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Hyperbolic geometric flow on reduced Berwald spaces: Short-time existence and uniqueness

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Abstract. In this paper, we introduce a new understanding tool, the Finsler hyperbolic geometric flow, and establish the short-time existence and uniqueness theorem for reduced Berwald spaces. This kind of flow is very natural to understand certain wave phenomena in physics as well as the geometry of Finsler manifolds. Also we illustrate the wave character of the metrics and curvatures of reduced Berwald manifolds.

1 Introduction

Geometric flows are important in many sections of mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated with some curvature. The well-known geometric flows in mathematics are the heat flow, the Ricci flow and the mean curvature flow. The subject of Hamilton's Ricci flow, $\frac{\partial}{\partial t}g(t)=-2Ric_{g(t)}$, lies in the more general field of geometric flows, which, in turn, lies in the even more general field of geometric analysis. As a fully nonlinear system of parabolic partial differential equations of second order, the Ricci flow in many respects appears to be a very natural equation. Similarly, since the hyperbolic equation or system is one of the most natural models in physics, we feel the hyperbolic geometric flow, introduced by Kong and Liu [1] in 2007, is also a very natural tool. Note that the elliptic, parabolic and hyperbolic partial differential equations have been successfully applied to differential geometry and physics.

The hyperbolic geometric flow is a system of nonlinear evolution partial differential equations of second order, and is useful to understand certain wave phenomena in physics as well as the geometry of manifolds, in particular, it describes the wave character of the metrics and curvatures of manifolds. A Riemannian geometry is defined on a manifold by a symmetric metric tensor and the corresponding Levi-Civita connection structure. However, the Finsler and Lagrange geometries are constructed from three fundamental and independent geometric objects: the nonlinear connection, metric and linear connection.

The hyperbolic geometric flow on a Riemannian manifold M with a Riemannian metric g_0 is defined by the family $g(t)$ of Riemannian metrics on M satisfying

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}g_{ij}(x,t)=-2(Ric)_{ij}(x,t)\\ g_{ij}(x,0)=(g_0)_{ij}(x), \qquad \dfrac{\partial g_{ij}}{\partial t}(x,0)=K^0_{ij}(x)\end{cases}, \tag{1}$$

where $x\in M$ and K^0_{ij} is a symmetric tensor on M, and Ric is the Ricci tensor of $g(t)$. Kong and Liu in [1] showed that there is a unique solution to this equation for an arbitrary smooth metric on a compact manifold over a sufficiently short time. In this paper we are going to study the hyperbolic geometric flow in the Finsler geometry. The Finsler hyperbolic geometric flow (FHGF) under consideration is the following evolution equation:

$$\frac{\partial^2 F^2}{\partial t^2}=-2F^2\mathcal{R}ic, \tag{2}$$

for a family of Finsler metrics $F(t)$ on M. In this paper, we prove the short-time existence and uniqueness theorem for reduced Berwald metrics, and drive the corresponding wave equations for the curvatures. The main difficulty to

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prove this theorem is that the Finsler hyperbolic geometric flow (2) is a system of nonlinear weakly hyperbolic partial differential equations of second order. As the Finsler hyperbolic geometric flow (2) is only weakly hyperbolic, the short-time existence and uniqueness result on a compact manifold does not come from the standard PDEs theory directly. In order to prove the short-time existence and uniqueness theorem, using the gauge fixing idea as in the Ricci flow, we can derive a system of nonlinear strictly hyperbolic partial differential equations of second order.

Preliminaries

1.1 Berwald metric

Let M be a connected n-dimensional manifold. Denote by T_xM the tangent space at $x \in M$, and $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. Any element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \longrightarrow M$ is given by $\pi(x, y) = x$. Denote the pull-back tangent bundle π^*TM by

$$\pi^*TM = \{(x, y, v) \mid y \in T_xM_0,\ v \in T_xM\},$$

where $TM_0 = TM \setminus \{0\}$, $\pi(v) = x$. A Finsler metric on a manifold M is a function $F : TM \longrightarrow [0, \infty)$, which has the following properties:

1) $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$
2) $F(x, y)$ is C^∞ on TM_0;
3) For any tangent vector $y \in T_xM$, the symmetric bilinear form $g_y : T_xM \times T_xM \longrightarrow \mathbb{R}$ on TM is positive definite, where

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial r}\left[F^2(x, y + su + rv)\right]\big|_{s=r=0}. \tag{3}$$

In the local coordinate system (x^i, y^i) we have $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$ and $(g^{ij}) := (g_{ij})^{-1}$. The pair (M, F) is called a Finsler manifold. The geodesics of F are characterized locally by

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + 2G^i\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0$$

$$G^i = \frac{1}{4} g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\} y^j y^k. \tag{4}$$

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and are given by

$$G^i = \frac{1}{4} g^{il}\left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\right], \quad y \in T_xM. \tag{5}$$

Finsler spaces for which the canonical parallel transport is a linear process are said to be of Berwald type. Thus, on Berwald spaces, the F-preserving diffeomorphisms, $\varphi_t : T_xM \setminus \{0\} \longrightarrow T_{\sigma(t)}M \setminus \{0\}$, generated by the canonical parallel transport become linear isometries between normed tangent spaces. For example, Riemannian spaces and locally Minkowskian spaces belong to this family.

