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Investigating dynamic characteristics of porous double-layered FG nanoplates in elastic medium via generalized nonlocal strain gradient elasticity

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Abstract. For the first time, a vibrating porous double-nanoplate system under in-plane periodic loads is modeled via the generalized nonlocal strain gradient theory (NSGT). Based on the proposed theory, one can examine both stiffness-softening and stiffness-hardening effects for a more accurate analysis of nanoplates. Nanopores or nanovoids are incorporated to the model based on a modified rule of mixture. Modeling of porous double-layered nanoplate is conducted according to a refined four-variable plate theory with fewer field variables than first-order plate theory. The governing equations and related classical and nonclassical boundary conditions are derived based on Hamilton's principle. These equations are solved for hinged nanoplates via Galerkin's method. It is shown that porosities, nonlocal parameter, strain gradient parameter, material gradation, interlayer stiffness, elastic foundation, side-to-thickness and aspect ratios have a notable impact on the vibration behavior of nanoporous materials.

1 Introduction

Porosities occurring inside the material structure during construction have a significant effect on the mechanical performance of inhomogeneous structures [1]. In the last few years, fabrication and synthesis of porous nanoplates have been performed by several researchers [2–4]. Functionally graded (FG) structures have an inhomogeneous nature, while their vibration behavior is affected by the porosity volume fraction [5]. These structures have excellent properties under environmental conditions such as thermal environments due to the gradation of material properties in the thickness direction which distinguishes them from conventional composites [6–8].

After the fabrication of functionally graded nanoplates [9–12], they have been used as structural components in nanoelectro-mechanical systems (NEMs) for sensing and actuating purposes [13–15]. Investigation of mechanical behavior of scale-free plates has been extensively conducted in the literature based on classical theories. However, these theories are impotent to describe the size effects on the nanostructures. This problem is resolved using the nonlocal elasticity theory of Eringen [16,17] in which small size effects are considered by introducing an additional scale parameter. According to the nonlocal stress field theory, the stress state at a given point depends on the strain states at all points. The nonlocal elasticity theory has been broadly applied to examine the static and dynamic behaviors of nanoscale structures [18–35]. A finite-element vibration analysis of FG nanosize plates based on classical plate theory (CPT) has been conducted by Natarajan *et al.* [36]. Based on the third-order plate theory, Daneshmehr and Rajabpoor [37] examined the buckling behavior of nonlocal graded nanoplates under different boundary conditions. The analysis of resonance frequencies of FG micro and nanoplates according to nonlocal elasticity and strain gradient theory has been performed by Nami and Janghorban [38]. They used nonlocal and strain gradient theories separately, and concluded that these theories have different mechanisms in the analysis of nanoplates. Application of three-dimensional nonlocal elasticity theory in static and vibration analysis of FG nanoplate has been investigated by Ansari *et al.* [39] based on classical plate model. Based on the generalized differential quadrature method (GDQM), Daneshmehr and Rajabpoor [40] analyzed the vibrational behavior of higher-order FG nanoplates using nonlocal stress field theory. Application of four-variable plate theory in vibration analysis of FG nanoplates is examined by Belkorissat *et al.* [41]. They stated that presented plate model have fewer field variables compared with first-order and third-order plate

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theories. Based on four-variable plate theory, the shear deformation effect is captured, while the governing equations are very similar to the classical plate theory. Wang and Li [42] examined nonlinear free vibration of nanotubes considering small-scale effects embedded in a viscoelastic medium. Wave propagation, buckling and vibration analyses of smart FG nanoplates under various physical fields have been carried out by Ebrahimi and Barati [43–45] using different plate theories. A comprehensive investigation of bending, buckling and vibrational behaviors of FG nanoplates on elastic medium is conducted by Sobhy [46]. Also, Khorshidi and Fallah [47] performed buckling analysis of FG nanoplates via a general nonlocal exponential shear deformation plate model. Sobhy and Radwan [48] presented a new quasi-3D nonlocal plate theory for vibration and buckling of FGM Nanoplates.

In the previous papers based on FG nanoplates, only the stiffness-softening effect of the nonlocal stress field was considered. Although the nonlocal elasticity theory (NET) of Eringen is a suitable theory for modeling of nanostructure, it has some shortcomings due to neglecting the stiffness-hardening mechanism reported in experimental works and strain gradient elasticity [49]. By using the nonlocal strain gradient theory (NSGT), Lim *et al.* [50] matched the dispersion curves of nanobeams with those of experimental data. They concluded that NSGT is more accurate for modeling and analysis of nanostructures by considering both stiffness reduction and enhancement effects. The application of NSGT in the wave dispersion analysis of FG nanobeams has been examined by Li *et al.* [51]. Also, some investigations have been performed using NSGT on vibration and buckling of nanorods, nanotubes and nanobeams [52–56]. Also, Farajpour *et al.* [57] presented buckling analysis of nanoplates via a nonlocal strain gradient plate model employing exact and differential quadrature methods. In another work, Farajpour *et al.* [58] presented nonlocal strain gradient modeling of nanomechanical vibrating piezoelectric mass sensors. Also, Ebrahimi *et al.* [59] applied NSGT for wave propagation analysis of FG nanoplates under thermal loading. Therefore, it is of great importance to analyze the vibration behavior of FG nanoplates via NSGT. Despite its significance, there is no study on the dynamic stability analysis of double-layered FG nanoplates supported by elastic medium.

The porosity-dependent dynamic analysis of double-layered nanoplates under in-plane periodic loads is presented for the first time according to the nonlocal strain gradient theory. In contrast to the nonlocal elasticity theory in which one scale parameter is used to describe the size effect, the present theory possesses two scale parameters for a better description of size effects. The material properties of a porous nanoplate are described via a new power-law function. Nonclassical boundary conditions related to nonlocal strain gradient theory as well as governing equations are obtained using Hamilton's principle. By solving the governing equations using Galerkin's method, natural frequencies of the nanoplate are obtained. The results show that the vibrational behavior of the nanoplate are significantly influenced by the porosities, nonlocality, strain gradient parameter, material composition, elastic foundation and geometrical parameters. The obtained frequencies can be used as benchmark results in the analysis of nanoplates modeled by nonlocal-strain gradient elasticity.

2 Nonlocal strain gradient nanoplate model

The proposed nonlocal strain gradient theory [50] takes into account both nonlocal stress field and the strain gradient effects by introducing two scale parameters. This theory defines the stress field as

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \nabla\sigma_{ij}^{(1)} \tag{1}$$

in which the stresses $\sigma_{ij}^{(0)}$ and $\sigma_{ij}^{(1)}$ are corresponding to strain ε_{ij} and strain gradient $\nabla\varepsilon_{ij}$, respectively as

$$\sigma_{ij}^{(0)} = \int_V C_{ijkl}\alpha_0(x, x', e_0 a)\varepsilon'_{kl}(x')\mathrm{d}x', \tag{2}$$

$$\sigma_{ij}^{(1)} = l^2 \int_V C_{ijkl}\alpha_1(x, x', e_1 a)\nabla\varepsilon'_{kl}(x')\mathrm{d}x', \tag{3}$$

in which C_{ijkl} are the elastic coefficients and $e_0 a$ and $e_1 a$ capture the nonlocal effects and l captures the strain gradient effects. When the nonlocal functions $\alpha_0(x, x', e_0 a)$ and $\alpha_1(x, x', e_1 a)$ satisfy the developed conditions by Eringen, the constitutive relation of nonlocal strain gradient theory has the following form:

$$[1 - (e_1 a)^2\nabla^2][1 - (e_0 a)^2\nabla^2]\sigma_{ij} = C_{ijkl}[1 - (e_1 a)^2\nabla^2]\varepsilon_{kl} - C_{ijkl}l^2[1 - (e_0 a)^2\nabla^2]\nabla^2\varepsilon_{kl}, \tag{4}$$

in which ∇^2 denotes the Laplacian operator. Considering $e_1 = e_0 = e$, the general constitutive relation in eq. (3) becomes

$$[1 - (ea)^2\nabla^2]\sigma_{ij} = C_{ijkl}[1 - l^2\nabla^2]\varepsilon_{kl}. \tag{5}$$

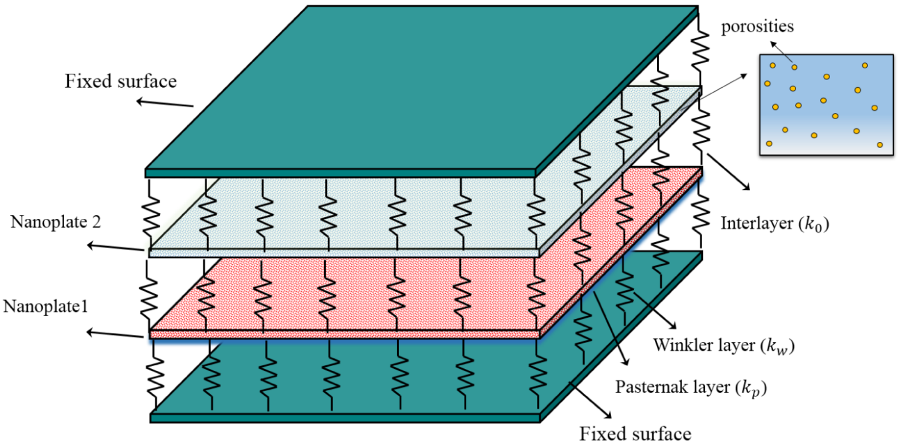

Fig. 1. Configuration of a nanoporous inhomogeneous double-layered nanoplate on elastic substrate.

