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Eigen solutions and entropic system for Hellmann potential in the presence of the Schrödinger equation

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Abstract. By using the supersymmetric approach, we studied the approximate analytic solutions of the three-dimensional Schrödinger equation with the Hellmann potential by applying a suitable approximation scheme to the centrifugal term. The solutions of other useful potentials, such as Coulomb potential and Yukawa potential, are obtained by transformation of variables from the Hellmann potential. Finally, we calculated the Tsallis entropy and Rényi entropy both in position and momentum spaces under the Hellmann potential using integral method. The effects of these entropies on the angular momentum quantum number are investigated in detail.

1 Introduction

A two-particle system interacting through a combination of the attractive Coulomb potential and Yukawa potential, given as

$$V(r) = -\frac{a}{r} + \frac{be^{-\delta r}}{r}, \tag{1}$$

is called the Hellmann potential. This has received considerable attention in theoretical physics over the years. In the potential (1) above, the parameters a and b characterize the strength of the Coulomb and the Yukawa potentials, respectively; δ is the screening parameter and r is the distance between the two particles. The Hellmann potential was first studied by Hellmann [1–3]. Thereafter, various authors worked on the potential, *e.g.*, Dutt *et al.* [4] investigated the bound state energies and the wave functions using the large N expansion technique. Ikhdair and Sever [5,6] investigated the energy levels of neutral atoms by applying an alternative perturbative scheme in solving the Schrödinger equation for the Yukawa potential model with a modified screening parameter, the bound states of the Hellmann potential with arbitrary strength b and screening parameter δ by using a perturbative approach. Das and Chakravarty [7] proposed that such a potential is suitable for the study of inner-shell ionization problems. Varshni and Shukla [8] used the potential model for alkali hydride molecules. Adamowski [9] studied the bound state energies of this potential for various sets of values of the strength and screening parameters (b and δ) in a variational framework using ten variational parameters. Hall and Katatbeh [10] used the potential envelopes method to analyze the bound state spectrum of the Schrödinger Hamiltonian with the potential. Roy *et al.* [11] studied the Hellmann problem using a generalized pseudospectral method. Nasser and Abdelmonem [12] using the J-matrix approach, studied the trajectories of the poles of the S-matrix for a Hellmann potential in the complex energy plane near the critical screening parameter. Hamzavi *et al.* [13] solved the approximate bound states solutions of the Hellmann potential using the generalized parametric Nikiforov-Uvarov method. Amlan *et al.* [14] investigated accurate calculation of the bound states of Hellmann potential using the generalized pseudospectral method. Rajabi and Hamzavi [15] obtained tensor coupling and relativistic spin and pseudospin symmetries with the Hellmann potential. Onate *et al.* [16] obtained approximate eigensolutions of the DKP and Klein-Gordon equations with the Hellmann potential. The Hellmann potential found its applications in the field of atomic and condensed matter physics, *e.g.*, electron-core [17,18], electron-ion [19] inner-shell ionization problem, alkali hydride molecules, solid state physics [20,21]. Despite its applications and various studies by different researchers, a study of entropic systems under the Hellmann potential is missing. Thus, leading to the motivation for this work.

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The essential inadequacy of the position and momentum concepts for a single particle in a physical system is quantum mechanically showed by variance-based Heisenberg relation [22,23] and its moment generalizations [24–26]. This however can be done in a much more appropriate and stringent manner by other position-momentum uncertainty relations which use information-theoretic quantities of global type as uncertainty measures: the entropic or Shannon-entropy–based, the Rényi-entropy–based and the Tsallis-entropy–based ones [27–30]. This gives another priority for this study.

The organization of this paper is as follows. In the next section, we report the bound state energy of the Hellmann potential, Coulomb potential and Yukawa potential. In sect. 3, we calculate the Tsallis entropy and Rényi entropy. In the last section, we discuss our result and give the concluding remark.

2 Bound state solution of the Schrödinger equation with Hellmann potential

The Schrödinger equation is given by [31,32]

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+V(r)-E\right)\psi(\boldsymbol{r})=0. \tag{2}$$

Setting the wave function $\psi(\boldsymbol{r})=\frac{R_{nl}(r)Y_{ml}(\theta,\phi)}{r}$, we obtain the radial part of the equation by the separation of variables as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2m}{\hbar^2}(E_{n,\ell}-V(r))-\frac{\ell(\ell+1)}{r^2}\right]R_{n,\ell}(r)=0, \tag{3}$$

where $E_{n,\ell}$ is the non-relativistic energy, $V(r)$ is the interacting potential, m is the particle mass, ℓ is the angular momentum number, $\hbar$ is the reduced Planck constant and $R_{n,\ell}(r)$ is the wave function. The Hellmann potential (1) and eq. (3) above, cannot be solved exactly due to the orbital centrifugal term $\frac{\ell(\ell+1)}{r^2}$. To obtain the approximate analytical solution, we have to apply a proper approximation scheme to deal with the orbital centrifugal term. It is noted that for a short-range potential, the relation

$$\frac{1}{r^2}\approx\frac{\delta^2}{(1-e^{-\delta r})^2} \tag{4}$$

is a good approximation to $\frac{1}{r^2}$, as proposed by Greene and Aldrich [33,34]. The implication is that eq. (4) is not a good approximation to the centrifugal barrier when the potential parameter δ becomes large. Thus, the approximation is valid when $\delta\ll 1$. Substituting potential (1) and approximation (4) into eq. (3), we obtain an equation of the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2mE_{n,\ell}+2ma\delta}{\hbar^2}-\ell(\ell+1)\delta^2+\frac{\left(\frac{2m\delta(a-b)}{\hbar^2}-\ell(\ell+1)\delta^2\right)e^{-\delta r}}{1-e^{-\delta r}}-\frac{\ell(\ell+1)\delta^2e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}\right]R_{n,\ell}(r)=0. \tag{5}$$

