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Charged cylindrically symmetric collapse in f(R) gravity

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Abstract. This paper investigates the charged cylindrically symmetric gravitational collapse with perfect fluid in $f(R)$ gravity. We formulate dynamical equations using the Misner-Sharp formalism and study the effects of charge, energy density and effective pressure on the cylindrical collapse. We also establish a relationship between the Weyl tensor, energy density and dark source terms. For constant Ricci scalar, it is concluded that the spacetime is conformally flat if and only if the energy density is homogeneous.

1 Introduction

Recent evidences from several sources, like cosmic microwave background, supernovae type Ia and weak gravitational lensing of distant galaxies, indicate an accelerated expansion of the universe. It is assumed that this increasing rate of cosmic expansion is due to a cryptical force called dark energy (DE). This is an unknown form of energy which possesses a large negative pressure and is considered to spread all of the space inclined to accelerate expansion of the universe. There are mainly two ways to study the ambiguous nature of DE: either by modifying the matter or geometric part of the Einstein-Hilbert action. One of these modified theories, which has received much attention, is the $f(R)$ theory obtained by replacing the Ricci scalar R by an arbitrary function in the Einstein-Hilbert action. In the last few years, many people have made theoretical developments in this gravity and compared their results with general relativity (GR).

The gravitational explosion of astronomical objects is considered an interesting phenomenon for astrophysicists. It is the basic mechanism for the structure formation of the universe and describes contraction of celestial objects. For massive stars, gravitational collapse occurs when the nuclear fuel is exhausted and there is not enough pressure to overcome the gravitational force. The collapsing phenomenon leads to the formation of new relativistic objects, such as neutron stars, white dwarfs and blackholes. The dynamics of radiating collapse is an interesting issue in GR. Chandrasekhar [1] initiated the work on gravitational collapse in 1936. Chan [2] explored the collapse of radiating stars in the presence of anisotropic fluid and described the effect of shear viscosity on the pressure anisotropy. Herrera and Santos [3] examined the dynamical processes for which dissipation occurs in terms of radiation density and heat flux. Sharif and Ahmed [4] discussed the spherical collapse for a perfect fluid with the cosmological constant and observed that the cosmological constant decreases the process of collapse.

It is observed that electromagnetic field acts as a Coulomb repulsive force, which decreases the collapsing phenomenon by hydrostatic equilibrium. If stellar objects contain non-zero electric charge, then the collapse of such astronomical objects leads to the formation of a Reissner-Nordström black hole. Benkenstein [5] analyzed the charged spherical collapse for a perfect fluid. Di Prisco *et al.* [6] investigated spherical collapse with an anisotropic fluid and formulated dynamical as well as causal transport equations. Sharif and Siddiqa [7] studied the plane symmetric collapse in the presence of a charge viscous fluid with non-adiabatic flow and formulated dynamical equation, Weyl tensor and heat transport equation.

The deviation from spherical to other symmetries is an incidental characteristic which reveals significant mathematical results about self-gravitating objects. There is a strong motivation for cylindrical symmetry due to the existence of cylindrical gravitational waves. Sharif and Abbas [8] analyzed charged cylindrical perfect fluid collapse and found that the collapsing process slows down when the gravitational and Coulomb forces balance each other due to hydrostatic

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equilibrium. The same authors [9] explored the charged spherical perfect fluid collapse in the presence of the cosmological constant. Sharif and his collaborators [10–12] analyzed the effects of the electromagnetic field on the dynamical instability of collapsing systems and found that the collapse rate decreases by increasing electric charge. Chakraborty and Chakraborty [13] studied cylindrical collapse in the presence of an anisotropic fluid using the Misner-Sharp formalism and concluded that collapsing matter is affected by radial pressure, which acts as a gravitational wave. Shah and Abbas [14] discussed the charged cylindrical collapse for the radiating stars and formulated heat transport equation.

The phenomenon of gravitational collapse has also been examined in $f(R)$ gravity by taking isotropic as well as anisotropic fluids. Sharif and Kausar [15] explored the spherical perfect fluid collapse in this gravity and examined the apparent horizon as well as time formation of singularity. Cembranos *et al.* [16] analyzed the collapsing phenomenon in modified theories and considered the collapse of dust particles for a particular $f(R)$ model. Chakrabarti and Banerjee [17] investigated spherical collapse with perfect fluid in $f(R)$ gravity and formulated time formation of singularity as well as apparent horizon.

In this paper, we study the dynamics of the charged cylindrical collapse with a perfect fluid in $f(R)$ gravity using the Misner-Sharp formalism. The format of the paper is as follows. In the next section, we formulate the Einstein-Maxwell field equations and obtain C-energy. We also discuss Darmois junction conditions. Section 3 deals with dynamical equations. In sect. 4, we establish the relationship between the Weyl tensor, energy density and dark source terms. Finally, we conclude our results in the last section.

2 Collapsing model and field equations

The action of $f(R)$ gravity is defined as [18]

$$\tilde{S} = \int \left[\frac{f(R)}{2\kappa} + \mathcal{L}_m\right]\sqrt{-g}\mathrm{d}^4x. \tag{1}$$

Taking variation of this action with respect to $g_{\mu\nu}$, we obtain the field equations in $f(R)$ gravity as

$$f_R R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R = 8\pi G T^{(m)}_{\mu\nu}, \tag{2}$$

where $f_R = \frac{\mathrm{d}f}{\mathrm{d}R}$ and $\Box$ is the d'Alembert operator. In terms of the Einstein tensor, the field equations can be written as

$$G_{\mu\nu} = \frac{\kappa}{f_R}(T^{(m)}_{\mu\nu} + T^{(D)}_{\mu\nu}), \tag{3}$$

where

$$T^{(D)}_{\mu\nu} = \frac{1}{\kappa}\left(\frac{f(R) - Rf_R}{2}g_{\mu\nu} + \nabla_\mu\nabla_\nu f_R - g_{\mu\nu}\Box f_R\right) \tag{4}$$

is the effective energy-momentum tensor, which gives contribution of curvature to the Einstein tensor. The stress-energy tensor for the perfect fluid is

$$T^{(m)}_{\mu\nu} = (\rho + p)v_\mu v_\nu + g_{\mu\nu}p,$$

where p and ρ represent pressure and density of the fluid, respectively. The four-velocity of the fluid particles (v_μ) is given by $v^\mu = A^{-1}\delta^\mu_0$ and, being it a unit timelike vector, it is normalized as $v_\mu v^\mu = -1$.