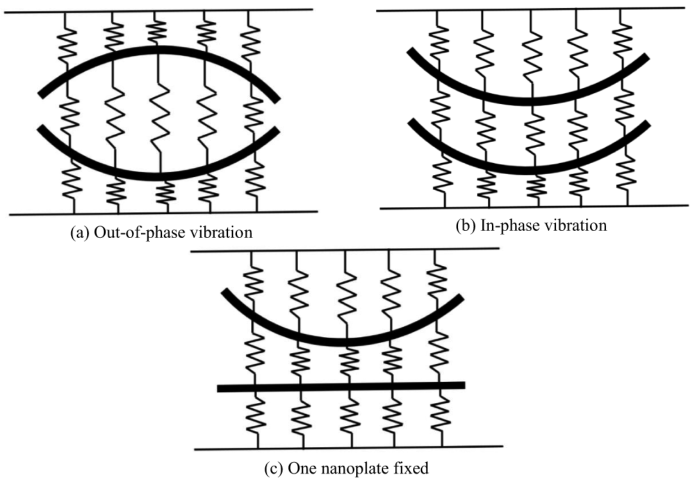

Fig. 2. Different types of motion for a double-layered nanoplate.

3 FG plate model based on neutral surface position

Consider a rectangular (a × b) porous nanoplate of uniform thickness h as shown in fig. 1. The double-layered nanoplate experiences different types of vibration as shown in fig. 2. A FG material can be specified by the variation in the volume fractions. Due to this variation, the neutral axis of the nanoplate may not coincide with its mid-surface which leads to bending-extension coupling. By using neutral axis, this coupling is eliminated. Based on the modified power-law model, Young' modulus E and mass density ρ are described as [5]

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m - \frac{\xi}{2}(E_c + E_m), \tag{6a}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m - \frac{\xi}{2}(\rho_c + \rho_m), \tag{6b}$$

in which c and m denote the material properties of the ceramic and metal phases, respectively and p is the inhomogeneity or the power-law index. Also, ξ is the porosity volume fraction. The displacement field according to the four-variable plate model considering an exact position of the neutral surface can be expressed by

$$u_1(x,y,z,t) = u(x,y,t) - (z - z^*)\frac{\partial w_b}{\partial x} - [f(z) - z^{**}]\frac{\partial w_s}{\partial x}, \tag{7a}$$

$$u_2(x,y,z,t) = v(x,y,t) - (z - z^*)\frac{\partial w_b}{\partial y} - [f(z) - z^{**}]\frac{\partial w_s}{\partial y}, \tag{7b}$$

$$u_3(x,y,z,t) = w(x,y,t) = w_b(x,y,t) + w_s(x,y,t), \tag{7c}$$

where

$$z^* = \frac{\int_{-h/2}^{h/2} E(z) z \mathrm{d}z}{\int_{-h/2}^{h/2} E(z) \mathrm{d}z}, \qquad z^{**} = \frac{\int_{-h/2}^{h/2} E(z) f(z) \mathrm{d}z}{\int_{-h/2}^{h/2} E(z) \mathrm{d}z}. \tag{8}$$

Also, u and v are in-plane displacements and w_b and w_s denote the bending and shear transverse displacement, respectively. The shape function of transverse shear deformation is considered as

$$f(z) = -\frac{z}{4} + \frac{5z^3}{3h^2}. \tag{9}$$

According to the present plate theory with four unknowns, the nonzero strains are obtained as

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} - (z - z^*)\frac{\partial^2 w_b}{\partial x^2} - [f(z) - z^{**}]\frac{\partial^2 w_s}{\partial x^2}, \\
\varepsilon_y &= \frac{\partial v}{\partial y} - (z - z^*)\frac{\partial^2 w_b}{\partial y^2} - [f(z) - z^{**}]\frac{\partial^2 w_s}{\partial y^2}, \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2(z - z^*)\frac{\partial^2 w_b}{\partial x \partial y} - 2[f(z) - z^{**}]\frac{\partial^2 w_s}{\partial x \partial y}, \\
\gamma_{yz} &= g(z)\frac{\partial w_s}{\partial y}, \qquad \gamma_{xz} = g(z)\frac{\partial w_s}{\partial x}.
\end{aligned} \tag{10}$$

Also, the extended Hamilton's principle express that:

$$\int_0^t \delta(U - T + V)\mathrm{d}t = 0, \tag{11}$$

here U is the strain energy, T is the kinetic energy and V is the work done by the external forces. The first variation of the strain energy can be calculated as

$$\begin{aligned}
\delta U = \int_V \Big(&\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xx}^{(1)}\delta\nabla\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{yy}^{(1)}\delta\nabla\varepsilon_{yy} + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xy}^{(1)}\delta\nabla\gamma_{xy} \\
&+ \sigma_{yz}\delta\gamma_{yz} + \sigma_{yz}^{(1)}\delta\nabla\gamma_{yz} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{xz}^{(1)}\delta\nabla\gamma_{xz}\Big)\,\mathrm{d}V,
\end{aligned} \tag{12}$$

in which σ are the components of the stress tensor and ε are the components of the strain tensor.

Substituting eqs. (8) and (10) into eq. (12) yields

$$\begin{aligned}
\delta U = \int_0^a \int_0^b \Bigg[&N_{xx}\left[\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right] - M_{xx}^b \frac{\partial^2 \delta w_b}{\partial x^2} - M_{xx}^s \frac{\partial^2 \delta w_s}{\partial x^2} + N_{yy}\left[\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right] \\
&- M_{yy}^b \frac{\partial^2 \delta w_b}{\partial y^2} - M_{yy}^s \frac{\partial^2 \delta w_s}{\partial y^2} + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right) - 2M_{xy}^b \frac{\partial^2 \delta w_b}{\partial x \partial y} \\
&- 2M_{xy}^s \frac{\partial^2 \delta w_s}{\partial x \partial y} + Q_{yz}\frac{\partial \delta w_s}{\partial y} + Q_{xz}\frac{\partial \delta w_s}{\partial x}\Bigg]\,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{13}$$

in which

$$
\begin{aligned}
N_{xx} &= \int_{-h/2}^{h/2} \left(\sigma_{xx}^{0} - \nabla\sigma_{xx}^{(1)}\right) \mathrm{d}z = N_{xx}^{(0)} - \nabla N_{xx}^{(1)},\\
N_{xy} &= \int_{-h/2}^{h/2} \left(\sigma_{xy}^{0} - \nabla\sigma_{xy}^{(1)}\right) \mathrm{d}z = N_{xy}^{(0)} - \nabla N_{xy}^{(1)},\\
N_{yy} &= \int_{-h/2}^{h/2} \left(\sigma_{yy}^{0} - \nabla\sigma_{yy}^{(1)}\right) \mathrm{d}z = N_{yy}^{(0)} - \nabla N_{yy}^{(1)},\\
M_{xx}^{b} &= \int_{-h/2}^{h/2} z\left(\sigma_{xx}^{0} - \nabla\sigma_{xx}^{(1)}\right) \mathrm{d}z = M_{xx}^{b(0)} - \nabla M_{xx}^{b(1)},\\
M_{xx}^{s} &= \int_{-h/2}^{h/2} f\left(\sigma_{xx}^{0} - \nabla\sigma_{xx}^{(1)}\right) \mathrm{d}z = M_{xx}^{s(0)} - \nabla M_{xx}^{s(1)},\\
M_{yy}^{b} &= \int_{-h/2}^{h/2} z\left(\sigma_{yy}^{0} - \nabla\sigma_{yy}^{(1)}\right) \mathrm{d}z = M_{yy}^{b(0)} - \nabla M_{yy}^{b(1)},\\
M_{yy}^{s} &= \int_{-h/2}^{h/2} f\left(\sigma_{yy}^{0} - \nabla\sigma_{yy}^{(1)}\right) \mathrm{d}z = M_{yy}^{s(0)} - \nabla M_{yy}^{s(1)},\\
M_{xy}^{b} &= \int_{-h/2}^{h/2} z\left(\sigma_{xy}^{0} - \nabla\sigma_{xy}^{(1)}\right) \mathrm{d}z = M_{xy}^{b(0)} - \nabla M_{xy}^{b(1)},\\
M_{xy}^{s} &= \int_{-h/2}^{h/2} f\left(\sigma_{xy}^{0} - \nabla\sigma_{xy}^{(1)}\right) \mathrm{d}z = M_{xy}^{s(0)} - \nabla M_{xy}^{s(1)},\\
Q_{xz} &= \int_{-h/2}^{h/2} g\left(\sigma_{xz}^{0} - \nabla\sigma_{xz}^{(1)}\right) \mathrm{d}z = Q_{xz}^{(0)} - \nabla Q_{xz}^{(1)},\\
Q_{yz} &= \int_{-h/2}^{h/2} g\left(\sigma_{yz}^{0} - \nabla\sigma_{yz}^{(1)}\right) \mathrm{d}z = Q_{yz}^{(0)} - \nabla Q_{yz}^{(1)},
\end{aligned}
\tag{14a}
$$