In order to solve eq. (5) using the methodology of supersymmetric quantum mechanics and shape invariance technique [35–37], we propose a supersymmetric superpotential [38,39]. The proposed superpotential is written in the form

$$W(r)=\alpha+\frac{\beta}{1-e^{-\delta r}}\,, \tag{6}$$

for the ground state $U_{0,\ell}(r)$, its logarithmic derivative $\bar{U}_{0,\ell}(r)$ is essentially the same as the superpotential [40–42] by the relation

$$U_{0,\ell}(r)=\exp\left(\int W(r)\mathrm{d}r\right)=\exp\left(\int\bar{U}_{0,\ell}(r)\mathrm{d}r\right), \tag{7}$$

corresponding to the two partner Hamiltonians

$$H_-=\hat{A}^\dagger\hat{A}=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_-(r),\quad H_+=\hat{A}\hat{A}^\dagger=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_+(r), \tag{8}$$

where

$$\hat{A}=\frac{\mathrm{d}}{\mathrm{d}r}-W(r),\quad \hat{A}^\dagger=-\frac{\mathrm{d}}{\mathrm{d}r}-W(r). \tag{9}$$

In this bound state solution, the radial part of the wave function must satisfy the boundary conditions that $U_{n,\ell}(r)/r$ becomes zero, as $r\to\infty$, and $U_{n,\ell}(r)/r$ is finite, at $r=0$. Relating eq. (5) to a non-linear Riccati equation of the form

$$\frac{\mathrm{d}^2R_{n,\ell}(r)}{\mathrm{d}r^2}=W^2(r)-\frac{\mathrm{d}W(r)}{\mathrm{d}r}\,, \tag{10}$$

we obtain the following relations:

$$\alpha^2 = -\frac{2mE_{n,\ell}}{\hbar^2} - \frac{2ma\delta}{\hbar^2} + \ell(\ell+1)\delta^2, \tag{11}$$

$$\ell(\ell+1)\delta^2 = \frac{2m\delta(a-b)}{\hbar^2}, \tag{12}$$

$$\beta = -(\ell+1)\delta, \tag{13}$$

$$\alpha = \frac{2\delta m(a-b) - \beta^2\hbar^2 - \ell(\ell+1)\delta^2\hbar^2}{2\beta\hbar^2}, \tag{14}$$

where $\beta < 0$. When $\beta = 0$ and $\delta \to 0$, $\alpha \to 0$. From relations (8) and (9), we can now construct the supersymmetric partner potentials $V_\pm(r) = W^2(r) \pm \frac{\mathrm{d}W(r)}{\mathrm{d}r}$,

$$V_+(r) = \alpha^2 + \frac{\beta^2}{(1-e^{-\delta r})^2} + \frac{2\alpha\beta}{1-e^{-\delta r}} - \frac{\delta\beta e^{-\delta r}}{(1-e^{-\delta r})^2} = \alpha^2 + \frac{2\alpha\beta}{1-e^{-\delta r}} + \frac{\beta^2(1+e^{-\delta r}) - \beta(\beta+\delta)e^{-\delta r}}{(1-e^{-\delta r})^2}, \tag{15}$$

$$V_-(r) = \alpha^2 + \frac{\beta^2}{(1-e^{-\delta r})^2} + \frac{2\alpha\beta}{1-e^{-\delta r}} - \frac{\delta\beta e^{-\delta r}}{(1-e^{-\delta r})^2} = \alpha^2 + \frac{2\alpha\beta}{1-e^{-\delta r}} + \frac{\beta^2(1+e^{-\delta r}) - \beta(\beta-\delta)e^{-\delta r}}{(1-e^{-\delta r})^2}, \tag{16}$$

from which we find that the family potentials $V_+(r)$ and $V_-(r)$ are shape-invariant and thus satisfy the shape invariance condition [43–45]

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1), \tag{17}$$

via mapping of the form $\beta \to \beta - \delta$, where $\beta = a_0$. It is deduced that $a_1 = F(a_0) \Rightarrow a_0 - \delta$, where a_1 is a new set of parameters uniquely determined from the old set a_0 and $R(a_1)$ is a residual term which is independent of the variable r. Since $a_1 = a_0 - \delta$, subsequently, $a_n = a_0 - n\delta$.

Now, using the shape invariance approach, we obtain [46,47]

$$R(a_1) = V_+(r, a_0) - V_-(r, a_1), \tag{18a}$$

$$R(a_2) = V_+(r, a_1) - V_-(r, a_2), \tag{18b}$$

$$R(a_3) = V_+(r, a_2) - V_-(r, a_3), \tag{18c}$$

$$R(a_n) = V_+(r, a_{n-1}) - V_-(r, a_n), \tag{19}$$

whose energy levels are given as

$$E_{n\ell} = \sum_{k=1}^{n} R(a_k) = V_+(r, a_0) - V_-(r, a_n). \tag{20}$$

Using eqs. (11), (13) and (14), the energy eigenvalue equation is obtain in the following form:

$$E_{n,\ell} = \delta\left(\frac{\delta\hbar^2\ell(\ell+1)}{2m} - a\right) - \frac{\hbar^2}{2m}\left[\frac{\frac{2m}{\hbar^2}(a-b) - \delta((\ell+n+1))^2 - \ell(\ell+1)\delta}{2(\ell+n+1)}\right]^2, \tag{21}$$

which is identical to eq. (24) of ref. [13].