The cylindrically symmetric metric (V^-) with comoving coordinates inside the hypersurface is of the form [19]

$$\mathrm{d}s^2_- = -A^2(t,r)\mathrm{d}t^2 + B^2(t,r)\mathrm{d}r^2 + C^2(t,r)\mathrm{d}\phi^2 + D^2(t,r)\mathrm{d}z^2, \tag{5}$$

where $-\infty \le t \le \infty$, $r \ge 0$, $0 \le \phi \le 2\pi$, $-\infty < z < +\infty$. For the exterior spacetime (V^+), we consider a charged cylindrically symmetric metric as [20]

$$\mathrm{d}s^2_+ = -H\mathrm{d}\hat{T}^2 + H\mathrm{d}\hat{R}^2 + \hat{R}^2(\mathrm{d}\phi^2 + \beta^2\mathrm{d}z^2), \tag{6}$$

where

$$H = \left(\frac{2Q^2}{\hat{R}^2} - \frac{4M}{\hat{R}}\right). \tag{7}$$

Here Q is the charge per unit length of the cylinder, M represents the mass and β is a constant having dimension of $\frac{1}{\mathrm{lenth}}$. In the presence of electromagnetic field, the field equations become

$$G_{\mu\nu} = \frac{\kappa}{f_R}\left(T^{(m)}_{\mu\nu} + T^{(D)}_{\mu\nu} + E_{\mu\nu}\right), \tag{8}$$

where $E_{\mu\nu}$ is the energy-momentum tensor for electromagnetic field given by

$$E_{\mu\nu} = \frac{1}{4\pi}\left(F^{\lambda}_{\mu}F_{\nu\lambda} - \frac{1}{4}F^{\lambda\gamma}F_{\lambda\gamma}g_{\mu\nu}\right). \tag{9}$$

The Maxwell field equations are

$$F_{\lambda\gamma} = \Phi_{\gamma,\lambda} - \Phi_{\lambda,\gamma}, \quad F^{\lambda\gamma}_{\gamma} = 4\pi J^{\lambda}, \tag{10}$$

where $F_{\lambda\gamma}$ is the Maxwell field tensor, Φ_{λ} denotes the four-potential and J_{λ} represents the four-current. Since in comoving coordinates, charge is considered to be at rest, so the magnetic field vanishes [21]. Consequently, the four-potential and four-current take the form

$$\Phi_{\lambda} = \Phi\delta^{0}_{\lambda}, \quad J^{\lambda} = \mu_o v^{\lambda}. \tag{11}$$

The charge conservation, $J^{\lambda}_{;\lambda} = 0$, yields

$$\check{q}(r) = 2\pi\int \mu_o BCD\mathrm{d}r, \tag{12}$$

which gives total charge distribution per unit length of the cylinder. The electric field intensity for the given cylindrical surface is defined as

$$\mathbb{E} = \frac{\check{q}(r)}{2\pi C}\,.$$

The corresponding Maxwell equations are given by

$$\Phi'' - \left(\frac{A'}{A} + \frac{B'}{B} - \frac{C'}{C} - \frac{D'}{D}\right)\Phi' = 4\pi\mu_o AB^2, \tag{13}$$

$$\dot{\Phi}' - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right) = 0, \tag{14}$$

where dot and prime denote differentiation with respect to t and r, respectively. Integrating eq. (13), we have

$$\Phi' = \frac{2\check{q}(r)AB}{CD}\,. \tag{15}$$

The C-energy for a cylindrically symmetric spacetime is defined by [22]

$$\hat{E} = m(t,r) = \frac{1}{8}\left(1 - L^{-2}\nabla^{\beta}r\nabla_{\beta}r\right), \tag{16}$$

where $\hat{E}$ is the gravitational energy per specific length of the cylinder. The specific length L, circumference radius ρ and areal radius r are expressed as $L^2 = \xi_{(3)\beta}\xi^{\beta}_{(3)}$, $\rho^2 = \xi_{(2)\beta}\xi^{\beta}_{(2)}$ and $r = \rho L$. For the interior spacetime, the C-energy becomes

$$\hat{E} = \frac{1}{8} + \frac{1}{8D^2}\left[\frac{1}{A^2}(C\dot{D} + \dot{C}D) - \frac{1}{B^2}(CD' + DC')\right].$$

In the presence of an electromagnetic field in the interior region of stars, the specific energy ($E^* = \hat{E}L$) of the cylinder is given as follows:

$$E^* = \frac{L}{8} + \frac{1}{8D}\left[\frac{1}{A^2}(C\dot{D} + \dot{C}D) - \frac{1}{B^2}(CD' + DC')\right] + \frac{\check{q}^2}{2C}\,. \tag{17}$$