where

$$
\begin{aligned}
N_{ij}^{(0)} &= \int_{-h/2}^{h/2} \left(\sigma_{ij}^{(0)}\right) \mathrm{d}z, & N_{ij}^{(1)} &= \int_{-h/2}^{h/2} \left(\sigma_{ij}^{(1)}\right) \mathrm{d}z,\\
M_{ij}^{b(0)} &= \int_{-h/2}^{h/2} z\left(\sigma_{ij}^{b(0)}\right) \mathrm{d}z, & M_{ij}^{b(1)} &= \int_{-h/2}^{h/2} z\left(\sigma_{ij}^{b(1)}\right) \mathrm{d}z,\\
M_{ij}^{s(0)} &= \int_{-h/2}^{h/2} f\left(\sigma_{ij}^{s(0)}\right) \mathrm{d}z, & M_{ij}^{s(1)} &= \int_{-h/2}^{h/2} f\left(\sigma_{ij}^{s(1)}\right) \mathrm{d}z,\\
Q_{xz}^{(0)} &= \int_{-h/2}^{h/2} g\left(\sigma_{xz}^{i(0)}\right) \mathrm{d}z, & Q_{xz}^{(1)} &= \int_{-h/2}^{h/2} g\left(\sigma_{xz}^{i(1)}\right) \mathrm{d}z,\\
Q_{yz}^{(0)} &= \int_{-h/2}^{h/2} g\left(\sigma_{yz}^{i(0)}\right) \mathrm{d}z, & Q_{yz}^{(1)} &= \int_{-h/2}^{h/2} g\left(\sigma_{yz}^{i(1)}\right) \mathrm{d}z,
\end{aligned}
\tag{14b}
$$

in which $(ij = xx, xy, yy)$. The first variation of the work done by the applied forces can be written as

$$
\begin{aligned}
\delta V = \int_0^a \int_0^b \Bigg(& N_x^0 \frac{\partial(w_b+w_s)}{\partial x}\frac{\partial\delta(w_b+w_s)}{\partial x} + N_y^0 \frac{\partial(w_b+w_s)}{\partial y}\frac{\partial\delta(w_b+w_s)}{\partial y}\\
&+ 2\delta N_{xy}^0 \frac{\partial(w_b+w_s)}{\partial x}\frac{\partial(w_b+w_s)}{\partial y} - k_w(w_b+w_s)\delta(w_b+w_s)\\
&+ k_p\left(\frac{\partial(w_b+w_s)}{\partial x}\frac{\partial\delta(w_b+w_s)}{\partial x} + \frac{\partial(w_b+w_s)}{\partial y}\frac{\partial\delta(w_b+w_s)}{\partial y}\right)\Bigg) \mathrm{d}y\,\mathrm{d}x,
\end{aligned}
\tag{15}
$$

where N_x^0, N_y^0, N_{xy}^0 are the in-plane applied loads; k_w and k_p are Winkler and Pasternak constants. However, shear loading is discarded in this article. The first variation of the kinetic energy can be written in the following form:

$$\begin{aligned}\delta K = \int_0^a \int_0^b \Bigg[& I_0 \left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \delta v}{\partial t} + \frac{\partial (w_b+w_s)}{\partial t}\frac{\partial \delta (w_b+w_s)}{\partial t} \right) - I_1 \left(\frac{\partial u}{\partial t}\frac{\partial \delta w_b}{\partial x \partial t} + \frac{\partial w_b}{\partial x \partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \delta w_b}{\partial y \partial t} \right.\\ & \left. + \frac{\partial w_b}{\partial y \partial t}\frac{\partial \delta v}{\partial t} \right) - I_3 \left(\frac{\partial u}{\partial t}\frac{\partial \delta w_s}{\partial x \partial t} + \frac{\partial w_s}{\partial x \partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \delta w_s}{\partial y \partial t} + \frac{\partial w_s}{\partial y \partial t}\frac{\partial \delta v}{\partial t} \right) + I_2 \left(\frac{\partial w_b}{\partial x \partial t}\frac{\partial \delta w_b}{\partial x \partial t} + \frac{\partial w_b}{\partial y \partial t}\frac{\partial \delta w_b}{\partial y \partial t} \right)\\ & + I_5 \left(\frac{\partial w_s}{\partial x \partial t}\frac{\partial \delta w_s}{\partial x \partial t} + \frac{\partial w_s}{\partial y \partial t}\frac{\partial \delta w_s}{\partial y \partial t} \right) + I_4 \left(\frac{\partial w_b}{\partial x \partial t}\frac{\partial \delta w_s}{\partial x \partial t} + \frac{\partial w_s}{\partial x \partial t}\frac{\partial \delta w_b}{\partial x \partial t} + \frac{\partial w_b}{\partial y \partial t}\frac{\partial \delta w_s}{\partial y \partial t} + \frac{\partial w_s}{\partial y \partial t}\frac{\partial \delta w_b}{\partial y \partial t} \right) \Bigg] \mathrm{d}y\,\mathrm{d}x, \end{aligned} \tag{16}$$

in which

$$(I_0, I_1, I_2, I_3, I_4, I_5) = \int_{-h/2}^{h/2} \left(1, z-z^*, (z-z^*)^2, f-z^{**}, (z-z^*)(f-z^{**}), (f-z^{**})^2\right) \rho(z)\mathrm{d}z. \tag{17}$$

By inserting eqs. (13)–(16) into eq. (11) and setting the coefficients of δu, δv, δw_b and δw_s to zero, the following Euler-Lagrange equations can be obtained:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^3 w_b}{\partial x \partial t^2} - I_3 \frac{\partial^3 w_s}{\partial x \partial t^2}, \tag{18}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v}{\partial t^2} - I_1 \frac{\partial^3 w_b}{\partial y \partial t^2} - I_3 \frac{\partial^3 w_s}{\partial y \partial t^2}, \tag{19}$$

$$\begin{aligned}&\frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} - k_w(w_b+w_s) + (k_p - N^0)\nabla^2(w_b+w_s),\\ &= I_0 \frac{\partial^2 (w_b+w_s)}{\partial t^2} + I_1 \left(\frac{\partial^3 u}{\partial x \partial t^2} + \frac{\partial^3 v}{\partial y \partial t^2} \right) - I_2 \nabla^2 \left(\frac{\partial^2 w_b}{\partial t^2} \right) - I_4 \nabla^2 \left(\frac{\partial^2 w_s}{\partial t^2} \right),\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} - k_w(w_b+w_s) + (k_p - N^0)\nabla^2(w_b+w_s) =\\ &I_0 \frac{\partial^2 (w_b+w_s)}{\partial t^2} + I_3 \left(\frac{\partial^3 u}{\partial x \partial t^2} + \frac{\partial^3 v}{\partial y \partial t^2} \right) - I_4 \nabla^2 \left(\frac{\partial^2 w_b}{\partial t^2} \right) - I_5 \nabla^2 \left(\frac{\partial^2 w_s}{\partial t^2} \right).\end{aligned} \tag{21}$$

The classical and nonclassical boundary conditions can be obtained in the derivation process when using the integrations by parts. Thus, we obtain classical boundary conditions at $x = 0$ or a and $y = 0$ or b as

$$\begin{aligned}&\text{Specify} \quad w_b \quad \text{or} \quad \left(\frac{\partial M_{xx}^b}{\partial x} + \frac{\partial M_{xy}^b}{\partial y} \right) n_x + \left(\frac{\partial M_{yy}^b}{\partial y} + \frac{\partial M_{xy}^b}{\partial x} \right) n_y = 0,\\ &\text{Specify} \quad w_s \quad \text{or} \quad \left(\frac{\partial M_{xx}^s}{\partial x} + \frac{\partial M_{xy}^s}{\partial y} + Q_{xz} \right) n_x + \left(\frac{\partial M_{yy}^s}{\partial y} + \frac{\partial M_{xy}^s}{\partial x} + Q_{yz} \right) n_y = 0,\\ &\text{Specify} \quad \frac{\partial w_b}{\partial n} \quad \text{or} \quad M_{xx}^b n_x^2 + n_x n_y M_{xy}^b + M_{yy}^b n_y^2 = 0,\end{aligned} \tag{22}$$

where $\frac{\partial ()}{\partial n} = n_x \frac{\partial ()}{\partial x} + n_y \frac{\partial ()}{\partial y}$; n_x and n_y are the x and y-components of the unit normal vector on the nanoplate boundaries, respectively and the nonclassical boundary conditions are:

$$\begin{aligned}&\text{Specify} \quad \frac{\partial^2 w_b}{\partial x^2} \quad \text{or} \quad M_{xx}^{b(1)} = 0,\\ &\text{Specify} \quad \frac{\partial^2 w_b}{\partial y^2} \quad \text{or} \quad M_{yy}^{b(1)} = 0,\\ &\text{Specify} \quad \frac{\partial^2 w_s}{\partial x^2} \quad \text{or} \quad M_{xx}^{s(1)} = 0,\\ &\text{Specify} \quad \frac{\partial^2 w_s}{\partial y^2} \quad \text{or} \quad M_{yy}^{s(1)} = 0.\end{aligned} \tag{23}$$

Based on the NSGT, the constitutive relations of the presented higher-order FG nanoplate can be stated as

$$(1-\mu\nabla^2)\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}=\frac{E(z)}{1-v^2}(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0 & 0 & 0\\ v & 1 & 0 & 0 & 0\\ 0 & 0 & (1-v)/2 & 0 & 0\\ 0 & 0 & 0 & (1-v)/2 & 0\\ 0 & 0 & 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}. \tag{24}$$