Now, let us consider some special cases, when $a = 0$, the potential (1) turns to the Yukawa potential and eq. (21) becomes

$$E_{n,\ell} = \frac{(\hbar\delta)^2\ell(\ell+1)}{2m} - \frac{\hbar^2}{2m}\left[\frac{-\frac{2mb}{\hbar^2} - \delta((\ell+n+1))^2 - \delta\ell(\ell+1)}{2(\ell+n+1)}\right]^2. \tag{22}$$

Now putting $b = 0$, the potential (1) becomes Coulomb potential and the energy equation (21) becomes

$$E_{n,\ell} = \frac{(\hbar\delta)^2\ell(\ell+1)}{2m} - a\delta - \frac{\hbar^2}{2m}\left[\frac{\frac{2ma}{\hbar^2} - \delta(\ell+n+1)^2 - \delta\ell(\ell+1)}{2(\ell+n+1)}\right]^2. \tag{23}$$

Now, let us obtain an unnormalized wave function by defining a variable of the form $y = \exp(-\delta r)$ and inserting it into eq. (5), we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1-y}{y(1-y)}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{Ay^2 + Py + Q}{(y(1-y))^2}\right] R_{n,\ell}(y) = 0, \tag{24}$$

where

$$A = 2\ell(\ell+1) + \frac{2m[a+(a-b)]}{\delta\hbar^2} - \frac{2mE_{n,\ell}}{\delta^2\hbar^2}\,, \tag{25}$$

$$P = \frac{2mE_{n,\ell}}{\delta^2\hbar^2} - \frac{2m[2a+(a-b)]}{\delta\hbar^2} - 4\ell(\ell+1), \tag{26}$$

$$Q = \frac{-2mE_{n,\ell}}{\delta^2\hbar^2} + \frac{2ma}{\delta\hbar^2} + \ell(\ell+1). \tag{27}$$

Analyzing the asymptotic behavior of eq. (24) at origin and at infinity, it can be tested when $r \to 0 (y \to 1)$ and when $r \to \infty (y \to 0)$ that eq. (24) has a solution

$$R_{n,\ell}(y) = (1-y)^{\ell+1} y^u, \tag{28}$$

where

$$u = \sqrt{\frac{2ma}{\delta\hbar^2} - \frac{2mE_{n,\ell}}{\delta^2\hbar^2} + \ell(\ell+1)}. \tag{29}$$

By taking the trial wave function of the form given in eq. (28) and inserting it into eq. (24), we have

$$f''(y) + f'(y)\left(\frac{(2v+1) - y(2(v+1+u)+1)}{y(1-y)}\right) - f(y)\left(\frac{(v+u)^2 + A}{y(1-y)}\right) = 0. \tag{30}$$

Equation (30) is a differential equation satisfied by the hypergeometric function. Thus, its solution is obtain as

$$f(y) = {}_2F_1(-n, n+2(v+u); 2v+1, y). \tag{31}$$

Replacing the function $f(z)$ with the hypergeometric function and write the complete wave function as

$$R_{n,\ell}(y) = N_{n,\ell}\, y^u (1-y)^{\ell+1}\; {}_2F_1(-n, n+2(u+\ell+1); 2u+1, y), \tag{32}$$

where $N_{n,\ell}$ is the normalization factor which, by using the normalization condition, is obtained as

$$N_{n,\ell} = \sqrt{\frac{n!\delta a \Gamma(n+a+v+1)}{\Gamma(n+a+1)\Gamma(n+v+1)}}. \tag{33}$$

Equation (32) can be written in terms of the Jacobi polynomial in the form

$$R_{n\ell}(y) = N_{n\ell}\, y^u (1-y)^{\ell+1} P_n^{(2u,2\ell+1)}(1-2y). \tag{34}$$

3 Hellmann potential and entropies

In this section, we calculate some entropies. Entropy is the measure of a system's thermal energy per unit temperature that is unavailable for doing useful work. The concept of entropy provides deep insight into the direction of spontaneous change for many phenomena. This study is limited to statistical entropy which is a probabilistic measure of uncertainty.

3.1 Tsallis entropy

The Tsallis entropy was introduced in 1988 by Constantino Tsallis as a basis for generalizing the standard statistical mechanics [48]. The Tsallis entropy is defined as [48–53]

$$T_q(\rho) = \frac{1}{q-1}\left(1 - 4\pi\int_0^\infty \rho(r)^q \mathrm{d}r\right), \tag{35}$$

where

$$\rho(y) = R_{n\ell}^2(y) = N_{n\ell}^2 y^a (1-y)^{v+1}\left[P_n^{(a,v)}(1-2y)\right]^2 \tag{36}$$

is called probability density with $a = 2u$ and $v = 2\ell + 1$. Now, to obtain the Tsallis entropy in position space, we define $y = e^{-\delta r}$ to have

$$T_q(\rho) = \frac{1}{q-1}\left(1 + \frac{4\pi}{\delta}\int_1^0 \rho(y)^q \frac{1}{y}\mathrm{d}y\right). \tag{37}$$

If we, again, define $s = 1 - y$, then we obtain

$$T_q(\rho) = \frac{1}{q-1}\left(1 - \frac{4\pi}{\delta}\int_0^1 \rho(s)^q \frac{1}{1-s}\mathrm{d}s\right). \tag{38}$$

Substituting for the probability density, we have the Tsallis entropy in position space as

$$T_q(\rho) = \frac{1}{q-1}\left[1 - \frac{12.568}{\delta} \times \left(\frac{n!a2^{v+a+1}\Gamma(n+a)\Gamma(n+2+v)}{(2n+a+v)\Gamma(n+a+1)\Gamma(n+1+v)\Gamma(n+1)}\right)^q\right]. \tag{39}$$