A Finsler metric F is called a Berwald metric, if $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ is quadratic in $y \in T_xM$ for any $x \in M$, where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{lj}}{\partial y^r} G^r_k + \frac{\partial g_{jk}}{\partial y^r} G^r_l - \frac{\partial g_{kl}}{\partial y^r} G^r_j\right). \tag{6}$$

For a Berwald metric we have $\Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$ and $G^i_j = \frac{\partial G^i}{\partial y^j}$. Put $A^i_{jk} = \frac{1}{2} g^{il}(\frac{\partial g_{lj}}{\partial y^r} G^r_k - \frac{\partial g_{jk}}{\partial y^r} G^r_l + \frac{\partial g_{kl}}{\partial y^r} G^r_j)$ [2] and define the following.

Definition 1. A Berwald metric F is called a reduced Berwald metric if A^k_{ij} is independent of y.

Example 1. Let $F(x, y) = \frac{(y^2)^2}{y^1}$ such that $y^1 y^2 \neq 0$, we have $g_{11} = 3(\frac{y^2}{y^1})^4$, $g_{22} = 6(\frac{y^2}{y^1})^2$ and $g_{12} = -(\frac{y^2}{y^1})^3 = g_{21}$. This is a non-Riemannian reduced Berwald metric.

For a vector $y \in T_xM_0$, the Berwald connection is a map $\nabla^y : T_xM \times C^\infty(TM) \longrightarrow T_xM$ defined by

$$\nabla^y_u V := \left\{u(V^i)(x) + V^j(x)\Gamma^i_{jk}(y)u^k\right\} \frac{\partial}{\partial x^i}\bigg|_x,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x \in T_xM$ and $V = V^i \frac{\partial}{\partial x^i} \in C^\infty(TM)$.

From now, for a vector $y \in T_xM_0$, we suppose that $\nabla = \nabla^y$. The coefficients of the Riemann curvature $R_y = R^i_k dx^i \otimes \frac{\partial}{\partial x^i}$ are given by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} \tag{7}$$

and

$$R^i_{jk} := \frac{1}{3}\left(\frac{\partial R^i_j}{\partial y^k} - \frac{\partial R^i_k}{\partial y^j}\right) \tag{8}$$

$$R^i_{jkl} := \frac{1}{3}\left(\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\right). \tag{9}$$

The Ricci scalar function of F is given by

$$\mathcal{R}ic := \frac{1}{F^2} R^i_i.$$

A companion of the Ricci scalar is the Ricci tensor,

$$Ric_{ij} := \left(\frac{1}{2}F^2 \mathcal{R}ic\right)_{y^i y^j}. \tag{10}$$

A Finsler metric is said to be Einstein if the Ricci scalar function is a function of x alone, equivalently $Ric_{ij} = R(x)$.

2 Finslerian hyperbolic geometric flow

In this section we state the short-time existence and uniqueness result for the Finsler hyperbolic geometric flow (2) on a compact n-dimensional reduced Berwald manifold M. We can show that it is a system of second-order nonlinear weakly hyperbolic PDE therefore we consider a modified system of evolution equations of the hyperbolic geometric flow, which is strictly hyperbolic, so that we can get a solution for a short time by solving the corresponding Cauchy problem. The solution of the system (2) comes from the solution of the modified equations.

More generally, one can provide lower bounds for the “life span” of solutions of second-order hyperbolic equations,

$$u_{tt} - G(Du, D_x Du) = 0, \tag{11}$$

where

$$Du = (u_t, u_{x_1}, \ldots, u_{x_n})$$
$$D_x Du = (u_{x_i x_j}), \quad i, j = 0, \ldots, n \quad i + j > 0$$

and, by convention, $u_{x_0} = u_t$.

We suppose G is a smooth function of $\tilde{\lambda} = ((\lambda_i),\ i = 0, \ldots, n, (\tilde{\lambda}_{ij}),\ i, j, = 0, \ldots, n, i + j > 0)$, in a neighborhood of $\tilde{\lambda} = 0$. Also $G(0, 0) = 0$ and

$$\sum_{i,j=1}^{n} G_{\tilde{\lambda}_{ij}}(0, 0)\xi_i\xi_j \geq m|\xi|^2 \quad m > 0, \quad \text{for any } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \tag{12}$$

which expresses the fact that (11) is hyperbolic near the origin.