Integrating eq. (24) over the plate's cross-section area, one can obtain the force-strain, and the moment-strain of the nonlocal refined FG plates can be obtained as follows:

$$(1-\mu\nabla^2)\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix}=A(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{Bmatrix}, \tag{25}$$

$$(1-\mu\nabla^2)\begin{Bmatrix}M_x^b\\ M_y^b\\ M_{xy}^b\end{Bmatrix}=D(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}-\frac{\partial^2 w_b}{\partial x^2}\\ -\frac{\partial^2 w_b}{\partial y^2}\\ -2\frac{\partial^2 w_b}{\partial x\partial y}\end{Bmatrix}+E(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}-\frac{\partial^2 w_s}{\partial x^2}\\ -\frac{\partial^2 w_s}{\partial y^2}\\ -2\frac{\partial^2 w_s}{\partial x\partial y}\end{Bmatrix}, \tag{26}$$

$$(1-\mu\nabla^2)\begin{Bmatrix}M_x^s\\ M_y^s\\ M_{xy}^s\end{Bmatrix}=E(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}-\frac{\partial^2 w_b}{\partial x^2}\\ -\frac{\partial^2 w_b}{\partial y^2}\\ -2\frac{\partial^2 w_b}{\partial x\partial y}\end{Bmatrix}+F(1-\lambda\nabla^2)\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v)/2\end{pmatrix}\begin{Bmatrix}-\frac{\partial^2 w_s}{\partial x^2}\\ -\frac{\partial^2 w_s}{\partial y^2}\\ -2\frac{\partial^2 w_s}{\partial x\partial y}\end{Bmatrix}, \tag{27}$$

$$(1-\mu\nabla^2)\begin{Bmatrix}Q_x\\ Q_y\end{Bmatrix}=A_{44}(1-\lambda\nabla^2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{Bmatrix}\frac{\partial w_s}{\partial x}\\ \frac{\partial w_s}{\partial y}\end{Bmatrix}, \tag{28}$$

in which

$$A=\int_{-h/2}^{h/2}\frac{E(z)}{1-v^2}\mathrm{d}z,\qquad D=\int_{-h/2}^{h/2}\frac{E(z)(z-z^*)^2}{1-v^2}\mathrm{d}z,\qquad E=\int_{-h/2}^{h/2}\frac{E(z)(z-z^*)(f-z^{**})}{1-v^2}\mathrm{d}z$$
$$F=\int_{-h/2}^{h/2}\frac{E(z)(f-z^{**})^2}{1-v^2}\mathrm{d}z,\qquad A_{44}=\int_{-h/2}^{h/2}\frac{E(z)}{2(1+v)}g^2\mathrm{d}z. \tag{29}$$

The governing equations in terms of the displacements for a NSGT refined four-variable FG nanoplate can be derived by substituting eqs. (25)–(28), into eqs. (18)–(21) as follows:

$$A(1-\lambda\nabla^2)\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{1-v}{2}\frac{\partial^2 u_1}{\partial y^2}+\frac{1+v}{2}\frac{\partial^2 v_1}{\partial x\partial y}\right)+(1-\mu\nabla^2)\left(-I_0\frac{\partial^2 u_1}{\partial t^2}+I_1\frac{\partial^3 w_{1,b}}{\partial x\partial t^2}+I_3\frac{\partial^3 w_{1,s}}{\partial x\partial t^2}\right)=0, \tag{30}$$

$$A(1-\lambda\nabla^2)\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{1-v}{2}\frac{\partial^2 v_1}{\partial x^2}+\frac{1+v}{2}\frac{\partial^2 u_1}{\partial x\partial y}\right)+(1-\mu\nabla^2)\left(-I_0\frac{\partial^2 v_1}{\partial t^2}+I_1\frac{\partial^3 w_{1,b}}{\partial y\partial t^2}+I_3\frac{\partial^3 w_{1,s}}{\partial y\partial t^2}\right)=0, \tag{31}$$

$$
\begin{aligned}
&-D(1-\lambda\nabla^2)\left(\frac{\partial^4 w_{1,b}}{\partial x^4}+2\frac{\partial^4 w_{1,b}}{\partial x^2\partial y^2}+\frac{\partial^4 w_{1,b}}{\partial y^4}\right)-E(1-\lambda\nabla^2)\left(\frac{\partial^4 w_{1,s}}{\partial x^4}+2\frac{\partial^4 w_{1,s}}{\partial x^2\partial y^2}+\frac{\partial^4 w_{1,s}}{\partial y^4}\right)\\
&+(1-\mu\nabla^2)\left(-I_0\frac{\partial^2(w_{1,b}+w_{1,s})}{\partial t^2}-I_1\left(\frac{\partial^3 u_1}{\partial x\partial t^2}+\frac{\partial^3 v_1}{\partial y\partial t^2}\right)+I_2\nabla^2\left(\frac{\partial^2 w_{1,b}}{\partial t^2}\right)\right.\\
&\left.+I_4\nabla^2\left(\frac{\partial^2 w_{1,s}}{\partial t^2}\right)-k_w(w_{1,b}+w_{1,s})+(k_p-N^0)\nabla^2(w_{1,b}+w_{1,s})\right)\\
&-k_0\left[(w_{1,b}+w_{1,s}-w_{2,b}-w_{2,s})-\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(w_{1,b}+w_{1,s}-w_{2,b}-w_{2,s})\right]=0, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&-E(1-\lambda\nabla^2)\left(\frac{\partial^4 w_{1,b}}{\partial x^4}+2\frac{\partial^4 w_{1,b}}{\partial x^2\partial y^2}+\frac{\partial^4 w_{1,b}}{\partial y^4}\right)-F(1-\lambda\nabla^2)\left(\frac{\partial^4 w_{1,s}}{\partial x^4}+2\frac{\partial^4 w_{1,s}}{\partial x^2\partial y^2}+\frac{\partial^4 w_{1,s}}{\partial y^4}\right)\\
&+A_{44}(1-\lambda\nabla^2)\left(\frac{\partial^2 w_{1,s}}{\partial x^2}+\frac{\partial^2 w_{1,s}}{\partial y^2}\right)+(1-\mu\nabla^2)\left(-I_0\frac{\partial^2(w_{1,b}+w_{1,s})}{\partial t^2}-I_3\left(\frac{\partial^3 u_1}{\partial x\partial t^2}+\frac{\partial^3 v_1}{\partial y\partial t^2}\right)\right.\\
&\left.+I_4\nabla^2\left(\frac{\partial^2 w_{1,b}}{\partial t^2}\right)+I_5\nabla^2\left(\frac{\partial^2 w_{1,s}}{\partial t^2}\right)-k_w(w_{1,b}+w_{1,s})+(k_p-N^0)\nabla^2(w_{1,b}+w_{1,s})\right)\\
&-k_0\left[(w_{1,b}+w_{1,s}-w_{2,b}-w_{2,s})-\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(w_{1,b}+w_{1,s}-w_{2,b}-w_{2,s})\right]=0, \qquad (33)
\end{aligned}
$$

4 Solution procedure

In this section, Galerkin's method is implemented to solve the governing equations of nonlocal strain gradient based double-layered nanoplates. The double-layered nanoplates experience three kinds of vibrations as indicate in fig. 2:

- Out of phase vibration: $w_b = w_{1,b} - w_{2,b} \neq 0$ and $w_s = w_{1,s} - w_{2,s} \neq 0$.
- In-phase vibration: $w_b = w_{1,b} - w_{2,b} = 0$ and $w_s = w_{1,s} - w_{2,s} = 0$.
- One nanoplate fixed: $w_b = w_{1,b} = 0$ and $w_s = w_{1,s} = 0$.

In the case of out-of-phase vibration, both nanoplates vibrate asynchronously, however, in the case of in-phase vibration both nanoplates vibrate synchronously. Thus, the displacement field can be calculated as:

$$u=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\frac{\partial X_m(x)}{\partial x}Y_n(y)e^{i\omega_n t}, \qquad (34)$$

$$v=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}X_m(x)\frac{\partial Y_n(y)}{\partial y}e^{i\omega_n t}, \qquad (35)$$

$$w_b=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{bmn}X_m(x)Y_n(y)e^{i\omega_n t}, \qquad (36)$$

$$w_s=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{smn}X_m(x)Y_n(y)e^{i\omega_n t}, \qquad (37)$$

where $(U_{mn}, V_{mn}, W_{bmn}, W_{smn})$ are the unknown coefficients and the functions X_m and Y_n satisfy the boundary conditions. The classical and nonclassical boundary condition for each nanoplate are

$$
\begin{aligned}
&w_b = w_s = 0,\\
&\frac{\partial^2 w_b}{\partial x^2}=\frac{\partial^2 w_s}{\partial x^2}=\frac{\partial^2 w_b}{\partial y^2}=\frac{\partial^2 w_s}{\partial y^2}=0,\\
&\frac{\partial^4 w_b}{\partial x^4}=\frac{\partial^4 w_s}{\partial x^4}=\frac{\partial^4 w_b}{\partial y^4}=\frac{\partial^4 w_s}{\partial y^4}=0. \qquad (38)
\end{aligned}
$$

By substituting eqs. (34)–(37) into eqs. (30)–(33), the matrix form of the governing equations of harmonically loaded nanoplate can be expressed by

$$[M]\{\ddot{\Lambda}\}+[[K]+N_0(t)[G]]\{\Lambda\}=0, \qquad (39)$$

where $[\mathbf{M}]$, $[\mathbf{K}]$ and $[\mathbf{G}]$ denote the mass, stiffness and geometric stiffness matrices, respectively, and $\{\Lambda\}$ is the displacement vector ($\{\Lambda\} = \{U_{mn}, V_{mn}, W_{bmn}, W_{smn}\}$). Also, the components of stiffness and mass matrices are listed in appendix A.