In this paper, we consider $q = 2$. Thus,

$$T_2(\rho) = 1 - 12.568\delta \times \left(\frac{n!a2^{v+a+1}\Gamma(n+a)\Gamma(n+2+v)}{(2n+a+v)\Gamma(n+a+1)\Gamma(n+1+v)\Gamma(n+1)}\right)^2, \tag{40}$$

where we have used an integral of the form

$$\int_0^1 x^t(1-x)^z \left[P_n^{(t,z)}(2x-1)\right]^2 \mathrm{d}x = \frac{2^{t+z+1}\Gamma(t+n+1)\Gamma(z+n+1)}{(2n+t+z)\Gamma(n+1)\Gamma(t+z+n+1)}. \tag{41}$$

The momentum Tsallis entropy is obtained by defining $z = 1 - 2y$. This turns eq. (37) to

$$T_q(\rho) = \frac{1}{q-1}\left(1 - \frac{2\pi}{\delta}\int_{-1}^1 \rho(z)^q \frac{2}{1-z}\mathrm{d}z\right), \tag{42}$$

where

$$\rho(r) = \gamma(r) = N_{n\ell}^2 \left(\frac{1-z}{2}\right)^{a-1} \left(\frac{1+z}{2}\right)^{v+1} \left[P_n^{(a,v)}(z)\right]^2. \tag{43}$$

Substituting eq. (43) into eq. (42), we have the Tsallis entropy in momentum space as

$$T_q(\gamma) = \frac{1}{q-1}\left[1 - \frac{6.284}{\delta} \times \left(\frac{\delta a 2^{v+a+1}\Gamma(n+v+2)\Gamma(n+a)}{(a-1)\Gamma(n+a+1)\Gamma(n+v+1)}\right)^q\right]. \tag{44}$$

When $q = 2$,

$$T_2(\gamma) = \left[1 - 6.284\delta \times \left(\frac{\delta a 2^{v+a+1}\Gamma(n+v+2)\Gamma(n+a)}{(a-1)\Gamma(n+a+1)\Gamma(n+v+1)}\right)^2\right], \tag{45}$$

where we have used integral of the form

$$\int_{-1}^1 \left(\frac{1-x}{2}\right)^{a-1} \left(\frac{1+x}{2}\right)^b \times \left[P_n^{(a,b)}(x)\right]^2 \mathrm{d}x = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!a\Gamma(a+b+n+1)}. \tag{46}$$

3.2 Rényi entropy

The Rényi entropy introduced by Rényi in 1960 [54] is a generalization of the Shannon entropy which depends on a parameter q. The Rényi entropy $R_q(\rho)$ is defined as [54–58]

$$R_q(\rho) = \frac{1}{1-q}\log 4\pi \int_0^\infty \rho(r)^q \mathrm{d}r. \tag{47}$$

Let us recall that $y = e^{-\delta r}$, then

$$R_q(\rho) = -\frac{1}{1-q}\log \frac{4\pi}{\delta}\int_1^0 \rho(y)^q \mathrm{d}y. \tag{48}$$

To get the Rényi entropy in the position space, we have already defined a function of the form $s = 1 - y$. Hence

$$R_q(\rho) = \frac{1}{1-q} \log \frac{4\pi}{\delta} \int_0^1 \rho(s)^q \frac{1}{1-s} \mathrm{d}s. \tag{49}$$

Now, substituting for the probability density, we easily have

$$R_q(\rho) = \frac{1}{1-q} \left[\log \frac{12.568}{\delta} + q \log \frac{n!\delta a 2^{v+a+1} \Gamma(n+a)\Gamma(n+v+2)}{(2n+a+v)\Gamma(n+1)\Gamma(n+a+1)\Gamma(n+v+1)} \right]. \tag{50}$$

For $q = 2$, the Rényi entropy in eq. (49) becomes

$$R_q(\rho) = -\log \frac{12.568}{\delta} - 2\log \frac{n!\delta a 2^{v+a+1} \Gamma(n+a)\Gamma(n+v+2)}{(2n+a+v)\Gamma(n+1)\Gamma(n+a+1)\Gamma(n+v+1)}, \tag{51}$$

where we have used integral in eq. (41). To have the Rényi entropy in momentum space, we recall the change of variable $z = 1 - 2y$ previously made. Thus, eq. (48) turns out to be

$$R_q(\gamma) = \frac{1}{1-q} \log \frac{2\pi}{\delta} \int_{-1}^1 \gamma(z)^q \frac{2}{1-z} \mathrm{d}z. \tag{52}$$

Substituting for the probability density into eq. (52) and by using the integral in eq. (46) gives the Rényi entropy in momentum space as

$$R_q(\gamma) = \frac{1}{1-q} \left[\log 6.284\delta + q \log \frac{2^{v+a+1} a \Gamma(n+a++v+2)\Gamma(n+a)}{(a-1)\Gamma(n+a+1)\Gamma(n+v+1)} \right]. \tag{53}$$

When $q = 2$,

$$R_2(\gamma) = -\log 6.284\delta - 2\log \frac{2^{v+a+1} a \Gamma(n+a++v+2)\Gamma(n+a)}{(a-1)\Gamma(n+a+1)\Gamma(n+v+1)}. \tag{54}$$

