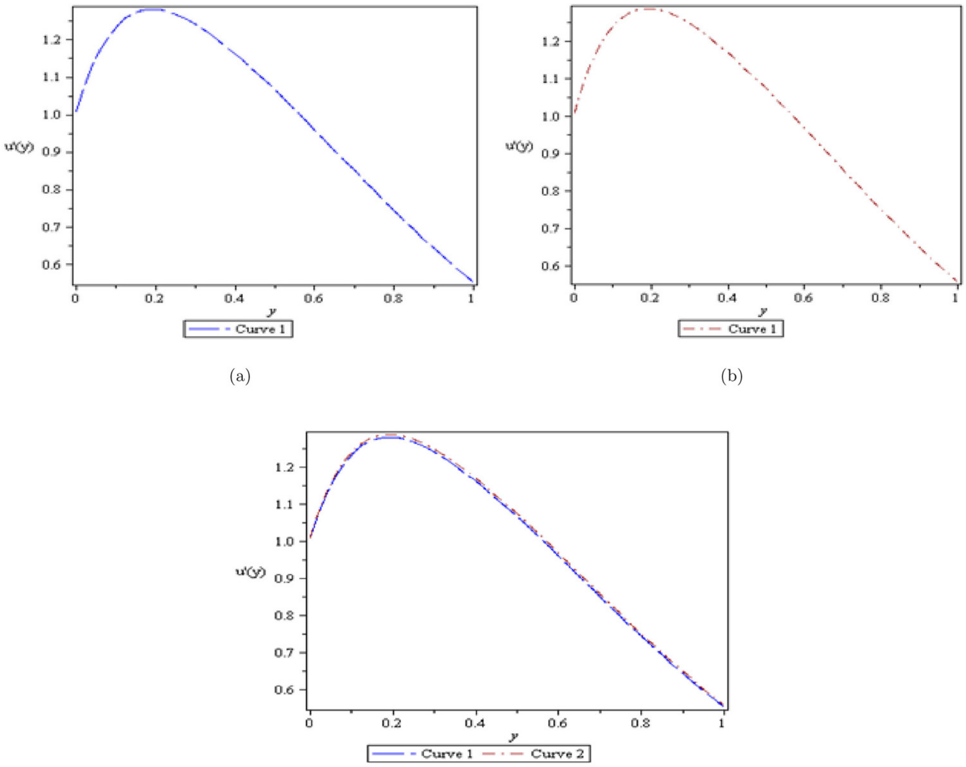


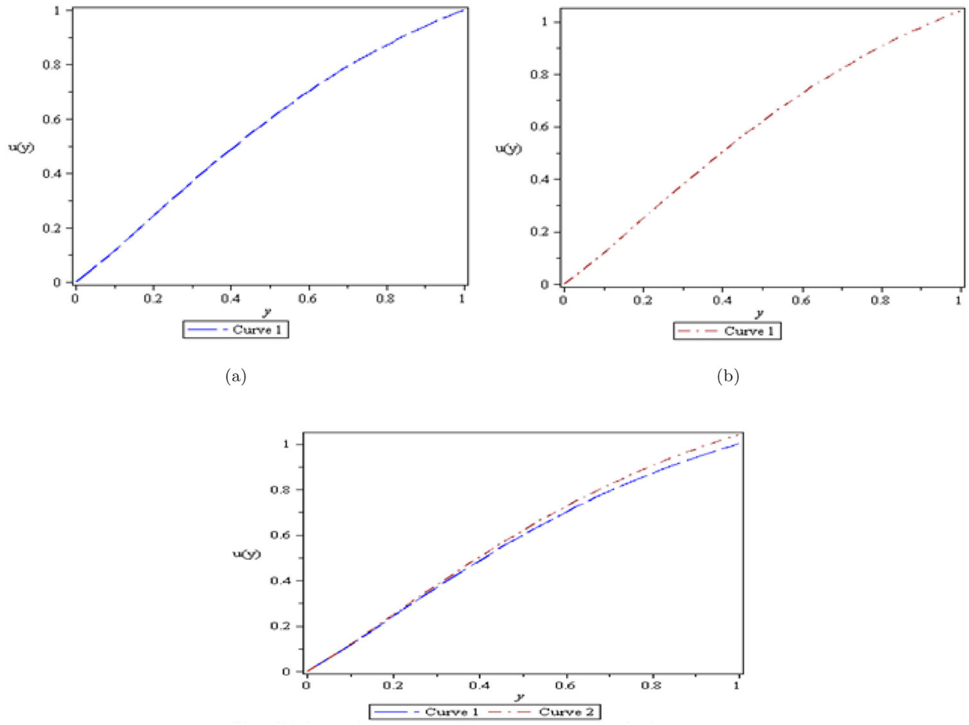
Fig. 3. (a) Exact solution for $u(y)$ when $v = 0.3$ and $\mathrm{d}p/\mathrm{d}x = -1$. (b) Results obtained by VIM for $u(y)$ when $v = 0.3$ and $\mathrm{d}p/\mathrm{d}x = -1$. (c) Comparison between VIM and exact solution for fully plane Couette-Poiseuille flow.