Proposition 1. *Let (M, g) be a compact Finsler manifold with reduced Berwald metric. Then the Finsler hyperbolic geometric flow on M is not strictly hyperbolic.*

Proof. From (2), (3), (7) and (10), we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g_{rs} &= -(F^2\mathcal{R}ic)_{y^r y^s}\\ &= \left(2\frac{\partial G^i}{\partial x^i} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^i} + 2G^j\frac{\partial^2 G^i}{\partial y^i\partial y^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^i}\right)_{y^r y^s}\\ &= g^{il}\left(-\frac{\partial^2 g_{rl}}{\partial x^s\partial x^i} - \frac{\partial^2 g_{sl}}{\partial x^r\partial x^i} + \frac{\partial^2 g_{rs}}{\partial x^i\partial x^l} + \frac{\partial^2 g_{il}}{\partial x^s\partial x^r}\right) + \text{lower-order terms.}\end{aligned}$$

If we put $\xi = (1, -1, 0, \dots, 0)$, it satisfies $\xi_1 = 1$, $\xi_2 = -1$ and $\xi_i = 0$ for $i > 2$, then $|\xi|^2 = 2$ and

$$\sum_{i,j=1}^{n} G_{\tilde{\lambda}_{ij}}(0,0)\xi_i\xi_j = G_{\tilde{\lambda}_{11}}(0,0) + G_{\tilde{\lambda}_{22}}(0,0) - 2G_{\tilde{\lambda}_{12}}(0,0).$$

By putting $r = 1$ and $s = 2$ a simple computation shows that

$$\frac{\partial^2}{\partial t^2}g_{12} = \sum_{i,l=1}^{n} g^{il}\left(-\frac{\partial^2 g_{1l}}{\partial x^2\partial x^i} - \frac{\partial^2 g_{2l}}{\partial x^1\partial x^i} + \frac{\partial^2 g_{12}}{\partial x^i\partial x^l} + \frac{\partial^2 g_{il}}{\partial x^2\partial x^1}\right) + \text{lower-order terms.}$$

therefore, $\sum_{i,j=1}^{n} G_{\tilde{\lambda}_{ij}}(0,0)\xi_i\xi_j = -4g^{12}$, if $g^{12} \geq 0$ then $-2g^{12} < m$. So FHGF is not strictly hyperbolic.

2.1 Short-time existence

Theorem 1. *Let M be a compact Finsler manifold with reduced Berwald metric $F_0(x,y)$, then there exist a constant $\varepsilon > 0$ and a smooth one-parameter family of reduced Berwald metrics $F(x,y,t)$, $t \in [0,\varepsilon)$, such that $F(x,y,t)$ is a solution of the initial value problem*

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}F^2(x,y,t) = -2F^2\mathcal{R}ic(x,y,t)\\ F(x,y,0) = F_0, \qquad \dfrac{\partial F}{\partial t}(x,y,0) = F^0(x,y)\end{cases}, \tag{13}$$

where $F^0(x,y)$ is a C^∞ function on TM_0, such that $F^0(x,\lambda y) = \lambda F^0(x,y)$ for all $\lambda > 0$. Moreover, the solution $F(x,y,t)$ is unique.

Remark 1. We use $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\partial y^j}(x,y)$ and we rewrite (13) as

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g_{ij}(x,y,t) &= -\left(F^2\mathcal{R}ic\right)_{y^i y^j}(x,y,t)\\ &= -2Ric_{ij},\end{aligned} \tag{14}$$

where F is a Finsler metric. So, if g_0 and $g(x,t)$ are Riemannian metrics on a compact manifold M, $F_0(x,y) = \sqrt{g_{0ij}(x)y^i y^j}$ and $F_t(x,y) = \sqrt{g_{ij}(t,x)y^i y^j}$ are Finsler metrics, then FHGF is the same as the hyperbolic geometric flow on the Riemannian manifold (M,g). Therefore the Finsler hyperbolic geometric flow is the natural extension of the hyperbolic geometric flow.

We now follow the strategy of proving short-time existence and uniqueness for Finsler hyperbolic geometric flow.

Suppose the reduced Berwald metric $\hat{g}_{ij}(x,y,t)$ is a solution of the Finslerian hyperbolic geometric flow (14) and $\psi_t : M \longrightarrow M$ is a family of diffeomorphism of M. Let $g_{ij}(x,y,t) = \psi_t^*\hat{g}_{ij}(x,y,t)$ be the pull-back metrics. We now want to find the evolution equations for the metrics $g_{ij}(x,y,t)$, and denote by $z(x,t) = \psi_t(x) = (z^1(x,t), \dots, z^n(x,t))$ in local coordinates. Then

$$g_{ij}(x,y,t) = \frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(z,y,t) \tag{15}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,y,t) &= \frac{\partial}{\partial t}\left[\hat{g}_{\alpha\beta}(z,y,t)\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\right]\\ &= \frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(z(x,t),y,t) + \hat{g}_{\alpha\beta}(z,y,t)\frac{\partial}{\partial t}\left(\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\right).\end{aligned}$$