4 Discussion

To examine the energy behavior, we plotted energy with $\ell = 1$, for $n = 1, 2, 3, 4$ and 5 as a function of the potential parameter (δ) as shown in figs. 1, 2 and 3, for the Hellmann potential, Yukawa potential and Coulomb potential, respectively. It is observed that as the potential parameter increases, the energy of the system also increases. Similarly, the energy increases as n increases. For the values of the potential parameter -2 to 1.5, the energy at all levels are equivalent. From the figures, it can be observed that the energy obtained from each of the potentials is equivalent. In figs. 4 and 5, we plotted the Tsallis entropy in momentum space and position space, respectively, against the angular momentum quantum number at the ground state. The Tsallis entropy decreases in the momentum space with increasing angular momentum quantum number but increases in the position space. In figs. 6 and 7, respectively, we plotted the Rényi entropy in momentum space against the angular momentum quantum number and the Rényi entropy in position space against the angular momentum quantum number. In the position space, the Rényi entropy increases as the angular momentum quantum number increases but decreases in the momentum space. In table 1, we numerically compared our results with the results from other methods. As can be seen from the table, our results agree with the results from the parametric Nikiforov-Uvarov method and the amplitude phase method. From table 1, one can see that the energy eigenvalue decreases as the potential parameter increases for all the state. Thus, the energy becomes more negative in value as the potential parameter increases positively. Hence a particle in this system becomes more attractive as the energy becomes more bound.

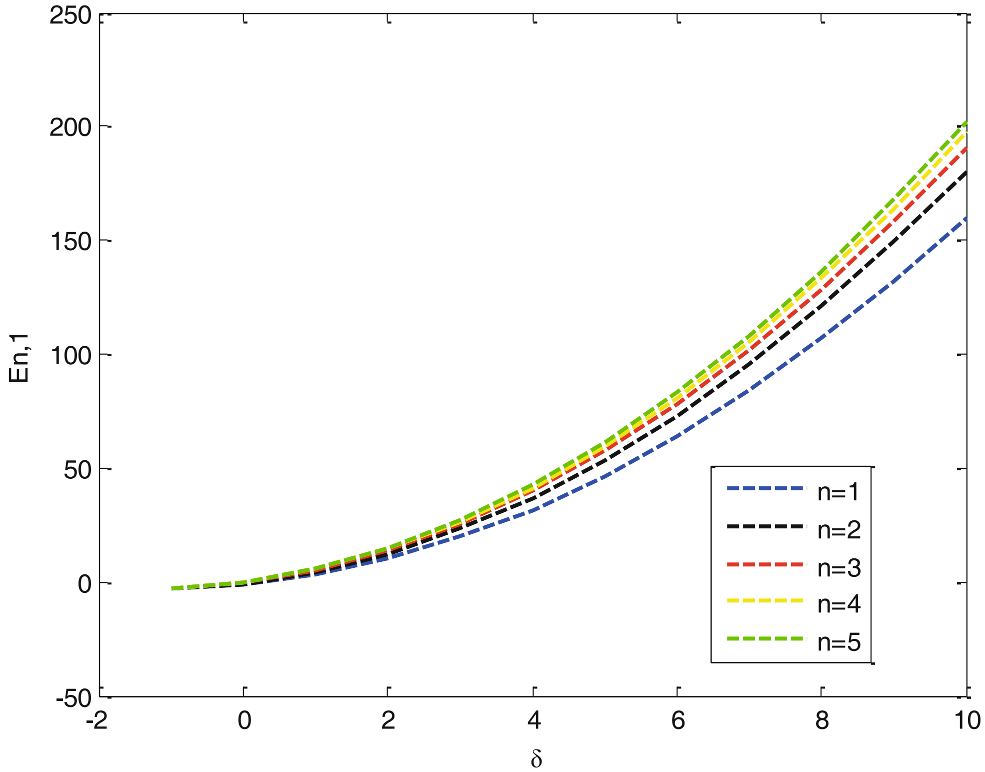

Fig. 1. $E_{n,1}$ *vs.* δ for the Hellmann potential with $a = -5$, $2m = \hbar = 1$ and $b = 2$.