(a) (b)

(c)

Fig. 4. (a) Exact solution for $u'(y)$ when $v = 0.3$ and $\mathrm{d}p/\mathrm{d}x = -1$. (b) Results obtained by VIM for $u'(y)$ when $v = 0.3$ and $\mathrm{d}p/\mathrm{d}x = -1$. (c) Comparison between VIM and exact solution for fully plane Couette-Poiseuille flow.



(a) (b)

(c)

Fig. 5. (a) Exact solution for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -2$. (b) Results obtained by VIM for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -2$. (c) Comparison between VIM and exact solution for fully plane Couette-Poiseuille flow.

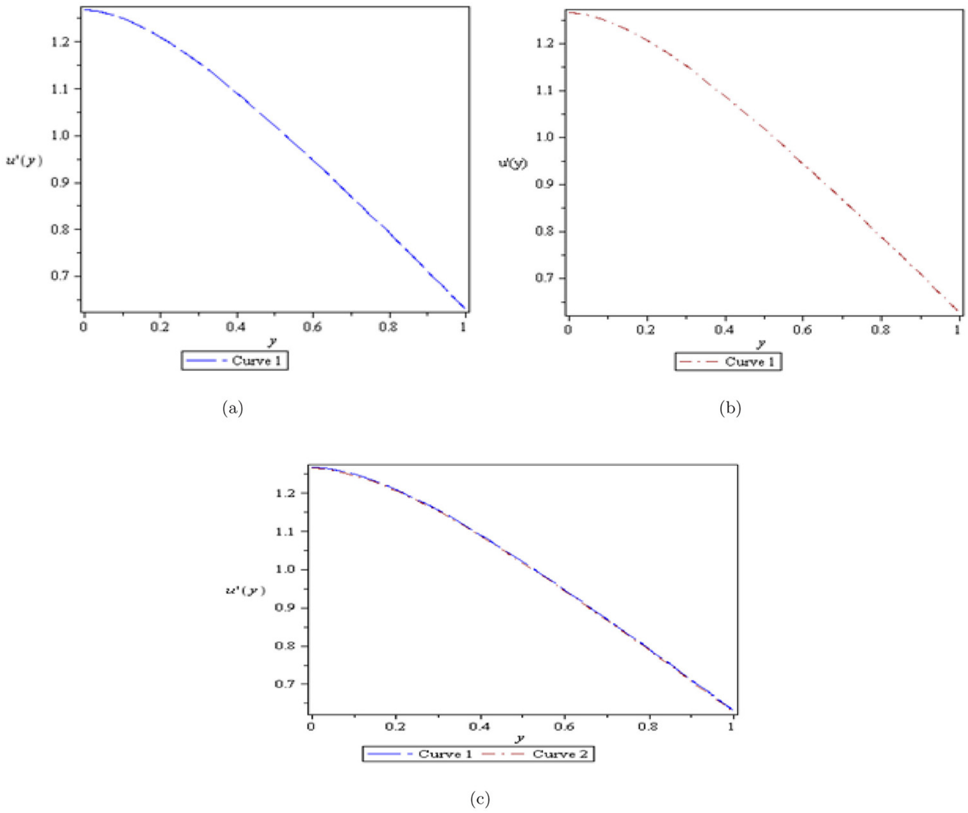


Fig. 6. (a) Exact solution for $u'(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -2$. (b) Results obtained by VIM for $u'(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -2$. (c) Comparison between VIM and exact solution for plane Couette-Poiseuille flow.