Lemma 1. *The second-order time derivative of* g_{ij} *is*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g_{ij}(x,y,t) = & -2\hat{R}_{\alpha\beta}(z,y,t)\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j} + \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial z^\lambda}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial z^\gamma}{\partial t}\frac{\partial z^\lambda}{\partial t} \\ & + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial t}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial z^\gamma}{\partial t} + \frac{\partial}{\partial x^i}\left(\hat{g}_{\alpha\beta}\frac{\partial z^\beta}{\partial x^j}\frac{\partial^2 z^\alpha}{\partial t^2}\right) + \frac{\partial}{\partial x^j}\left(\hat{g}_{\alpha\beta}\frac{\partial z^\beta}{\partial x^i}\frac{\partial^2 z^\alpha}{\partial t^2}\right) \\ & + \left[\frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}\left(\frac{\partial z^\beta}{\partial x^j}\hat{g}_{\beta\gamma}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial x^i}\hat{g}_{\beta\gamma}\right)\right]\frac{\partial^2 z^\gamma}{\partial t^2} \\ & + 2\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial z^\beta}{\partial x^j}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\gamma}{\partial t}\right) \\ & + 2\frac{\partial z^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right)\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}\right) \\ & + 2\hat{g}_{\alpha\beta}\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right).\end{aligned}$$

Proof. It follows easily that:

$$\frac{\partial \hat{g}_{\alpha\beta}}{\partial t}(z(x,t),y,t) = \frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\gamma}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial t}$$

and

$$\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial t^2}(z(x,t),y,t) = \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial z^\lambda}\frac{\partial z^\gamma}{\partial t}\frac{\partial z^\lambda}{\partial t} + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial t}\frac{\partial z^\gamma}{\partial t} + \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial t^2} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial^2 z^\gamma}{\partial t^2}.$$

So, from (15) we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g_{ij}(x,y,t) = & \frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\mathrm{d}^2 \hat{g}_{\alpha\beta}}{\mathrm{d}t^2}(z(x,t),y,t) + \frac{\partial}{\partial x^i}\left(\frac{\partial^2 z^\alpha}{\partial t^2}\right)\frac{\partial z^\beta}{\partial x^j}\hat{g}_{\alpha\beta} \\ & + \frac{\partial z^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial^2 z^\beta}{\partial t^2}\right)\hat{g}_{\alpha\beta} + 2\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial z^\beta}{\partial x^j}\frac{\mathrm{d}\hat{g}_{\alpha\beta}}{\mathrm{d}t} \\ & + 2\frac{\partial z^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right)\frac{\mathrm{d}\hat{g}_{\alpha\beta}}{\mathrm{d}t} + 2\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}.\end{aligned}$$

The proof follows from the fact that the reduced Berwald metric $\hat{g}_{ij}(x,y)$ is a solution of FHGF.

Lemma 2. *If we choose the normal coordinates (see* [3]*) around a fixed point* $p \in M$*, i.e.* $\frac{\partial g_{ik}}{\partial x^j} = 0$ *at* p*, then*

$$\frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^k} - \frac{\partial}{\partial x^i}\left(\frac{\partial z^\beta}{\partial x^k}\hat{g}_{\beta\gamma}\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial z^\beta}{\partial x^i}\hat{g}_{\beta\gamma}\right) = 0 \quad \forall i,k,\gamma = 1,\ldots,n.$$

Proof. See [4], p. 6.

Using lemmas 1 and 2, then

$$\begin{aligned}\frac{\partial^2 g_{ij}}{\partial t^2}(x,y,t) = & -2R_{ij}(x,y,t) \\ & + \frac{\partial}{\partial x^i}\left(g_{mj}\frac{\partial x^m}{\partial z^\alpha}\frac{\partial^2 z^\alpha}{\partial t^2}\right) + \frac{\partial}{\partial x^j}\left(g_{mi}\frac{\partial z^\alpha}{\partial x^m}\frac{\partial^2 z^\alpha}{\partial t^2}\right) \\ & + \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial z^\lambda}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial z^\gamma}{\partial t}\frac{\partial z^\lambda}{\partial t} + 2\frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial z^\gamma \partial t}\frac{\partial z^\gamma}{\partial t}\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j} \\ & + 2\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial z^\beta}{\partial x^j}\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\gamma}{\partial t}\right) \\ & + 2\frac{\partial z^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right)\left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial t} + \frac{\partial \hat{g}_{\alpha\beta}}{\partial z^\gamma}\frac{\partial z^\gamma}{\partial t}\right) \\ & + 2\hat{g}_{\alpha\beta}\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial z_\beta}{\partial t}\right).\end{aligned}$$

We define a time-dependent vector field on the slit tangent bundle TM_0 as

$$V^k = g^{jl}\left(\Gamma^k_{jl} - {\Gamma^0}^k_{jl}\right) \tag{16}$$

and define $z(x,t) = \psi_t(x)$ by the following initial value problem:

$$\begin{cases} \dfrac{\partial^2 z^\alpha}{\partial t^2} = \dfrac{\partial z^\alpha}{\partial x^k}V^k - g^{jl}\hat{\Gamma}^\alpha_{\beta\gamma}\dfrac{\partial z^\beta}{\partial x^j}\dfrac{\partial z^\gamma}{\partial x^i} \\ z^i(x,0) = x^i, \qquad \dfrac{\partial}{\partial t}z^\alpha(x,0) = \mathcal{Z}^\alpha(x) \end{cases}, \tag{17}$$

where Γ^k_{jl} and ${\Gamma^0}^k_{jl}$ are the connection coefficients corresponding to the reduced Berwald metrics $g_{ij}(x,y,t)$ and $g_{ij}(x,y,0)$ and $\mathcal{Z}^\alpha(x)$ $(\alpha = 1,2,\ldots,n)$ are arbitrary C^∞ smooth functions on the manifold M.