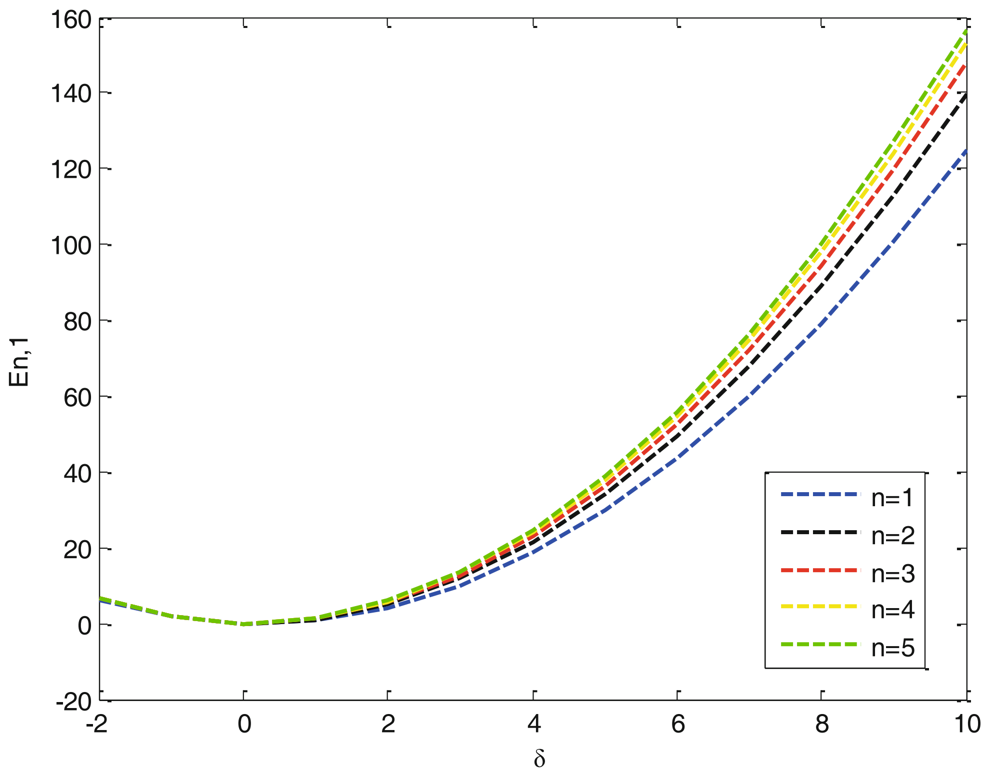

Fig. 2. $E_{n,1}$ against δ for the Yukawa potential with $b = 2$, $\delta = 0.1$, and $2m = \hbar = 1$.

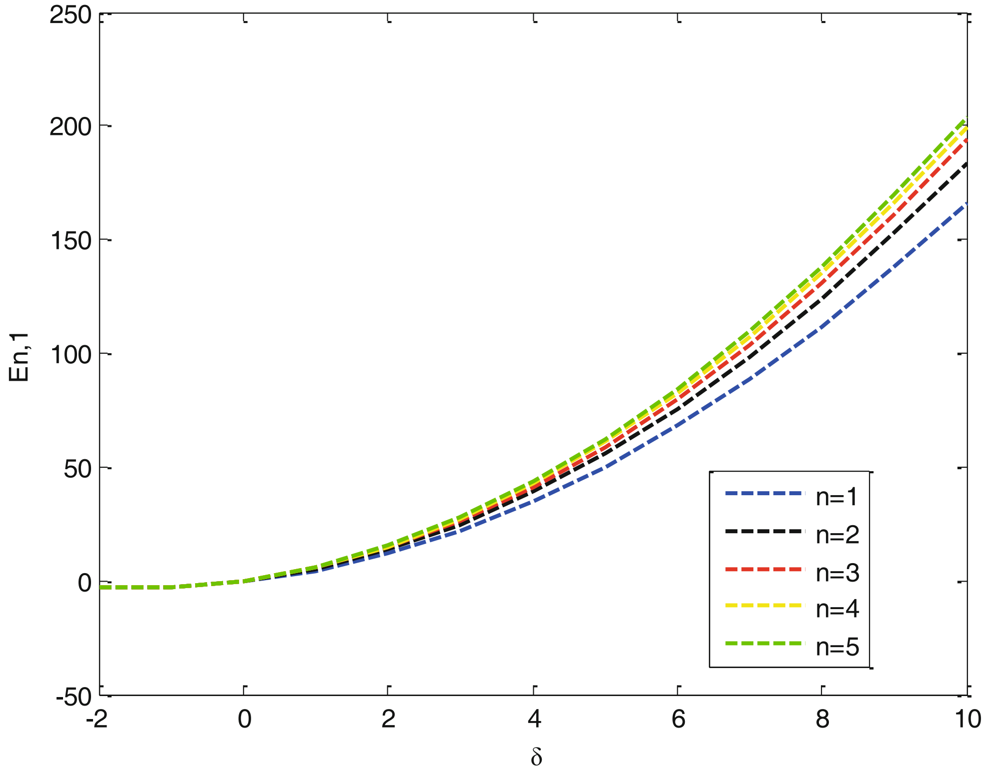

Fig. 3. $E_{n,1}$ against δ for the Coulomb potential with $a = -5$ and $2m = \hbar = 1$.

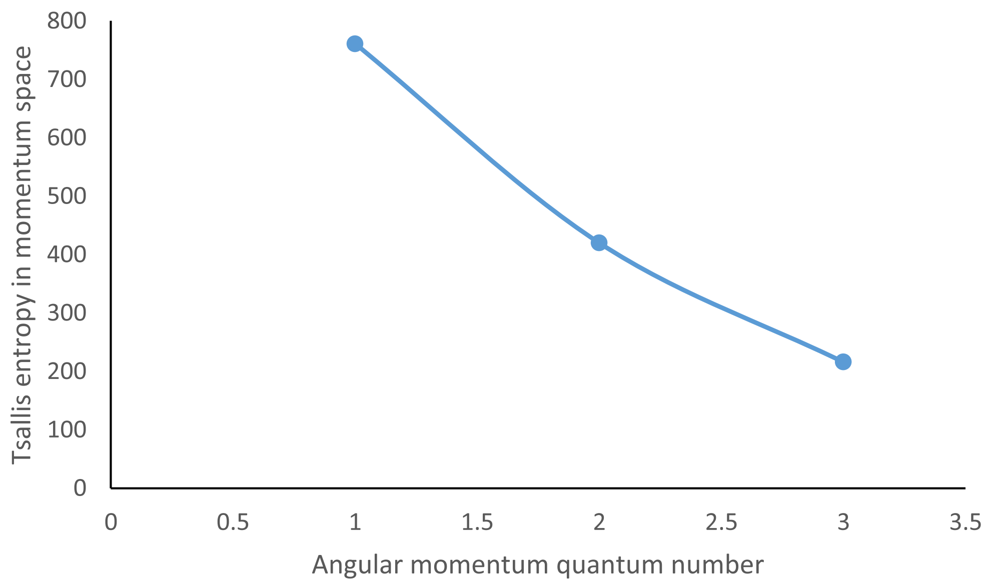

Fig. 4. Tsallis entropy in momentum space against the angular momentum quantum number at the ground state with $a = 1 = \delta$ and $b = -1$.