The corresponding Einstein-Maxwell field equations (8) give

$$\frac{\kappa}{f_R}\left(\rho+\frac{T_{00}^{(D)}}{A^2}-\frac{2\pi\mathbb{E}^2}{D^2}\right)A^2=\frac{A^2}{B^2}\left(-\frac{C''}{C}-\frac{D''}{D}+\frac{B'C'}{BC}+\frac{B'D'}{BD}-\frac{C'D'}{CD}\right)+\left(\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{B}\dot{D}}{BD}+\frac{\dot{C}\dot{D}}{CD}\right), \tag{18}$$

$$\frac{\kappa}{f_R}\left(p+\frac{T_{11}^{(D)}}{B^2}-\frac{2\pi\mathbb{E}^2}{D^2}\right)B^2=-\frac{B^2}{A^2}\left(\frac{\ddot{C}}{C}+\frac{\ddot{D}}{D}+\frac{\dot{C}\dot{D}}{CD}-\frac{\dot{A}\dot{C}}{AC}-\frac{\dot{A}\dot{D}}{AD}\right)+\left(\frac{C'D'}{CD}+\frac{A'C'}{AC}+\frac{A'D'}{AD}\right), \tag{19}$$

$$\frac{\kappa}{f_R}\left(p+\frac{T_{22}^{(D)}}{C^2}+\frac{2\pi\mathbb{E}^2}{D^2}\right)C^2=\frac{C^2}{B^2}\left(\frac{A''}{A}+\frac{D''}{D}+\frac{A'D'}{AD}-\frac{A'B'}{AB}-\frac{D'B'}{DB}\right)-\frac{C^2}{A^2}\left(\frac{\ddot{B}}{B}+\frac{\ddot{D}}{D}-\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{A}\dot{D}}{AD}+\frac{\dot{B}\dot{D}}{BD}\right), \tag{20}$$

$$\frac{\kappa}{f_R}\left(p+\frac{T_{33}^{(D)}}{D^2}+\frac{2\pi\mathbb{E}^2}{D^2}\right)D^2=\frac{D^2}{B^2}\left(\frac{A''}{A}+\frac{C''}{C}+\frac{A'C'}{AC}-\frac{A'B'}{AB}-\frac{C'B'}{CB}\right)-\frac{D^2}{A^2}\left(\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}\right), \tag{21}$$

$$\frac{\kappa}{f_R}\left(T_{01}^{(D)}\right)=\left(-\frac{\dot{C}'}{C}-\frac{\dot{D}'}{D}+\frac{C'\dot{B}}{CB}+\frac{D'\dot{B}}{DB}+\frac{A'\dot{C}}{AC}+\frac{A'\dot{D}}{AD}\right). \tag{22}$$

Equations (18)–(22) describe how the spacetime is affected by the electromagnetic field and how it is curved due to the gravitation, mass and energy of the system. The values of $T_{00}^{(D)}$, $T_{11}^{(D)}$, $T_{22}^{(D)}$ and $T_{33}^{(D)}$ are given in appendix A. Equations (A.1)–(A.5) indicate that they contain third-order derivatives of f. The term $(\rho+\frac{T_{00}^{(D)}}{A^2})$ represents the effective energy density, while the terms $(p+\frac{T_{11}^{(D)}}{B^2})$, $(p+\frac{T_{22}^{(D)}}{C^2})$ and $(p+\frac{T_{33}^{(D)}}{D^2})$ give effective pressure.

Now we apply Darmois junction conditions [23,24] between interior (V^-) and exterior (V^+) spacetimes to find the mass function. These conditions are given as follows.
i) Over the hypersurface (Σ), the continuity of the first fundamental form gives

$$\mathrm{d}s_-^2=\mathrm{d}s_+^2=\mathrm{d}s^2. \tag{23}$$

ii) The continuity of the second fundamental form yields

$$[K_{jl}]_\Sigma=K_{jl}^+-K_{jl}^-=0, \tag{24}$$

where

$$K_{jl}^\pm=-n_\delta^\pm\left(\frac{\partial^2x_\pm^\delta}{\partial\xi^j\partial\xi^l}+\Gamma_{jl}^\delta\frac{\partial x_\pm^\delta}{\partial\xi^j\partial\xi^l}\right),\quad(\delta,j,l=0,2,3), \tag{25}$$

is the extrinsic curvature over the hypersurface, $n_\delta^\pm$ represents the outward unit normal to the hypersurface and $\xi^j=(\tau,\phi,z)$. The equations of hypersurface are

$$f^-(t,r)=r-\hat{r}_\Sigma=0, \tag{26}$$

$$f^+(\hat{T},\hat{R})=\hat{R}-\hat{R}_\Sigma(\hat{T})=0, \tag{27}$$

where $\hat{r}_\Sigma$ is a constant. Substituting eq. (27) in (6) and eq. (26) in (5), we obtain exterior and interior metrics on hypersurface, respectively,

$$(\mathrm{d}s_+^2)_\Sigma=-\left[H(\hat{R}_\Sigma)-(H(\hat{R}_\Sigma))^{-1}\left(\frac{\mathrm{d}\hat{R}_\Sigma}{\mathrm{d}\hat{T}}\right)^2\right]\mathrm{d}\hat{T}^2+\hat{R}^2(\mathrm{d}\phi^2+\beta^2\mathrm{d}z^2), \tag{28}$$

$$(\mathrm{d}s_-^2)_\Sigma=-A^2(t,\hat{r}_\Sigma)\mathrm{d}t^2+C^2(t,\hat{r}_\Sigma)\mathrm{d}\phi^2+D^2(t,\hat{r}_\Sigma)\mathrm{d}z^2. \tag{29}$$

The first fundamental form yields

$$C = \frac{1}{\beta} D = \hat{R}_{\Sigma}, \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{A}, \tag{30}$$

$$\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau} = \left[H - H^{-1}\left(\frac{\mathrm{d}\hat{R}_{\Sigma}}{\mathrm{d}\hat{T}}\right)^2\right]^{-\frac{1}{2}}. \tag{31}$$