Using lemma 2, we get the following evolution equation for the pull-back metric:

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial t^2}(x,y,t) = &-2R_{ij}(x,y,t) + \frac{\partial}{\partial x^i}V_j + \frac{\partial}{\partial x^j}V_i \\
&+ \frac{\partial^2}{\partial z^\gamma \partial z^\lambda}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right)\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial z^\gamma}{\partial t}\frac{\partial z^\lambda}{\partial t} \\
&+ 2\frac{\partial^2}{\partial z^\gamma \partial t}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right)\frac{\partial z^\alpha}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\frac{\partial z^\gamma}{\partial t} \\
&+ 2\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial z^\beta}{\partial x^j}\left(\frac{\partial}{\partial t}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right) + \frac{\partial}{\partial z^\gamma}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right)\frac{\partial z^\gamma}{\partial t}\right) \\
&+ 2\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right)\frac{\partial z^\alpha}{\partial x^i}\left(\frac{\partial}{\partial t}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right) + \frac{\partial}{\partial z^\gamma}\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right)\frac{\partial z^\gamma}{\partial t}\right) \\
&+ 2\left(\frac{\partial x^k}{\partial z^\alpha}\frac{\partial x^l}{\partial z^\beta}g_{kl}\right)\frac{\partial}{\partial x^i}\left(\frac{\partial z^\alpha}{\partial t}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial z^\beta}{\partial t}\right) \\
&- \frac{\partial}{\partial x^i}\left(g_{rj}g^{sl}\left(\frac{\partial x^l}{\partial z^\alpha}\frac{\partial x^k}{\partial z^\gamma}\frac{\partial z^\alpha}{\partial x^k}\Gamma^k_{lk} + \frac{\partial z^\alpha}{\partial x^l}\frac{\partial^2 x^l}{\partial z^\alpha \partial z^\gamma}\right)\frac{\partial z^\gamma}{\partial x^s}\right) \\
&- \frac{\partial}{\partial x^j}\left(g_{ri}g^{sl}\left(\frac{\partial x^l}{\partial z^\alpha}\frac{\partial x^k}{\partial z^\gamma}\frac{\partial z^\alpha}{\partial x^k}\Gamma^k_{lk} + \frac{\partial z^\alpha}{\partial x^l}\frac{\partial^2 x^l}{\partial z^\alpha \partial z^\gamma}\right)\frac{\partial z^\gamma}{\partial x^s}\right) \\
= &-2R_{ij}(x,y,t) + \frac{\partial}{\partial x^i}V_j + \frac{\partial}{\partial x^j}V_i + I(Dz, D_tD_xz),
\end{aligned}$$

where $g_{ij}(x,y,0) = g^0_{ij}(x,y)$, $\frac{\partial}{\partial t}g_{ij}(x,y,0) = k^0_{ij}(x,y)$, $Dz = (\frac{\partial z^\alpha}{\partial t}, \frac{\partial z^\alpha}{\partial x^i})$ and $D_tD_xz = (\frac{\partial^2 z^\alpha}{\partial x^i \partial t})$ for $(\alpha, i = 1,2,\ldots,n)$.

Let $\hat{\lambda} = (\frac{\partial z^\alpha}{\partial t}, \frac{\partial z^\alpha}{\partial x^i}, \frac{\partial^2 z^\alpha}{\partial x^i \partial t})$, $(\alpha, i = 1,2,\ldots,n)$, the nonlinear term $I = I(\hat{\lambda}) = I(Dz, D_tD_xz)$ is smooth and $I(\hat{\lambda}) = O(|\hat{\lambda}|^2)$ holds.