5 Concluding remark

In this article, we studied the bound state solutions of the Schrödinger equation, Tsallis entropy and Rényi entropy with the Hellmann potential for angular momentum quantum number $\ell \neq 0$. We obtained eigenvalue equation and unnormalized radial wave functions using supersymmetric method. The energy equation for the Yukawa potential and the Coulomb potential is obtained by putting $a = 0$ and $b = 0$, respectively, in the Hellmann potential energy equation. To test the accuracy of our results, we obtained numerically energy eigenvalues of the Hellmann potential for various

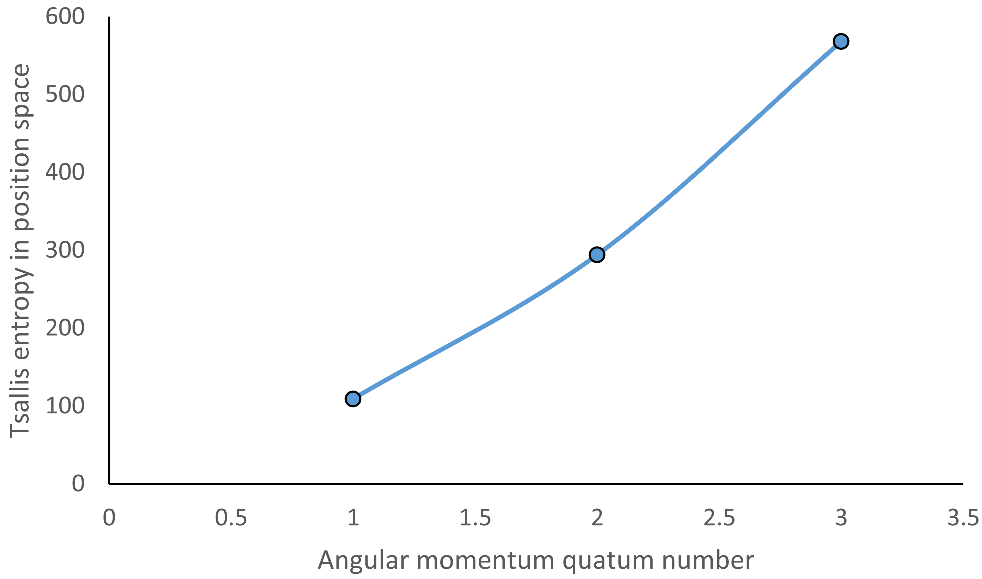

Fig. 5. Tsallis entropy in position space against the angular momentum quantum number at the ground state with $a = 1 = \delta$ and $b = -1$.

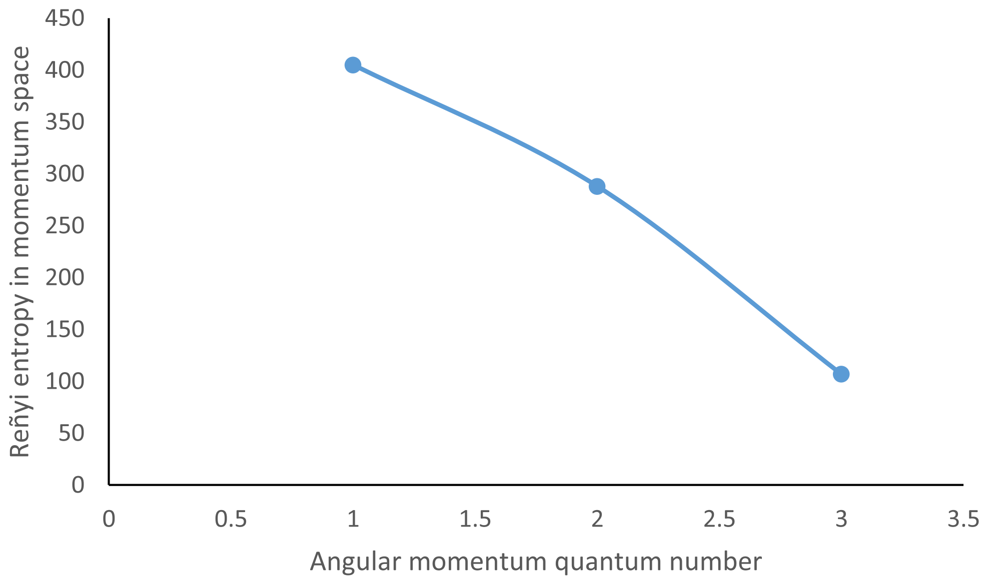

Fig. 6. Rényi entropy in momentum space against the angular momentum quantum number at the ground state with $a = 1 = \delta$ and $b = -1$.

states using eq. (21) and compared it with the results of Nikiforov-Uvarov (NU) method [13] and amplitude phase (AP) method [13] as shown in table 1. It is found that as the screening parameter goes to zero, the energy levels approach to the family pure Coulomb potential energy levels. In addition, the concepts of area law for the entropy of the black hole which prescribes the microscopic states close to the horizon and number of states had been to grow rapidly with area by Bekenstein [59]. Consequently, t Hooft [60] has studied the entropy of quantum black hole using the brick wall model. Most recently, Govindaraja and Munõz-Castañeda [61] modeled a quantum black hole using bond states with singular potentials. As pointed out in their paper, the existence of correct hehaviour of localized bound states on the boundary is a strong requirement for the correct entropy, we however believe that the present paper on the Tsallis and Rényi entropy calculations can lead to the black hole entropy in proper boundary conditions are applied, since it had been remarked that the boundaries are the creation of the devil.