The outward unit normal for interior and exterior spacetimes are of the form

$$\overset{-}{n_r} = (0, B, 0, 0), \quad \overset{+}{n_r} = \left(\frac{\mathrm{d}\hat{R}_{\Sigma}}{\mathrm{d}\tau}, \frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}, 0, 0\right).$$

The non-zero components of extrinsic curvature $K^{\pm}_{jl}$ are

$$K^{-}_{00} = -\frac{A'}{AB}, \quad K^{-}_{22} = \frac{1}{\beta^2} K^{-}_{33} = -\frac{CC'}{B}, \tag{32}$$

$$K^{+}_{00} = \left[\frac{\mathrm{d}^2\hat{T}}{\mathrm{d}\tau^2}\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tau} - \frac{\mathrm{d}^2\hat{R}}{\mathrm{d}\tau^2}\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau} - \frac{H}{2}\frac{\mathrm{d}H}{\mathrm{d}\hat{R}}\left(\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right)^3 + \frac{3}{2H}\frac{\mathrm{d}H}{\mathrm{d}\hat{R}}\left(\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right], \tag{33}$$

$$K^{+}_{22} = \frac{1}{\beta^2} K^{+}_{33} = \left[H\hat{R}\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right]. \tag{34}$$

The second fundamental form leads to

$$-\frac{A'}{AB} = \left[\frac{\mathrm{d}^2\hat{T}}{\mathrm{d}\tau^2}\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tau} - \frac{\mathrm{d}^2\hat{R}}{\mathrm{d}\tau^2}\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau} - \frac{H}{2}\frac{\mathrm{d}H}{\mathrm{d}\hat{R}}\left(\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right)^3 + \frac{3}{2H}\frac{\mathrm{d}H}{\mathrm{d}\hat{R}}\left(\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right], \tag{35}$$

$$-\frac{CC'}{B} = \left[H\hat{R}\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau}\right]. \tag{36}$$

Using eq. (31), it follows that

$$\frac{\mathrm{d}\hat{T}}{\mathrm{d}\tau} = \frac{1}{H}\left[H + \left(\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tau}\right)^2\right]^{\frac{1}{2}}. \tag{37}$$

Substituting eq. (37) in (36), we obtain

$$H = \left(\frac{C'}{B}\right)^2 - \left(\frac{\dot{C}}{A}\right)^2. \tag{38}$$

From eqs. (7) and (38), the mass function becomes [25, 26]

$$M = \frac{C}{4}\left(\frac{\dot{C}^2}{B^2} - \frac{C'^2}{A^2}\right) + \frac{Q^2}{2C}. \tag{39}$$

The following relationships are satisfied over the hypersurface:

$$E^* - \frac{L}{8} = M \quad \Longleftrightarrow \quad \check{q} = Q, \quad \lambda = \frac{1}{2},$$
$$p + \frac{T^{(D)}_{11}}{B^2} = -\frac{T^{(D)}_{01}}{AB} \quad \Longleftrightarrow \quad E^* = 0. \tag{40}$$

This provides a relationship between mass function and C-energy of the cylinder over the hypersurface, while the second equation represents the relationship between the $f(R)$ term and the pressure. In general, $T^{(D)}_{22} \neq T^{(D)}_{33}$, but for perfect fluid, the hypersurface condition, $\beta C = D$, leads to $T^{(D)}_{22} = \beta^{-2} T^{(D)}_{33}$.

3 Dynamical equations

In this section, we use the Misner-Sharp formalism [27] to construct dynamical equations. In the Misner-Sharp technique we discuss the dynamics of collapsing fluid under the $f(R)$ gravity. We discuss the effects of charge, properties of fluid and dark source terms on the rate of collapse. Bianchi identities yield

$$(T^{(m)\mu\nu}+T^{(D)\mu\nu}+E^{\mu\nu})_{;\nu}v_\mu=0,$$
$$(T^{(m)\mu\nu}+T^{(D)\mu\nu}+E^{\mu\nu})_{;\nu}\chi_\mu=0,$$

where $\chi^\mu=B^{-1}\delta^\mu_1=(0,B,0,0)$. Using the above two equations, we obtain the following dynamical equations:

$$\left(\frac{T^{(D)}_{00}}{A^2}\right)^{\cdot}-\frac{1}{AB}\left(\frac{T^{(D)}_{01}}{AB}\right)'-\frac{T^{(D)}_{01}}{A^2B^2}\left(\frac{2A'}{A}+\frac{2C'}{C}\right)+\frac{\dot{B}}{A^2B}\left(\rho+p+\frac{T^{(D)}_{00}}{A^2}+\frac{T^{(D)}_{11}}{B^2}\right)$$
$$+\frac{2\dot{C}}{A^2C}\left(\rho+p+\frac{T^{(D)}_{00}}{A^2}+\frac{T^{(D)}_{22}}{C^2}\right)+\frac{2\dot{A}}{A}\left(\frac{T^{(D)}_{00}}{A^4}\right)+\frac{\dot{\rho}}{A^2}=0,$$
$$\left[p+\frac{T^{(D)}_{11}}{B^2}\right]'+\left[\rho+p+\frac{T^{(D)}_{00}}{A^2}+\frac{T^{(D)}_{11}}{B^2}\right]\frac{A'}{A}-\frac{B}{A}\left[\left(\frac{T^{(D)}_{01}}{AB}\right)^{\cdot}+2\frac{\dot{C}}{A}\frac{T^{(D)}_{01}}{BC}\right]$$
$$+\frac{2C'}{C}\left[\frac{T^{(D)}_{11}}{B^2}-\frac{T^{(D)}_{22}}{C^2}\right]-\frac{aa'}{\beta^2\pi C^4}=0. \tag{41}$$