Considering periodic axial excitation compressive load $N_0(t) = -[\alpha + \beta \cos(\varpi t)]N_{cr}$ which is consist of static and dynamical components, the governing equation can be expressed by

$$[M]\{\ddot{\Lambda}\} + [[K] - \{\alpha + \beta \cos(\varpi t)\}N_{cr}[G]]\{\Lambda\} = 0, \tag{40}$$

where ϖ and N_{cr} denote excitation frequency and buckling load, respectively; α and β denote the static and dynamic load factors. To calculate dimensionless excitation frequency, the following relation is adopted:

$$\Omega = \varpi a\sqrt{\frac{\rho_c}{E_c}}\,. \tag{41}$$

The instability boundaries considering periodic coefficients of the Mathieu-Hill type can be formed by periodic T_0 and $2T_0$ in which $T_0 = 2\pi/\varpi$. It is reported that the boundaries of instability regions with period T_0 are less important compared to those with period $2T_0$. The solution with respect to period $2T_0$ can be obtained by the following equation:

$$[[K] - N_{cr}\{\alpha \pm 0.5\beta\}[G] - 0.25\varpi[M]]\{\Lambda\} = 0. \tag{42}$$

The nontrivial solution of eq. (20) gives

$$\det\begin{vmatrix} [\bar{K}] - (0.5\beta)N_{cr}[G] - (0.25\varpi)[M] & 0 \\ 0 & [\bar{K}] + (0.5\beta)N_{cr}[G] - (0.25\varpi)[M] \end{vmatrix} = 0, \tag{43}$$

in which $[\bar{K}] = [K] - \alpha N_{cr}[G]$. For a given value of α, the plots of eigenfrequency Ω with respect to β provide stability regions of the doble-layered FGM nanoplates. Also, nondimensional parameters are defined as

$$\mu = \frac{ea}{a}\,, \qquad \lambda = \frac{l}{a}\,, \qquad K_w = \frac{k_w a^4}{D_c}\,, \qquad K_0 = \frac{k_0 a^4}{D_c}\,, \qquad K_p = \frac{k_p a^2}{D_c}\,, \qquad D_c = \frac{E_c h^3}{12(1-v_c^2)}\,. \tag{44}$$

Finally, setting the coefficient matrix to zero gives the natural frequencies. The function X_m for simply-supported boundary conditions is defined by

$$X_m(x) = \sin(\lambda_m x), \qquad \lambda_m = \frac{m\pi}{a}\,. \tag{45}$$

The function Y_n can be obtained by replacing x, m and a, respectively by y, n and b.

5 Numerical results and discussions

In this section, results are presented for dynamic study of size-dependent and double-layered porous FGM nanoplates modeled via a 4-unknown plate model considering the exact position of neutral surface. First of all, the frequency response of the present study is validated with those of the classical plate theory obtained by Natarajan *et al.* (2012) through the finite-element approach. These results are tabulated in table 1 for fully simply-supported and fully clamped edge conditions and a good agreement is observed. The length of nanoplate is assumed as $a = 50\,\text{nm}$. Also, the material properties of nanoplate (alumina and aluminum) are considered as:

- $E_c = 380\,\text{GPa}$, $\rho_c = 3800\,\text{kg/m}^3$, $v_c = 0.3$,
- $E_m = 70\,\text{GPa}$, $\rho_m = 2707\,\text{kg/m}^3$, $v_m = 0.3$.

The variation of natural frequency of double-layered nanoplates with respect to nonlocal and strain gradient parameters is presented in fig. 3 when $a/h = 10$, $p = 1$, $K_w = 25$, $K_p = 5$ and $K_0 = 25$. It is clear that the natural frequency of double-layered nanoplates reduces with the increase of the nonlocal parameter for every value of the strain gradient parameter. But, the vibration frequency increases at a fixed nonlocal parameter and inhomogeneity index. Due to the lack of a strain gradient parameter in the previous vibration analyses of nanoplates, only the softening effect due to nonlocality was concluded. Therefore, the material instability and heterogeneous deformation due to the strain gradient could not be considered within the framework of the nonlocal elasticity theory. These observations are valid for every types of vibration. However, at fixed nonlocal and strain gradient parameters, the out-of-phase vibration of the system has larger frequencies compared with the in-phase motion. Also, when one nanoplate is fixed, the frequencies are always between those obtained for in-phase and out-of-phase motion.

Table 1. Comparison of nondimensional fundamental natural frequency $\hat{\omega} = \omega h\sqrt{\rho_c/G_c}$ of the nanoplates with simply-supported and clamped boundary conditions ($p = 5$).

a/h	μ	$a/b = 1$		$a/b = 2$	
		Natarajan *et al.* (2012)	Present study	Natarajan *et al.* (2012)	Present study
10	0	0.0441	0.043823	0.1055	0.104329
	1	0.0403	0.04007	0.0863	0.085493
	2	0.0374	0.037141	0.0748	0.074174
	4	0.0330	0.032806	0.0612	0.060673
20	0	0.0113	0.011256	0.0279	0.027756
	1	0.0103	0.010288	0.0229	0.022722
	2	0.0096	0.009534	0.0198	0.019704
	4	0.0085	0.008418	0.0162	0.016110

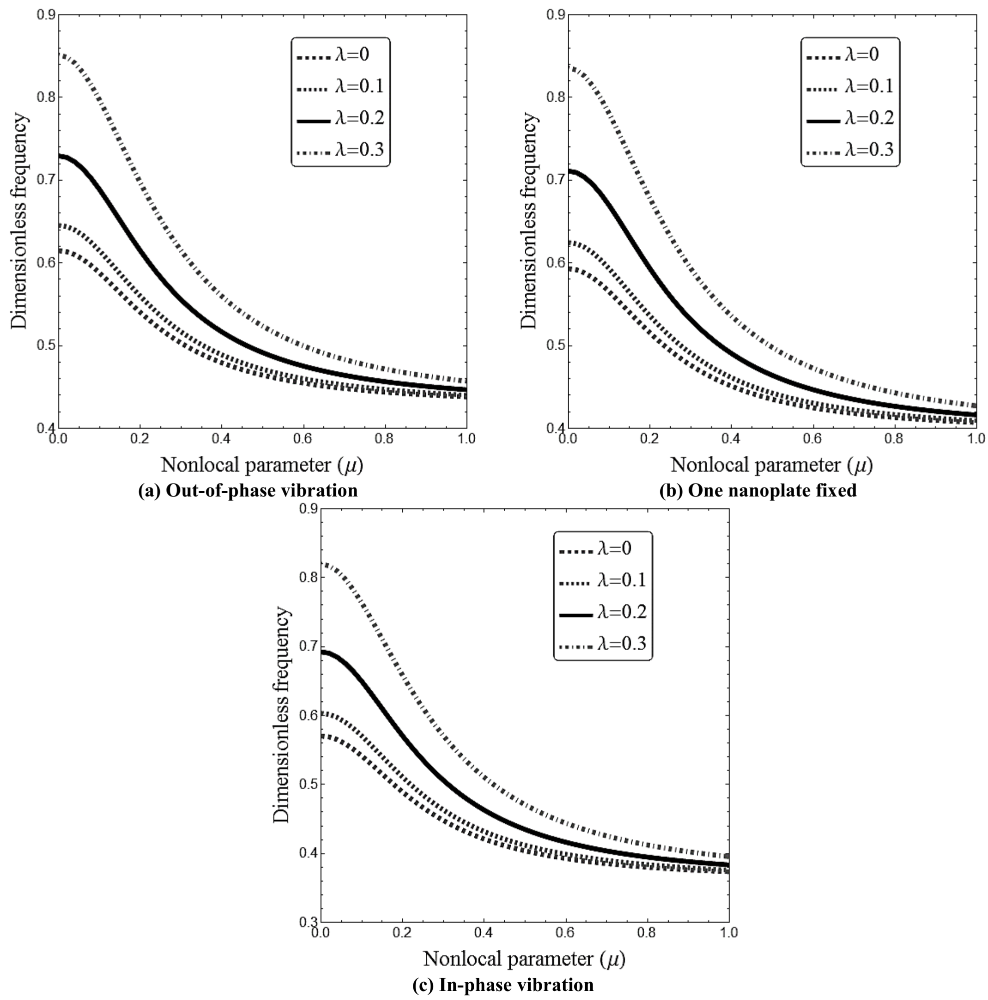

Fig. 3. Variation of the dimensionless frequency of a double-nanoplate system *versus* nonlocal and strain gradient parameters ($a/h = 10$, $p = 1$, $K_w = 25$, $K_p = 5$, $K_0 = 25$).