Since $\Gamma^k_{ji} = \frac{\partial z^\alpha}{\partial x^j}\frac{\partial z^\beta}{\partial x^i}\frac{\partial x^k}{\partial z^\gamma}\hat{\Gamma}^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial z^\alpha}\frac{\partial^2 z^\alpha}{\partial x^j \partial x^i}$, the initial value problem (17) can be written as

$$\begin{cases} \dfrac{\partial^2 z^\alpha}{\partial t^2} = g^{jl}\left(\dfrac{\partial^2 z^\alpha}{\partial x^j \partial x^l} - {\Gamma^0}^k_{jl}\dfrac{\partial z^\alpha}{\partial x^k}\right) \\ z^\alpha(x,0) = (x^\alpha), \qquad \dfrac{\partial}{\partial t}z^\alpha(x,0) = \mathcal{Z}^\alpha(x) \end{cases}. \tag{18}$$

We calculate $-2R_{ij}(x,y,t) + \frac{\partial}{\partial x^i}V_j + \frac{\partial}{\partial x^j}V_i$ using $Ric_{rs} := (\frac{1}{2}F^2\mathcal{R}ic)_{y^ry^s}$, where

$$F^2\mathcal{R}ic = 2\frac{\partial G^i}{\partial x^i} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^i} + 2G^j\frac{\partial^2 G^i}{\partial y^i \partial y^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^i}. \tag{19}$$

If g is a reduced Berwald metric, then

$$G^i = \frac{1}{4}g^{il}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k, \tag{20}$$

where $\frac{1}{4}g^{il}(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$ is independent of y. Using (19) and (20), then we have

$$\begin{aligned}\left(\frac{\partial G^i}{\partial x^i}\right)_{y^r y^s} &= \frac{1}{4}\frac{\partial g^{il}}{\partial x^i}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)(\delta^j_r\delta^k_s + \delta^j_s\delta^k_r)\\ &\quad + \frac{1}{4}g^{il}\left(2\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}\right)(\delta^j_r\delta^k_s + \delta^j_s\delta^k_r)\\ &= \frac{1}{2}g^{il}\frac{\partial^2 g_{rl}}{\partial x^i\partial x^s} - \frac{1}{2}g^{il}\frac{\partial^2 g_{rs}}{\partial x^i\partial x^l} + \frac{1}{2}g^{il}\frac{\partial^2 g_{sl}}{\partial x^i\partial x^r} + \text{lower-order terms}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 G^i}{\partial x^m\partial y^i} &= \frac{1}{4}\frac{\partial g^{il}}{\partial x^m}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)(\delta^j_i y^k + \delta^k_i y^j)\\ &\quad + \frac{1}{4}g^{il}\left(2\frac{\partial g_{jl}}{\partial x^m\partial x^k}\frac{\partial g_{jk}}{\partial x^m\partial x^l}\right)(\delta^j_i y^k + \delta^k_i y^j)\\ \left(y^m\frac{\partial^2 G^i}{\partial x^m\partial y^i}\right)_{y^r y^s} &= \left[\left(\frac{1}{2}\frac{\partial g^{jl}}{\partial x^m}\frac{\partial g_{jl}}{\partial x^k} - \frac{1}{4}\frac{\partial g^{jl}}{\partial x^m}\frac{\partial g_{jk}}{\partial x^l}\right)y^m y^k + \left(\frac{1}{2}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial g_{jl}}{\partial x^k} - \frac{1}{4}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial g_{jk}}{\partial x^l}\right)y^m y^j\right.\\ &\quad \left. + \left(\frac{1}{2}g^{jl}\frac{\partial^2 g_{jl}}{\partial x^m\partial x^k} - \frac{1}{4}g^{jl}\frac{\partial^2 g_{jk}}{\partial x^m\partial x^l}\right)y^m y^k + \left(\frac{1}{2}g^{kl}\frac{\partial^2 g_{jl}}{\partial x^m\partial x^k} - \frac{1}{4}g^{kl}\frac{\partial^2 g_{jk}}{\partial x^m\partial x^l}\right)y^m y^j\right]_{y^r y^s}\\ &= \frac{1}{2}g^{jl}\frac{\partial^2 g_{jl}}{\partial x^r\partial x^s} - \frac{1}{4}g^{jl}\frac{\partial^2 g_{js}}{\partial x^r\partial x^l} + \frac{1}{2}g^{jl}\frac{\partial^2 g_{jl}}{\partial x^s\partial x^r}\\ &\quad - \frac{1}{4}g^{jl}\frac{\partial^2 g_{jr}}{\partial x^s\partial x^l} + \frac{1}{2}g^{kl}\frac{\partial^2 g_{sl}}{\partial x^r\partial x^k} - \frac{1}{4}g^{kl}\frac{\partial^2 g_{sk}}{\partial x^r\partial x^l}\\ &\quad + \frac{1}{2}g^{kl}\frac{\partial^2 g_{rl}}{\partial x^s\partial x^k} - \frac{1}{4}g^{kl}\frac{\partial^2 g_{rk}}{\partial x^s\partial x^l} + \text{lower-order terms}\\ &= g^{jl}\frac{\partial^2 g_{jl}}{\partial x^r\partial x^s} + \text{lower-order terms.}\end{aligned}$$

Lemma 3. *If F is a reduced Berwald metric, then* $\frac{\partial^2 G^i}{\partial y^i\partial y^m}(G^m)_{y^r y^s}$ *has not any second derivative term.*