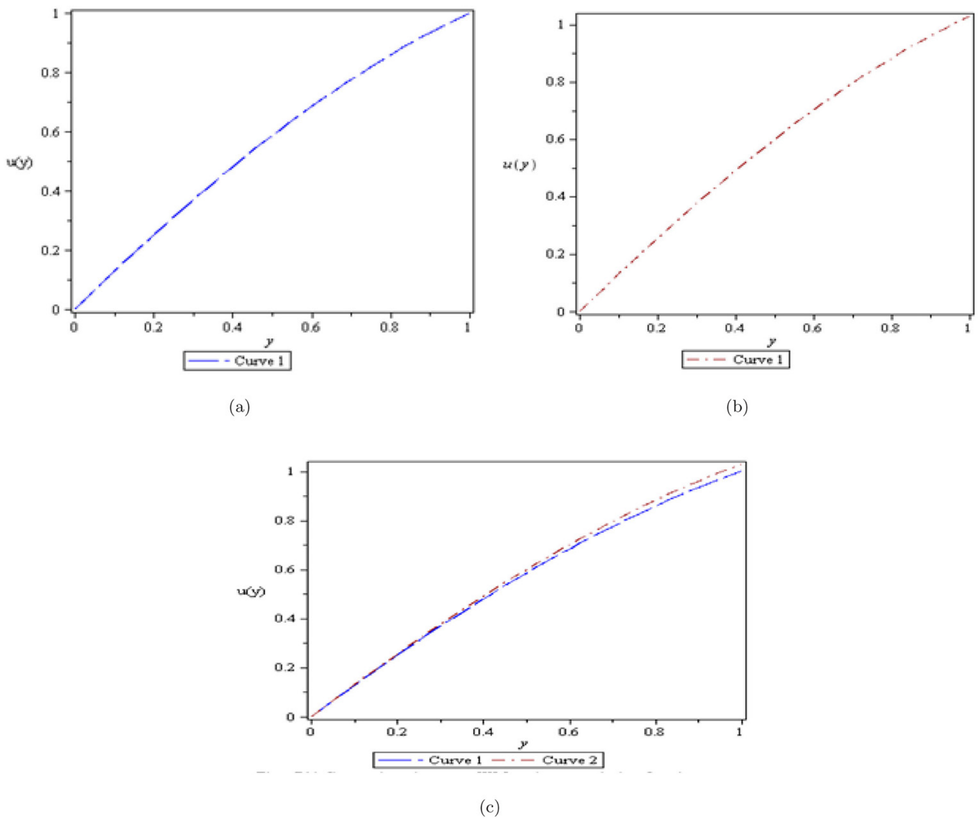


Fig. 7. (a) Exact solution for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -3$. (b) Results obtained by VIM for $u(y)$ when $v = 0.1$ and $\mathrm{d}p/\mathrm{d}x = -3$. (c) Comparison between VIM and exact solution for plane Couette-Poiseuille flow.

6 Conclusion

In the present work, the variational iteration method (VIM) has been efficiently applied to find the analytical solution of the resulting nonlinear differential equation. By using this technique analytical solutions are obtained for different values of nondimensional parameters of the given problem. The attained results are shown graphically and then these results are compared with the exact solution. The obtained results prove the efficiency and consistency of the technique. It is noticed that the variational iteration method is a better technique and very effective, accurate and close to the exact solution existing in literature. Hence, to sum up the above discussion we can say that VIM is a powerful mathematical tool for solving nonlinear problems in engineering and science.

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