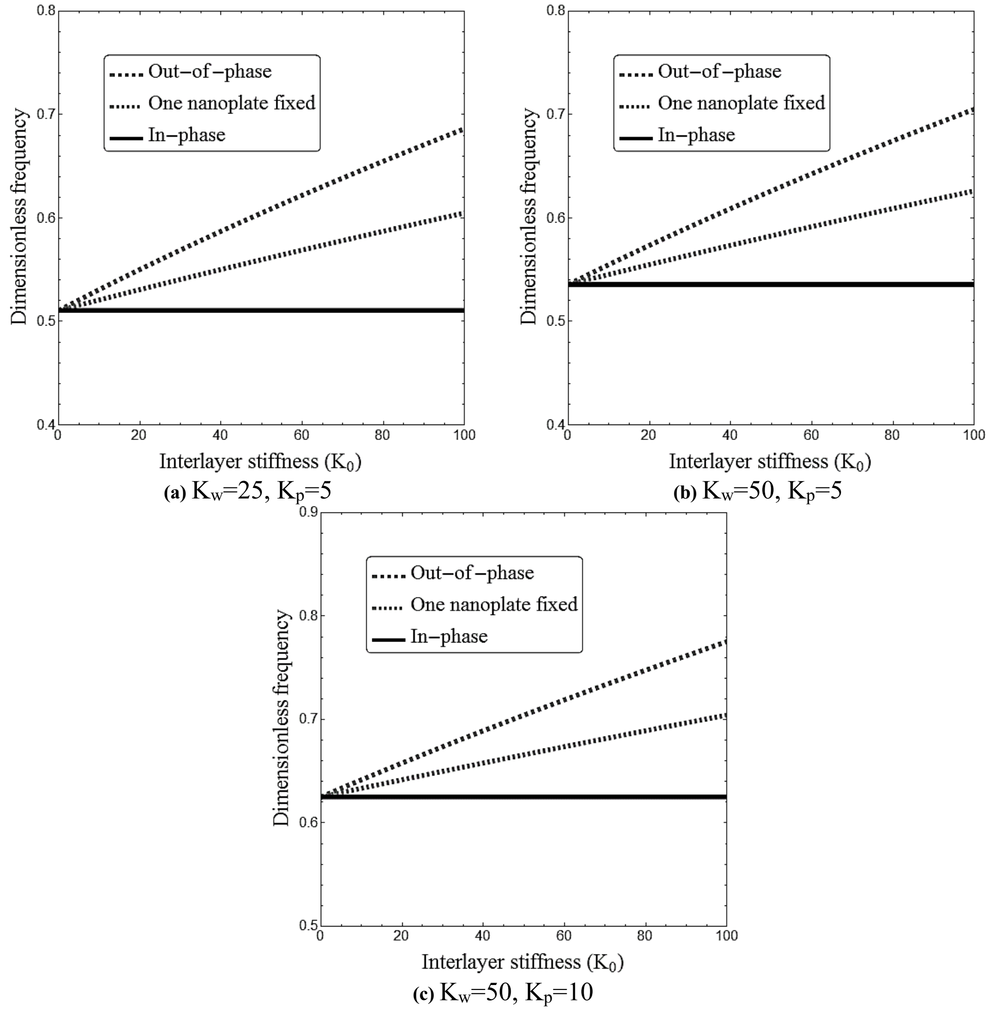

Fig. 4. Variation of the dimensionless frequency of a double-nanoplate system *versus* interlayer stiffness for various elastic foundation parameters ($a/h = 10$, $p = 1$, $\mu = 0.2$, $\lambda = 0.1$).

The influences of interlayer stiffness (K_0) and Winkler-Pasternak coefficients on the vibrational frequencies of double-layered nanoplates are illustrated in fig. 4 at $a/h = 10$, $p = 1$, $\mu = 0.2$ and $\lambda = 0.1$. As previously stated, the in-phase vibration of double-layered nanoplates is not influenced by the interlayer springs. However, the frequencies of out-of-phase and one nanoplate fixed vibrations enlarge with the increase of interlayer stiffness. Also, it is seen that the out-of-phase vibration of double-layered nanoplates is more influenced by the interlayer stiffness than when one nanoplate is fixed.

It is also obvious that the vibrational behavior of double-layered nanoplates rely on the magnitudes of both Winkler and Pasternak parameters. It should be mentioned that Pasternak layer has a continuous interaction with nanoplates, while Winkler layer has a discontinuous interaction with the nanoplate. The enlargement of Winkler and Pasternak coefficients yields the enhancement of the bending rigidity and natural frequency of the system.

Figure 5 shows the influence of the static load factor (α) and nonlocality parameter (μ) on the dynamic stability characteristics of the size-dependent FGM nanoplates at $a/h = 10$, $p = 1$ and $K_w = K_p = 0$. It can be observed in the figure that when the nonlocal parameter (μ) increases, the dynamic buckling boundaries are degraded. It means that the parametric instability can be enhanced by the nanoscale. However, the starting point ($\beta = 0$) is reduced with the increase of the nonlocal parameter. The reason is that the existence of nonlocality diminishes the bending rigidity of the FG nanoplates leading to the reduction in the frequencies. Hence, the nonlocal FGM plate model gives lower excitation frequency compared to local one. According to this figure, when the static load factor rises, the boundaries of the dynamic instability region reduce at a fixed nonlocal parameter. This is due to the fact that compressive static load degrades the flexibility of the FGM nanoplate, and leads to smaller excitation frequencies. One can see that the instability region of FGM nanoplates becomes closer to the origin by increasing the magnitude of the static load factor.

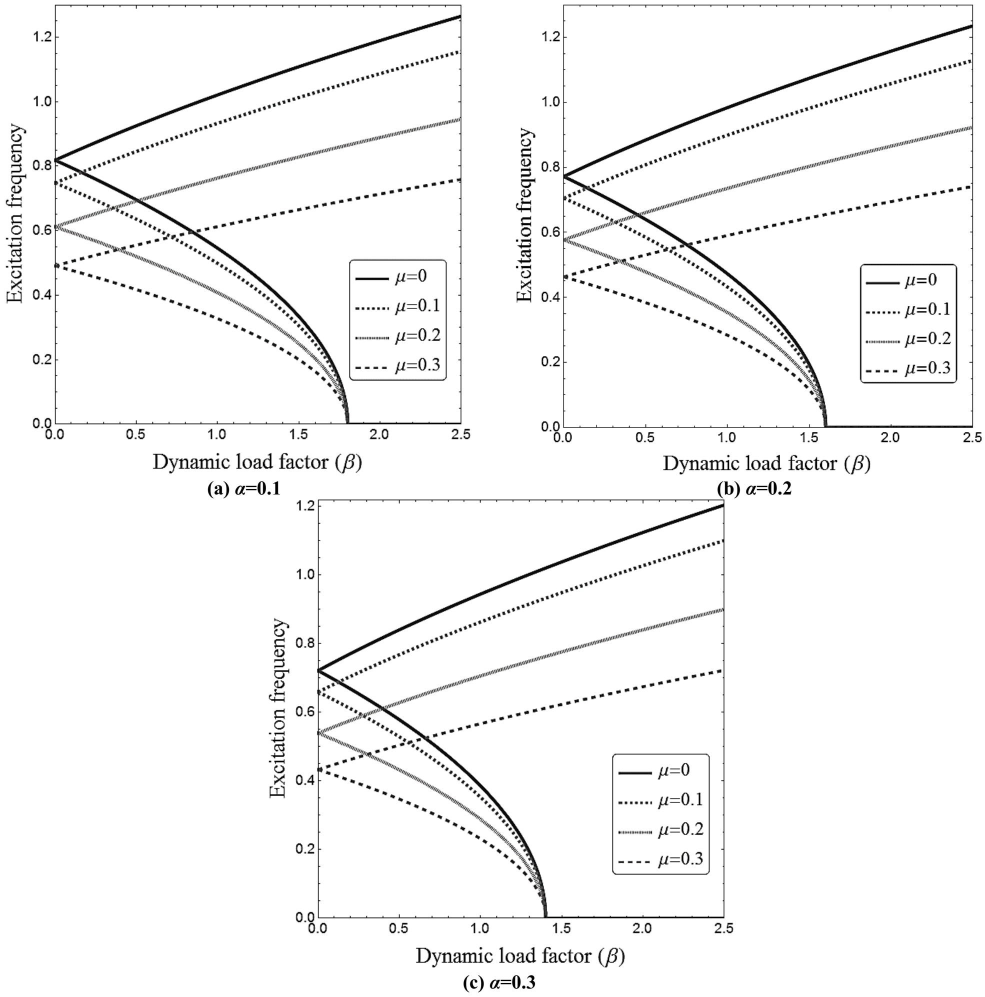

Fig. 5. Dimensionless frequency of the nanoplate *versus* the dynamic load factor for different nonlocal parameters an static load factors ($a/h = 10$, $p = 1$, $\xi = 0$, $K_w = 0$, $K_p = 0$).

In the case of in-phase vibration, the porosity effect on the stability boundaries of FG nanoplates with respect to the dynamic load factor is presented in fig. 6 at $\mu = 0.2$, $\alpha = 0.3$, $K_w = 0$ and $K_p = 0$ for different material inhomogeneity index (p). Porosities inside the material lead to smaller frequencies by reducing the stiffness of the nanoplate. However, the instability region becomes smaller with the increase of porosity volume fraction. Therefore, a porous FG nanoplate under periodic in-plane loads is more stable than a perfect one. It can be also deduced that by reducing the gradient index, the width of the instability region is increased. Also, it is observable that as the gradient index rises, the magnitude of nondimensional excitation frequency increases at a fixed dynamic load factor. Therefore, material gradation has a major role on the unstable region and should be considered in the dynamic analysis of nanoplates.