Proof. Direct calculating shows

$$\begin{aligned}(G^m)_{y^r y^s} &= \frac{1}{4}g^{ml}\left(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)(\delta^j_r\delta^k_s + \delta^j_s\delta^k_r)\\ &= \frac{1}{2}g^{ml}\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^l}\right)\end{aligned}$$

so

$$\begin{aligned}(G^m)_{y^r y^s}\frac{\partial^2 G^i}{\partial y^i\partial y^m} &= \frac{1}{2}g^{ml}\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^l}\right)\left[g^{jk}\left(\frac{1}{2}\frac{\partial g_{jt}}{\partial x^m} - \frac{1}{4}\frac{\partial g_{jm}}{\partial x^t}\right)\right.\\ &\quad \left. + g^{kt}\left(\frac{1}{2}\frac{\partial g_{mt}}{\partial x^k} - \frac{1}{4}\frac{\partial g_{mk}}{\partial x^t}\right)\right]\end{aligned}$$

has not any second derivative term.

For a reduced Berwald metric, we have $\Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$, so

$$\begin{aligned}\frac{\partial}{\partial x^r} V_s &= \frac{\partial}{\partial x^r}\left(g_{sk} g^{jl}(\Gamma^k_{jl} - {\Gamma^0}^k_{jl})\right)\\ &= g_{sk} g^{jl} \frac{\partial}{\partial x^r}\left(\frac{\partial^2 G^k}{\partial y^j \partial y^l}\right) + \text{lower-order terms}\\ &= g_{sk} g^{jl} \frac{\partial}{\partial x^r}\left(\frac{1}{2} g^{km}\left(\frac{\partial g_{jm}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^m} + \frac{\partial g_{lm}}{\partial x^j}\right)\right) + \text{lower-order terms}\\ &= \frac{1}{2} g_{sk} g^{km} g^{jl}\left(\frac{\partial^2 g_{jm}}{\partial x^l \partial x^r} - \frac{\partial^2 g_{jl}}{\partial x^m \partial x^r} + \frac{\partial^2 g_{lm}}{\partial x^j \partial x^r}\right) + \text{lower-order terms}\\ &= g^{jl}\left(\frac{\partial^2 g_{js}}{\partial x^l \partial x^r} - \frac{1}{2}\frac{\partial^2 g_{jl}}{\partial x^s \partial x^r}\right) + \text{lower-order terms.}\end{aligned}$$

Proof of theorem (1). By (14), (18), (19) and lemma 3, the second-order time derivative of g_{rs} is

$$\begin{aligned}-(F^2 \mathcal{R}ic)_{y^r y^s} + \frac{\partial}{\partial x^s} V_r + \frac{\partial}{\partial x^r} V_s &= \left(-2\frac{\partial G^i}{\partial x^i} + y^j \frac{\partial^2 G^i}{\partial x^j \partial y^i} - 2G^j \frac{\partial^2 G^i}{\partial y^i \partial y^j} + \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^i}\right)_{y^r y^s} + \frac{\partial}{\partial x^r} V_s + \frac{\partial}{\partial x^s} V_r\\ &= -g^{il}\frac{\partial^2 g_{rl}}{\partial x^i \partial x^s} - g^{il}\frac{\partial^2 g_{sl}}{\partial x^i \partial x^r} + g^{il}\frac{\partial^2 g_{rs}}{\partial x^i \partial x^l}\\ &\quad + g^{il}\frac{\partial^2 g_{il}}{\partial x^s \partial x^r} + g^{il}\frac{\partial^2 g_{ir}}{\partial x^s \partial x^l} - \frac{1}{2} g^{il}\frac{\partial^2 g_{il}}{\partial x^s \partial x^r}\\ &\quad + g^{il}\frac{\partial^2 g_{is}}{\partial x^r \partial x^l} - \frac{1}{2} g^{il}\frac{\partial^2 g_{il}}{\partial x^r \partial x^s} + \text{lower-order terms}\\ &= g^{il}\frac{\partial^2 g_{rs}}{\partial x^i \partial x^l} + \text{lower-order terms.}\end{aligned}$$

Thereby, the initial value problem (18) can be written as

$$\begin{cases}\dfrac{\partial^2 g_{rs}}{\partial t^2}(x,y,t) = g^{kl}\dfrac{\partial^2 g_{rs}}{\partial x^k \partial x^l} + I(Dz, D_t D_x z) + H(g, D_x g)\\ g_{rs}(x,y,0) = g^0_{rs}(x,y), \qquad \dfrac{\partial}{\partial t} g_{rs}(x,y,0) = k^0_{rs}(x,y)\end{cases}, \tag{21}$$

where $g = (g_{rs})$, $D_x g = (\frac{\partial g_{rs}}{\partial x^k})$, $(r,s,k = 1,2,\ldots,n)$ and $k^0_{rs}(x,y)$ is a bilinear form on M. Let $\hat{\mu} = (g_{rs}, \ \frac{\partial g_{rs}}{\partial x^k})$, $(r,s,k = 1,2,\ldots,n)$. The nonlinear term $H = H(\hat{\mu}) = H(g, D_x g)$ in (21) is smooth and quadratic with respect to $D_x g$. We observe that (21) and (18) are clearly strictly hyperbolic systems. Since the manifold M is compact, it follows from the standard theory of hyperbolic equations (see [5] and [6]) that the system united by