We define the proper time and radial derivatives as

$$D_{\hat{T}}=v^\mu\frac{\partial}{\partial X^{-\mu}}=\frac{1}{A}\frac{\partial}{\partial t},\quad D_{\hat{R}}=\frac{1}{\hat{R}'}\frac{\partial}{\partial\hat{R}}\,. \tag{42}$$

Let $\tilde{U}$ be the velocity of fluid particles, which is negative for the collapsing phenomenon, defined as [28]

$$\tilde{U}=D_{\hat{T}}(C)=D_{\hat{T}}(\hat{R})<0. \tag{43}$$

Using eq. (32), we obtain

$$\tilde{E}=\frac{C'}{B}=\left[\tilde{U}^2-\frac{2}{\beta C}\left(E^*-\frac{L}{8}\right)+\frac{\check{q}^2}{\beta C^2}\right]. \tag{44}$$

Using eqs. (18)–(22) and (42), the time rate of change of C-energy is given as

$$D_{\hat{T}}(E^*)=-\frac{4\pi\hat{R}^2\beta}{f_R}\left[\left(p+\frac{T^{(D)}_{11}}{B^2}-\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)\tilde{U}+\frac{T^{(D)}_{01}}{AB}\tilde{E}\right]-\frac{\check{q}^2}{2\hat{R}^2}\tilde{U}+\frac{\tilde{U}\beta}{8}\,. \tag{45}$$

This shows the behavior of energy with the contribution of dark source terms and matter variables. Since $\tilde{U}<0$, in the first square bracket on the right-hand side the term $(p+\frac{T^{(D)}_{11}}{B^2}-\frac{2\pi E^2}{(\beta\hat{R})^2})$ indicates that the energy will increase if

$$\left(p+\frac{T^{(D)}_{11}}{B^2}\right)>\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\,.$$

This indicates that work done by the effective pressure is greater than effects of charge on the system. The second dark source term $(\frac{T^{(D)}_{01}}{AB}\tilde{E})$ represents the outgoing energy of the system, while the third term $\frac{\check{q}^2}{\hat{R}^2}$ is just like the Coulomb force which can increase the energy of the collapsing system, since $\tilde{U}<0$ and the last term $\frac{\tilde{U}\beta}{8}$ indicate the collapsing phenomenon. This term will increase or decrease the energy of the system for $\beta<0$ or $\beta>0$.

Now we discuss radial derivative of the C-energy inside the hypersurface. Using eqs. (18)–(22) along with (42) and (17), it follows that

$$D_{\hat{R}}(E^*)=\frac{4\pi\hat{R}^2\beta}{f_R}\left[\left(\rho+\frac{T^{(D)}_{00}}{A^2}+\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)-\frac{T^{(D)}_{01}}{AB}\frac{\tilde{U}}{\tilde{E}}\right]+\frac{\check{q}}{\hat{R}}D_{\hat{R}}(\check{q})-\frac{\check{q}^2}{2\hat{R}^2}+\frac{\beta}{8}\,. \tag{46}$$

This shows the variation of energy between adjacent cylindrical surfaces. The first term in the square bracket indicates the work done due to effective energy density and effects of charge on the system. Thus energy between cylindrical surfaces grows up due to this work done by energy density. The second term in the square bracket shows the release of energy, which occurs in the inward direction as this term remains positive due to $\frac{T_{01}^{(D)}}{AB}\frac{\tilde{U}}{\tilde{E}}<0$ (due to collapse). The next two terms are due to the electromagnetic field and the last term causes an increase or decrease of energy inside the collapsing cylinder provided $\beta>0$ or $\beta<0$, respectively. Integrating the above equation with respect to $\hat{R}$, we obtain

$$E^*=\int_0^{\hat{R}}\frac{4\pi\hat{R}^2\beta}{f_R}\left[\left(\rho+\frac{T_{00}^{(D)}}{A^2}+\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)-\frac{T_{01}^{(D)}}{AB}\frac{\tilde{U}}{\tilde{E}}\right]\mathrm{d}\hat{R}+\frac{2}{\beta}\int_0^{R}\frac{\check{q}^2}{\hat{R}^2}\mathrm{d}R+\frac{\hat{R}\beta}{8}\,. \tag{47}$$

Using eqs. (42) and (43), we find the acceleration of collapsing matter inside the hypersurface as

$$D_{\hat{T}}(\tilde{U})=-\frac{4\pi\hat{R}}{f_R}\left(p+\frac{T_{11}^{(D)}}{B^2}-\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)-\frac{1}{\beta\hat{R}^2}\left(E^*-\frac{L}{8}\right)+\frac{A'}{A}\frac{\tilde{E}}{B}+\frac{\check{q}^2}{2\beta\hat{R}^3}\,. \tag{48}$$

Replacing the value of $\frac{A'}{A}$ from eq. (41) in the above equation, we have

$$\begin{aligned}&\left(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{11}^{(D)}}{B^2}\right)D_{\hat{T}}(\tilde{U})=-\left(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{11}^{(D)}}{B^2}\right)\\&\times\left[\frac{4\pi\hat{R}}{f_R}\left(p+\frac{T_{11}^{(D)}}{B^2}-\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)+\frac{M}{\beta\hat{R}^2}-\frac{\check{q}^2}{2\beta\hat{R}^3}\right]-\tilde{E}^2\left[D_{\hat{R}}\left(p+\frac{T_{11}^{(D)}}{B^2}\right)\right.\\&\left.-\frac{\check{q}}{\pi\hat{R}^4\beta^2}(D_R\check{q})+\frac{2}{\hat{R}}\left(\frac{T_{11}^{(D)}}{B^2}-\frac{T_{22}^{(D)}}{C^2}\right)\right]+\tilde{E}\left[D_{\hat{T}}\left(\frac{T_{01}^{(D)}}{AB}\right)+\frac{2\tilde{U}}{\hat{R}}\frac{T_{01}^{(D)}}{AB}\right].\end{aligned} \tag{49}$$