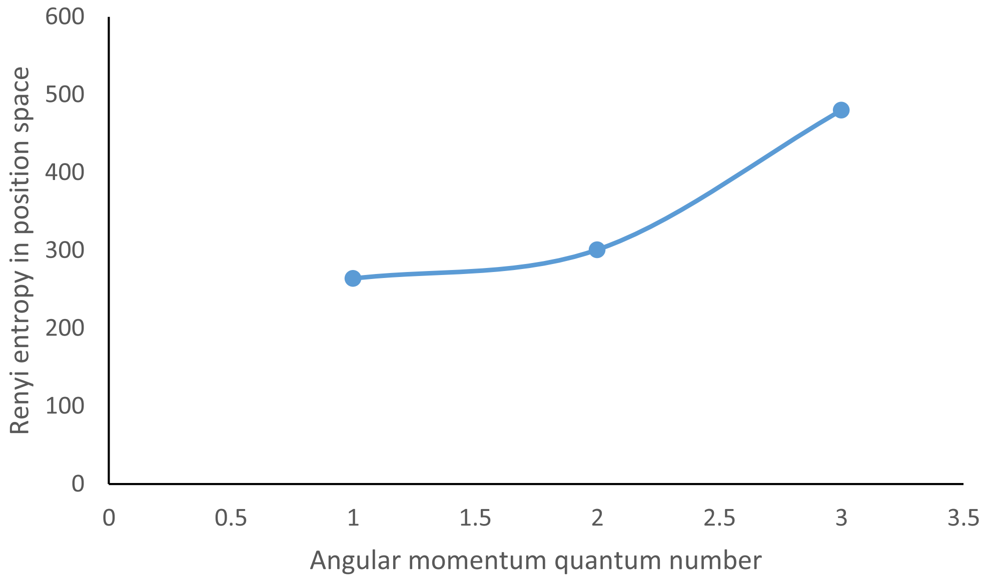

Fig. 7. Rényi entropy in position space against the angular momentum quantum number at the ground state with $a = 1 = \delta$ and $b = -1$.

Table 1. Ro-vibrational energy spectrum ($-E_{n,\ell}$) for $1s, 2s, 2p, 3s, 3p, 3d, 4s, 4p, 4d$ and $4f$ with $2m = \hbar = 1$ and $a = 2$.

States	δ	SUSY $b = 1$	NU [13] $b = 1$	AP [13] $b = 1$	SUSY $b = -1$	NU [13] $b = -1$	AP [13] $b = -1$
	0.001	0.251 500	0.251 500	0.250 969	2.250 500	2.250 500	2.248 981
1s	0.005	0.257 506	0.257 506	0.254 933	2.252 510	2.252 506	2.244 993
	0.010	0.265 025	0.265 025	0.259 823	2.255 020	2.255 025	2.240 030
	0.001	0.064 250	0.064 001	0.063 243	0.563 750	0.563 001	0.561 502
2s	0.005	0.071 256	0.070 025	0.067 106	0.568 756	0.565 025	0.557 549
	0.010	0.080 025	0.077 600	0.071 689	0.575 025	0.567 600	0.552 697
	0.001	0.063 999	0.064 000	0.063 495	0.562 999	0.563 000	0.561 502
2p	0.005	0.069 975	0.070 000	0.067 377	0.564 975	0.565 000	0.557 541
	0.010	0.077 400	0.077 500	0.072 020	0.567 400	0.567 500	0.552 664
	0.001	0.029 611	0.029 280	0.028 283	0.251 500	0.250 502	0.249 004
3s	0.005	0.036 951	0.035 334	0.031 993	0.257 506	0.252 556	0.245 110
	0.010	0.046 136	0.043 003	0.036 142	0.265 025	0.255 225	0.240 435
	0.001	0.029 499	0.029 279	0.028 765	0.251 165	0.250 501	0.249 004
3p	0.005	0.036 356	0.035 309	0.032 480	0.255 801	0.252 531	0.245 102
	0.010	0.044 869	0.042 903	0.036 645	0.261 536	0.255 125	0.240 404
	0.001	0.029 274	0.029 388	0.028 767	0.250 496	0.250 833	0.249 003
3d	0.005	0.035 184	0.035 817	0.032 526	0.252 406	0.254 151	0.245 086
	0.010	0.042 403	0.043 825	0.036814	0.254 625	0.258 269	0.240 341
	0.001	0.017 500	0.029 280	0.016 130	0.142 250	0.141 129	0.139 633
4s	0.005	0.025 006	0.035 334	0.019 646	0.148 756	0.143 225	0.135 819
	0.010	0.034 400	0.043 003	0.023 280	0.156 900	0.146 025	0.131 380
	0.001	0.017 436	0.017 128	0.016 602	0.142 061	0.141 128	0.139 632
4p	0.005	0.024 652	0.023 200	0.020 100	0.147 777	0.143 200	0.135 811
	0.010	0.033 606	0.030 925	0.023 711	0.154 856	0.145 925	0.131 350
	0.001	0.017 308	0.017 180	0.016 604	0.141 683	0.141 314	0.139 632
4d	0.005	0.023 952	0.023 464	0.020 142	0.145 827	0.144 089	0.135 795
	0.010	0.032 056	0.031 356	0.023 857	0.150 806	0.147 606	0.131 290
	0.001	0.017 117	0.017 311	0.016 607	0.141 117	0.141 686	0.139 631
4f	0.005	0.022 925	0.024 027	0.020 206	0.142 925	0.145 902	0.135 772
	0.010	0.029 825	0.032 356	0.024 072	0.144 825	0.151 106	0.131 200

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