Figure 7 illustrates the dimensionless frequency of the double-layered nanoplate *versus* the dynamic load factor for different cases of motion when $a/h = 10$, $p = 1$, $\xi = 0.1$, $K_w = 25$, $K_p = 5$, $K_0 = 50$, $\mu = 0.2$, $\lambda = 0.1$. As previously stated, the boundaries of the dynamic instability region reduce with the increase of the static load factor. This is due to the fact that a compressive static load degrades the flexibility of the FGM nanoplate, and leads to smaller excitation frequencies. It is also seen that the instability region of FGM nanoplates becomes closer to the origin by increasing the magnitude of the static load factor. Also, the stability region in the case of one nanoplate fixed is placed between the regions of in-phase and out-of-phase vibrations. It can be observed in the figure that the maximum frequency when $\beta = 0$ is observed for out-of-phase vibration.

Investigation of porosity and inhomogeneity effects on the free vibrational behavior of double-layered nanoplates is plotted in fig. 8 when $a/h = 10$, $K_w = 10$, $K_p = 0.5$, $K_0 = 10$ and $\mu = 0.2$. It is observed that the presence of porosities inside the material leads to lower plate stiffness and natural frequency. So, the obtained frequencies are overestimated by neglecting the porosity effect. Also, all these observations are affected by the gradation of material properties or inhomogeneity index (p). In fact, the increase of the inhomogeneity index (p) is proportional to higher metal constituent which leads to smaller frequencies. Another observation is that the effect of porosity on vibration frequencies becomes more significant at larger inhomogeneity indices.

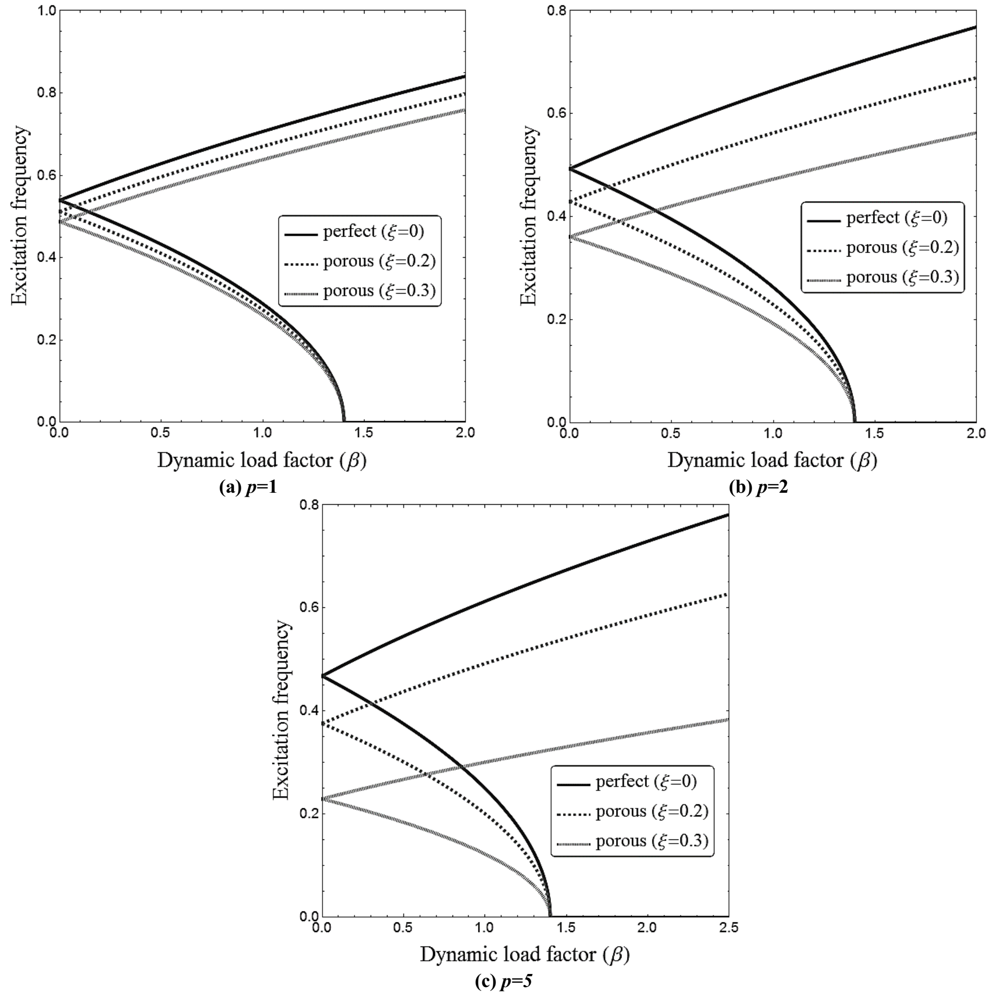

Fig. 6. Dimensionless frequency of the nanoplate *versus* the dynamic load factor for different gradient index and porosity volume fractions ($a/h = 10$, $\alpha = 0.3$, $K_w = 0$, $K_p = 0$, $\mu = 0.2$).

6 Conclusions

A new nanoplate model based on the nonlocal strain gradient theory was presented for the dynamic analysis of double-layered porous nanoplates. The proposed model introduced two scale parameters for the prediction of vibration frequencies of nanoplates very accurately. The formulation of the nanoplate was based on a higher-order shear deformation theory with four field variables. A power-law function was employed to describe the graded material properties. Employing extended Hamilton's principle, the governing equations of the nanoplate were derived. These equations were solved via Galerkin's method to obtain the frequencies. It was shown that porosities inside the material significantly affect the stability regions of the nanoplates. It was also reported that both nonlocal and strain gradient parameters should be considered in the modeling of double-layered nanoplates to capture both stiffness-softening and stiffness-hardening effects. Also, by reducing the gradient index, the width of the instability region is increased.

Appendix A.

$$
\begin{aligned}
k_{1,1} = {} & A\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^5 X_m}{\partial x^5}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.\\
& \left.\left.+\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right) + A\frac{1-v}{2}\left(\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& \left.-\lambda\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y+\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^4 Y_n}{\partial y^4}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),
\end{aligned}
\tag{A.1}
$$

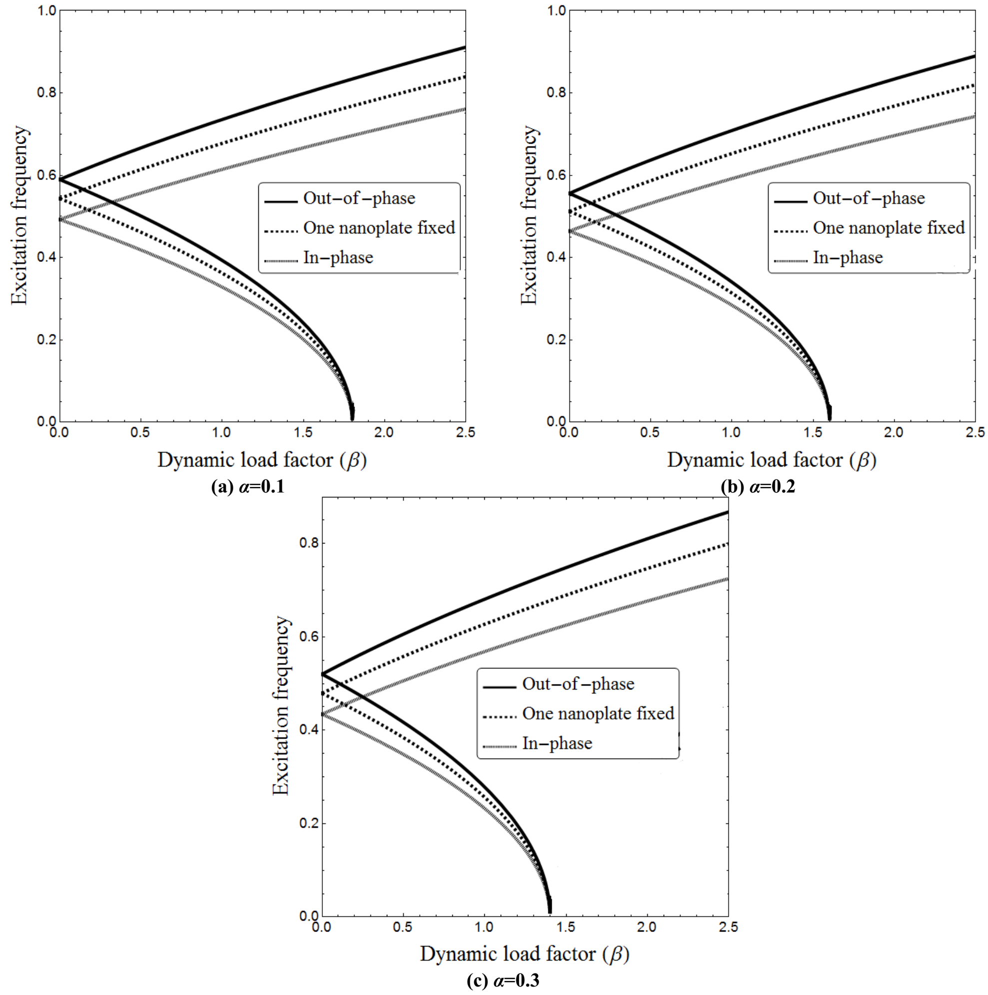

Fig. 7. Dimensionless frequency of the double-layered nanoplate *versus* the dynamic load factor for different cases of motion ($a/h = 10$, $p = 1$, $\xi = 0.1$, $K_w = 25$, $K_p = 5$, $K_0 = 50$, $\mu = 0.2$, $\lambda = 0.1$).