This expression is like the Newtonian force (force = mass density × acceleration) and shows that there are some forces which affect the fluid in the collapsing system. The term appearing on the left-hand side denotes inertial mass including effective pressure and effective density, whereas $D_{\hat{T}}\tilde{U}$ represents acceleration of the fluid particles. The first term on the right-hand side in the round bracket is the same as that on the left-hand side. By the equivalence principle, the factor in the round bracket indicates active gravitational mass. The first square bracket expresses the gravitational force (F_{grav}), whose Newtonian part is M. In this bracket, the term $\frac{4\pi\hat{R}}{f_R}(p+\frac{T_{11}^D}{B^2})$ represents the relativistic part and $(\frac{\check{q}^2}{2\beta\hat{R}^3})$ shows the effect of the electromagnetic field on the collapsing rate. This bracket describes how the active gravitational mass is affected by variables. The second square bracket denotes hydrodynamical force (F_{hyd}) in which the first term indicates resistance for the collapse. This resistance is due to the negative gradient of the effective pressure and work done by this gradient lies in the outward direction, hence decreases the process of collapse. The next term shows electromagnetic effects and the third term $(\frac{T_{11}^{(D)}}{B^2}-\frac{T_{22}^{(D)}}{C^2})$ is $f(R)$ terms. For the constant Ricci scalar, we have $\frac{T_{11}^D}{B^2}=\frac{T_{22}^D}{C^2}=-C_0$, where $C_0=\frac{1}{2\kappa}(R_0f_R(R_0)-f(R_0))$, so this term does not affect hydrodynamical equilibrium.

The last square bracket describes the role of dark source terms on the rate of collapse. The first term is the proper time derivative of dark source, and for a constant Ricci scalar, we have $\frac{T_{01}^D}{AB}=0$. Thus the first term does not contribute in the rate of collapse whereas the second term $\frac{2\tilde{U}}{\hat{R}}\frac{T_{01}^{(D)}}{AB}$ is negative as $\tilde{U}<0$ and this term always helps the collapsing phenomenon. Substituting the value of E^* from eq. (47) into (49), it follows that:

$$\begin{aligned}&\left(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{11}^{(D)}}{B^2}\right)D_{\hat{T}}(\tilde{U})=-\left(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{11}^{(D)}}{B^2}\right)\\&\times\left[\frac{1}{\hat{R}^2}\int_0^{\hat{R}}\left(\frac{4\pi\hat{R}^2}{f_R}\left(\rho+\frac{T_{00}^{(D)}}{A^2}+\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)-\frac{T_{01}^{(D)}}{AB}\frac{\tilde{U}}{\tilde{E}}\right)\mathrm{d}\hat{R}-\frac{2\check{q}^2}{\beta^2\hat{R}^3}\right.\\&\left.+\frac{2}{\beta^2\hat{R}^2}\int_0^{\hat{R}}\frac{\check{q}^2}{\hat{R}^2}\mathrm{d}\hat{R}+\frac{4\pi\hat{R}}{f_R}\left(p+\frac{T_{11}^{(D)}}{B^2}-\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)\right]\\&-\tilde{E}^2\left[D_{\hat{R}}\left(p+\frac{T_{11}^{(D)}}{B^2}\right)-\frac{\check{q}}{\pi\hat{R}^4\beta^2}(D_{\hat{R}}a)+\frac{2}{\hat{R}}\left(\frac{T_{11}^{(D)}}{B^2}-\frac{T_{22}^{(D)}}{C^2}\right)\right]+\tilde{E}\left[D_{\hat{T}}\left(\frac{T_{01}^{(D)}}{AB}\right)+\frac{2\tilde{U}}{\hat{R}}\frac{T_{01}^{(D)}}{AB}\right].\end{aligned} \tag{50}$$

The first square bracket on the right-hand side indicates that the active gravitational mass term $(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{11}^{(D)}}{B^2})$ increases only if

$$\frac{2}{\beta^2\hat{R}^2}\int_0^{\hat{R}}\frac{\check{q}^2}{\hat{R}^2}\mathrm{d}\hat{R}>\frac{2\check{q}^2}{\beta^2\hat{R}^3}.$$

Differentiating the above inequality, we have

$$\frac{\check{q}}{\hat{R}}>D_{\hat{R}}\check{q}.$$

If total charge of the cylinder is greater than the charge variation of the cylinder with respect to its radius, then the charge will enhance the active gravitational mass.

4 Relation between the Weyl Tensor and Matter Variables

Here we develop a relationship between the Weyl tensor, charge of the cylinder, energy density, effective pressure and dark source terms. The Weyl tensor is a measure of curvature of the spacetime and it gives distortion in the shape of body by the tidal force. It can be defined as [28]

$$\mathcal{C}=\frac{1}{3}R^2-2R^{\mu\nu}R_{\mu\nu}+\mathbf{R}, \tag{51}$$

where $\mathbf{R}$ is the Kretschmann scalar, R represents the Ricci scalar and $R_{\mu\nu}$ denotes the Ricci tensor. The Kretschmann scalar is defined by

$$\mathbf{R}=R^{\mu\nu\gamma\delta}R_{\mu\nu\gamma\delta}.$$

Substituting the values of the Reimann tensor from eqs. (A.6)–(A.10), this takes the form

$$\begin{aligned}\mathbf{R}=&\frac{48}{\beta^2C^6}\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right)^2-\frac{16}{\beta C^3}\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right)\left(\frac{G_{00}}{A^2}-\frac{G_{11}}{B^2}+\frac{G_{22}}{C^2}\right)\\&+4\left[\left(\frac{G_{22}}{C^2}\right)^2-\left(\frac{G_{01}}{AB}\right)^2\right]+3\left[\left(\frac{G_{00}}{A^2}\right)^2+\left(\frac{G_{11}}{B^2}\right)^2\right]-2\frac{G_{00}G_{11}}{A^2B^2}+4\left(\frac{G_{00}}{A^2}-\frac{G_{11}}{B^2}\right)\frac{G_{22}}{C^2}.\end{aligned} \tag{52}$$

Inserting eq. (52) in (51) and making use of eqs. (18)–(22), it follows that

$$\hbar=E^*-\frac{L}{8}-\frac{4\pi\hat{R}^3\beta}{3f_R}\left(\rho+\frac{T_{00}^{(D)}}{A^2}-\frac{T_{11}^{(D)}}{B^2}+\frac{T_{22}^{(D)}}{C^2}+\frac{6\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)-\frac{\check{q}^2}{2\hat{R}}, \tag{53}$$

where $\hbar=\frac{\mathcal{C}\hat{R}^3\beta}{\sqrt{48}}$.

Taking proper time derivative of eq. (53) and using (42), we have

$$\begin{aligned}D_{\hat{T}}\hbar=&-\frac{4\pi\hat{R}^2\beta}{f_R}\left[\left(\rho+p+\frac{T_{00}^{(D)}}{A^2}+\frac{T_{22}^{(D)}}{C^2}+\frac{4\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)\tilde{U}+\frac{T_{01}^{D}}{AB}\tilde{E}\right]\\&-\frac{4\pi\hat{R}^3\beta}{3f_R}D_{\hat{T}}\left(\rho+\frac{T_{00}^{(D)}}{A^2}-\frac{T_{11}^{(D)}}{B^2}+\frac{T_{22}^{(D)}}{C^2}+\frac{6\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right).\end{aligned} \tag{54}$$

Using eq. (42) in (53), the radial derivative of $\hbar$ becomes

$$\begin{aligned}D_{\hat{R}}\hbar=&-\frac{4\pi\hat{R}^3\beta}{3}D_{\hat{R}}\left[\frac{1}{f_R}\left(\rho+\frac{T_{00}^{(D)}}{A^2}+\frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)\right]-\frac{4\pi R^2\beta}{f_R}\frac{\tilde{U}}{\tilde{E}}\frac{T_{01}^{(D)}}{AB}\\&+D_{\hat{R}}\left[\frac{4\pi\hat{R}^3\beta}{3f_R}\left(\frac{T_{11}^{(D)}}{B^2}-\frac{T_{22}^{(D)}}{C^2}-\frac{4\pi\mathbb{E}^2}{(\beta\hat{R})^2}\right)\right]-\left(\frac{\check{q}^2}{\beta\hat{R}^2}\right).\end{aligned}$$

This provides a relationship between the Weyl tensor, dark source terms, electromagnetic field and energy density in $f(R)$ gravity. For zero charge distribution, it follows that

$$D_{\hat{R}}\hbar+\frac{4\pi\hat{R}^3\beta}{3}D_{\hat{R}}\left[\frac{1}{f_R}\left(\rho+\frac{T_{00}^{(D)}}{A^2}\right)\right]=D_{\hat{R}}\left[\frac{4\pi\hat{R}^3\beta}{3f_R}\left(\frac{T_{11}^{(D)}}{B^2}-\frac{T_{22}^{(D)}}{C^2}\right)\right]+\frac{4\pi\hat{R}^2\beta}{f_R}\frac{\tilde{U}}{\tilde{E}}\frac{T_{01}^{(D)}}{AB}. \tag{55}$$

For the constant Ricci scalar, $R = R_0$, we obtain

$$D_{\hat{R}}\hbar + \frac{4\pi\hat{R}^3\beta}{3f_R(R_0)} D_{\hat{R}} \left[\rho + \frac{1}{2\kappa}\left(R_0 f_R(R_0) - f(R_0)\right)\right] = 0. \tag{56}$$

Taking $C_0 = \frac{1}{2\kappa}(R_0 f_R(R_0) - f(R_0))$, this implies that

$$D_{\hat{R}}\left(\frac{\mathcal{C}\hat{R}^3\beta}{\sqrt{48}}\right) + \frac{4\pi\hat{R}^3\beta}{3f_R(R_0)} D_{\hat{R}}(\rho + C_0) = 0.$$

This indicates that $\hbar = 0 \Longleftrightarrow D_R\rho = 0$ (using regular axis condition), where $D_R\rho = 0$ implies homogeneity in the energy density.

5 Concluding remarks

In this paper, we have studied the charged cylindrical collapse with perfect fluid in $f(R)$ gravity. At the hypersurface, we matched the static interior and non-static exterior spacetimes. We have developed a relationship between gravitational mass, C-energy and specific length of the cylinder with the help of junction conditions. We have formulated dynamical equations using the Misner-Sharp formalism and discussed the effect of charge, energy density, effective pressure as well as dark source terms on the rate of collapse. Finally, we have developed a relationship between the Weyl tensor, matter variables and dark source terms which provides energy density inhomogeneity. The summary of the results is given as follows.