$$k_{1,2} = A\frac{1+v}{2}\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$
$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right), \tag{A.2}$$

$$k_{2,1} = A\frac{1+v}{2}\left(\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$
$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^4 Y_n}{\partial y^4}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right), \tag{A.3}$$

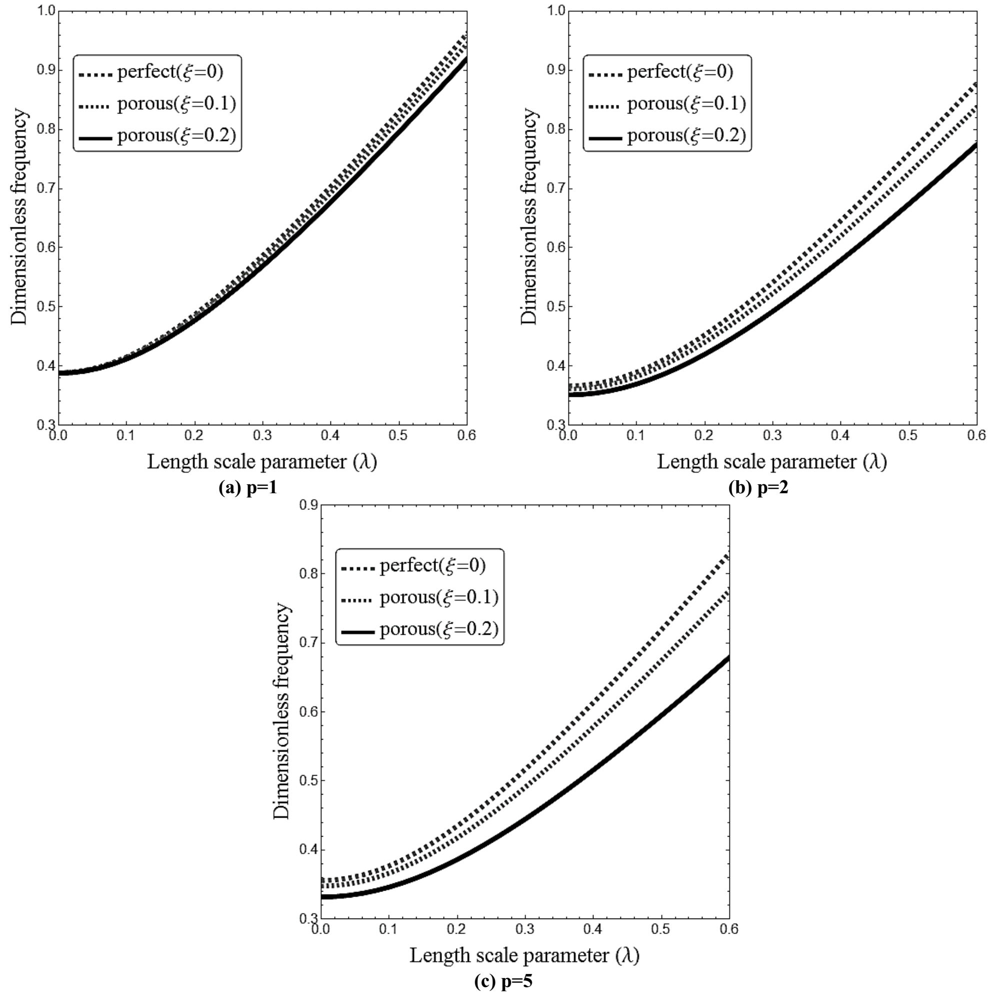

Fig. 8. Variation of the in-phase frequency of the double-nanoplate system *versus* the strain gradient parameter for different power-law indices and aspect ratios ($a/h = 10$, $K_w = 10$, $K_p = 0.5$, $K_0 = 10$, $\mu = 0.2$).

$$k_{2,2} = A\left(\int_0^b\int_0^a\left(X_m\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^5 Y_n}{\partial y^5}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right) + A\frac{1-v}{2}\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right), \tag{A.4}$$

$$k_{2,3} = k_{3,2} = -E\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_n X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^6 X_m}{\partial x^6}Y_n X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y + 3\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial^2 Y_n}{\partial y^2}X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y + 3\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^4 Y_n}{\partial y^4}X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^6 Y_n}{\partial y^6}X_m Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right), \tag{A.5}$$

$$\begin{aligned}
k_{3,3} = {} & -D\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^6 X_m}{\partial x^6}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + 3\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 3\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\\
& \left.\left. + \int_0^b\int_0^a\left(X_m\frac{\partial^6 Y_n}{\partial y^6}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right) - (K_w + K_0)\left(\int_0^b\int_0^a(X_mY_nX_mY_n)\mathrm{d}x\,\mathrm{d}y\right.\\
& \left.- \mu\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right)\\
& + K_p\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& - \mu\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& \left.\left. + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
k_{3,4} = k_{4,3} = {} & -E\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^6 X_m}{\partial x^6}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + 3\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 3\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\\
& \left.\left. + \int_0^b\int_0^a\left(X_m\frac{\partial^6 Y_n}{\partial y^6}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right) - (K_w + K_0)\left(\int_0^b\int_0^a(X_mY_nX_mY_n)\mathrm{d}x\,\mathrm{d}y\right.\\
& \left.- \mu\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right)\\
& + K_p\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + \int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& - \mu\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& \left.\left. + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
k_{4,4} = {} & -F\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + \int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y - \lambda\left(\int_0^b\int_0^a\left(\frac{\partial^6 X_m}{\partial x^6}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\\
& + 3\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y + 3\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\\
& \left.\left. + \int_0^b\int_0^a\left(X_m\frac{\partial^6 Y_n}{\partial y^6}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right) + A_{44}\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.
\end{aligned}$$

$$+\int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y-\lambda\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.$$

$$\left.+\,2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y+\int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)$$

$$-(K_w+K_0)\left(\int_0^b\int_0^a(X_mY_nX_mY_n)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right)+K_p\left(\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.$$

$$+\int_0^b\int_0^a\left(X_m\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(\frac{\partial^4 X_m}{\partial x^4}Y_nX_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right.$$

$$\left.\left.+\,2\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial^2 Y_n}{\partial y^2}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y+\int_0^b\int_0^a\left(X_m\frac{\partial^4 Y_n}{\partial y^4}X_mY_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.8}$$

$$m_{1,1}=+I_0\left(\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.9}$$

$$m_{1,2}=+I_0\left(\int_0^b\int_0^a\left(X_m\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(X_m\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.10}$$

$$m_{3,1}=-I_1\left(\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.11}$$

$$m_{4,1}=-I_3\left(\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(\frac{\partial^3 X_m}{\partial x^3}Y_n\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial X_m}{\partial x}\frac{\partial^2 Y_n}{\partial y^2}\frac{\partial X_m}{\partial x}Y_n\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.12}$$

$$m_{3,2}=-I_1\left(\int_0^b\int_0^a\left(X_m\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(X_m\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.13}$$

$$m_{4,2}=-I_3\left(\int_0^b\int_0^a\left(X_m\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y-\mu\left(\int_0^b\int_0^a\left(X_m\frac{\partial^3 Y_n}{\partial y^3}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right.\right.$$

$$\left.\left.+\int_0^b\int_0^a\left(\frac{\partial^2 X_m}{\partial x^2}\frac{\partial Y_n}{\partial y}X_m\frac{\partial Y_n}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right)\right),\tag{A.14}$$

$$m_{3,3} = +I_0 \left(\int_0^b \int_0^a (X_m Y_n X_m Y_n) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.\right.$$
$$\left.\left. + \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right) - I_2 \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$+ \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^4 X_m}{\partial x^4} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$\left.\left. + 2 \int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y + \int_0^b \int_0^a \left(X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right), \tag{A.15}$$

$$m_{3,4} = m_{4,3} = +I_0 \left(\int_0^b \int_0^a (X_m Y_n X_m Y_n) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.\right.$$
$$\left.\left. + \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right) - I_4 \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$+ \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^4 X_m}{\partial x^4} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$\left.\left. + 2 \int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y + \int_0^b \int_0^a \left(X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right), \tag{A.16}$$

$$m_{4,4} = +I_0 \left(\int_0^b \int_0^a (X_m Y_n X_m Y_n) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.\right.$$
$$\left.\left. + \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right) - I_5 \left(\int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$+ \int_0^b \int_0^a \left(X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y - \mu \left(\int_0^b \int_0^a \left(\frac{\partial^4 X_m}{\partial x^4} Y_n X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right.$$
$$\left.\left. + 2 \int_0^b \int_0^a \left(\frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y + \int_0^b \int_0^a \left(X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n \right) \mathrm{d}x\,\mathrm{d}y \right)\right). \tag{A.17}$$

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