- The variation of energy with respect to different radii indicates that the work done due to the effective energy density leads to an increase in the energy between cylindrical surfaces. The internal energy of the cylindrical surface is increased due to the release of energy in the inward direction. Also, energy will increase or decrease according to $\beta > 0$ or $\beta < 0$ between adjacent cylindrical layers.
- The variation of energy with respect to time shows the following results. The outgoing energy is greater than the ingoing energy of the system and for $\tilde{U} < 0$, the energy will increase only if $(p + \frac{T_{11}^{(D)}}{B^2}) > \frac{2\pi\mathbb{E}^2}{(\beta\hat{R})^2}$, *i.e.*, the work done by effective pressure is greater than the electric field strength of the system.
- The force in eq. (49) is just like the Newtonian force which affects the collapse due to gravitational and hydrodynamical forces.
- The hydrodynamical force prevents the collapse completely if $\frac{T_{11}^D}{B^2} = \frac{T_{22}^D}{C^2}$; this condition is satisfied only when the Ricci scalar is constant.
- The pressure helps to keep the stars in equilibrium position through the pressure gradient, which is balanced by the gravitational force. However, if this negative gradient of effective pressure dominates, then it decreases the collapsing phenomenon as this does the work in outward direction.
- The Coulomb force may lead to stop the collapse because it shows the repulsive effect just like the cosmological constant and is balanced by gravitational force.
- The inertial mass density of the system increases only for $\frac{\check{q}}{\hat{R}} > D_{\hat{R}}\check{q}$ which causes rapid collapsing phenomenon.
- For constant Ricci scalar, $R = R_0$, energy density is homogeneous if and only if the metric is conformally flat.

We can also conclude that gravitational collapse will be slower in $f(R)$ gravity as compared to GR.

Appendix A.

$$T_{00}^{(D)} = \frac{A^2}{\kappa}\left[\frac{Rf_R - f}{2} + \frac{1}{B^2}\left(f_R'' - \frac{B\dot{B}\dot{f}_R}{A^2} - \frac{B'f_R'}{B}\right) + \frac{1}{C^2}\left(\frac{CC'f_R'}{B^2} - \frac{C\dot{C}\dot{f}_R}{A^2}\right) + \frac{1}{C^2}\left(\frac{DD'f_R'}{B^2} - \frac{D\dot{D}\dot{f}_R}{A^2}\right)\right], \tag{A.1}$$

$$T_{11}^{(D)} = \frac{B^2}{\kappa}\left[\left(\frac{f-Rf_R}{2}\right)+\frac{1}{A^2}\left(\ddot{f}_R-\frac{\dot{A}\dot{f}_R}{A}-\frac{AA'f_R'}{B^2}\right)-\frac{1}{C^2}\left(\frac{CC'f_R'}{B^2}-\frac{C\dot{C}\dot{f}_R}{A^2}\right)\right.$$
$$\left.-\frac{1}{D^2}\left(\frac{DD'f_R'}{B^2}-\frac{D\dot{D}\dot{f}_R}{A^2}\right)\right], \tag{A.2}$$

$$T_{22}^{(D)} = \frac{C^2}{\kappa}\left[\left(\frac{f-Rf_R}{2}\right)+\frac{1}{A^2}\left(\ddot{f}_R-\frac{\dot{A}\dot{f}_R}{A}-\frac{AA'f_R'}{B^2}\right)-\frac{1}{B^2}\left(f_R''-\frac{B\dot{B}\dot{f}_R}{A^2}-\frac{B'f_R'}{B}\right)\right.$$
$$\left.-\frac{1}{D^2}\left(\frac{DD'f_R'}{B^2}-\frac{D\dot{D}\dot{f}_R}{A^2}\right)\right], \tag{A.3}$$

$$T_{33}^{(D)} = \frac{D^2}{\kappa}\left[\left(\frac{f-Rf_R}{2}\right)+\frac{1}{A^2}\left(\ddot{f}_R-\frac{\dot{A}\dot{f}_R}{A}-\frac{AA'f_R'}{B^2}\right)-\frac{1}{B^2}\left(f_R''-\frac{B\dot{B}\dot{f}_R}{A^2}-\frac{B'f_R'}{B}\right)\right.$$
$$\left.-\frac{1}{C^2}\left(\frac{CC'f_R'}{B^2}-\frac{C\dot{C}\dot{f}_R}{A^2}\right)\right], \tag{A.4}$$

$$T_{01}^{(D)} = \frac{1}{\kappa}\left[\dot{f}_R'-\frac{A'\dot{f}_R}{A}-\frac{\dot{B}f_R'}{B}\right]. \tag{A.5}$$

The Kretschmann scalar yields

$$\mathbf{R} = 4\left[\frac{1}{(AB)^4}(R_{1010})^2+\frac{2}{(AC)^4}(R_{2020})^2-\frac{4}{(AB)^2C^4}(R_{0212})^2+\frac{2}{(BC)^4}(R_{1212})^2+\frac{1}{C^8}(R_{2323})^2\right].$$

The Reimann tensors in terms of the Einstein tensor can be written as

$$R_{1010} = (AB)^2\left[\frac{G_{00}}{2A^2}-\frac{G_{11}}{2B^2}+\frac{G_{22}}{C^2}-\frac{2}{\beta C^3}\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right)\right], \tag{A.6}$$

$$R_{0202} = (AC)^2\left[\frac{G_{11}}{2B^2}+\frac{1}{\beta C^3}\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right)\right], \tag{A.7}$$

$$R_{1212} = (BC)^2\left[\frac{G_{00}}{2A^2}-\frac{1}{\beta C^3}\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right)\right], \tag{A.8}$$

$$R_{0212} = \frac{C^2}{2}G_{01}\,, \tag{A.9}$$

$$R_{2323} = 2C\beta\left(E^*-\frac{L}{8}-\frac{a^2}{2C}\right), \tag{A.10}$$

where $\beta^2 R_{0202} = R_{0303}$, $\beta^2 R_{1212} = R_{1313}$, $\beta^2 R_{0212} = R_{0313}$.

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