$$\begin{cases}\dfrac{\partial^2 g_{rs}}{\partial t^2}(x,y,t) = g^{kl}\dfrac{\partial^2 g_{rs}}{\partial x^k \partial x^l} + I(Dz, D_t D_x z) + H(g, D_x g)\\ \dfrac{\partial^2 z^\alpha}{\partial t^2} = g^{jl}\left(\dfrac{\partial^2 z^\alpha}{\partial x^j \partial x^l} - {\Gamma^0}^k_{jl}\dfrac{\partial z^\alpha}{\partial x^k}\right)\\ g_{rs}(x,y,0) = g^0_{rs}(x,y), \qquad \dfrac{\partial}{\partial t} g_{rs}(x,y,0) = k^0_{rs}(x,y)\\ z^\alpha(x,0) = (x^\alpha), \qquad \dfrac{\partial}{\partial t} z^\alpha(x,0) = \mathcal{Z}^\alpha(x)\end{cases} \tag{22}$$

is strictly hyperbolic and hence has a unique smooth solution for a short time.

Finally we prove the uniqueness result for FHGF. Suppose there are solutions $\hat{g}_1(t)$ and $\hat{g}_2(t)$ of FHGF, with initial condition $\hat{g}_1(0) = \hat{g}_2(0)$ and $\frac{\partial \hat{g}_1}{\partial t}(0) = \frac{\partial \hat{g}_2}{\partial t}(0)$. Taking $\psi_t : M \longrightarrow M$ as a family of diffeomorphisms of M, corresponding to the solutions z^α for (22) we produce solutions $g_1(t)$ and $g_2(t)$ for (22). Since $g_1(t)$ and $g_2(t)$ are solutions of the same initial value problem for a system of partial differential equations, by uniqueness of solutions of the system (22), $g_1(t) = g_2(t)$ for all t in their common interval of existence, so $\hat{g}_1(t) = {\psi_t^{-1}}^* g_1(t) = {\psi_t^{-1}}^* g_2(t) = \hat{g}_2(t)$ proving uniqueness for the FHGF.

Equation (13) has a unique solution with the initial reduced Berwald metric g_0, but it is not obvious whether FHGF metrics remain reduced Berwald along the flow. We show the FHGF metrics evolve in the space of reduced Berwald metrics, for this purpose we use [7] and check three coditions.

In reduced Berwald space we have $G^i(y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$, where

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{lj}}{\partial y^r}G^r_k + \frac{\partial g_{jk}}{\partial y^r}G^r_l - \frac{\partial g_{kl}}{\partial y^r}G^r_j\right)$$

and

$$Ric_{ij} = \frac{\partial}{\partial x^k}\Gamma^k_{ij} - \frac{1}{2}\frac{\partial}{\partial x^i}\Gamma^k_{jk} - \frac{1}{2}\frac{\partial}{\partial x^j}\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^s_{ks} - \Gamma^k_{si}\Gamma^s_{jk}.$$

In the above equation any term is only in terms of x, therefore $\frac{\partial Ric_{ij}}{\partial y^k} = 0$. Similarly $\frac{\partial Ric_{ik}}{\partial y^j} = 0$. Since A^i_{jk} is independent of y,

$$\frac{\partial}{\partial y^k}\left(\frac{\partial^2}{\partial t^2}g_{ij}(x,y,t)\right) = -\frac{\partial}{\partial y^k}(Ric_{ij}(x,y,t)) = 0, \tag{23}$$

it implies that

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial y^k}g_{ij}(x,y,t)\right) = 0$$

and

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial y^k}g_{ij}(x,y,t)\right) = \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial y^j}g_{ik}(x,y,t)\right) \quad \forall t \in [0,\varepsilon),$$

put

$$\frac{\partial}{\partial y^k}g_{ij}(x,y,t) - \frac{\partial}{\partial y^j}g_{ik}(x,y,t) = h(x,y,t) \quad \forall t \in [0,\varepsilon),$$

so in $t = 0$, we have

$$\frac{\partial}{\partial y^k}g_{ij}(x,y,0) - \frac{\partial}{\partial y^j}g_{ik}(x,y,0) = 0,$$

therefore $h(x,y,t) = 0$ for all $t \in [0,\varepsilon)$ and

$$\frac{\partial}{\partial y^k}g_{ij}(x,y,t) = \frac{\partial}{\partial y^j}g_{ik}(x,y,t).$$

Since Ric_{ij} is positively homogeneous of degree zero (that is, invariant under positive rescaling) we have

$$\frac{\partial^2}{\partial t^2}g_{ij}(x,\lambda y,t) = -2Ric_{ij}(x,\lambda y,t) = -2Ric_{ij}(x,y,t)$$

so

$$\frac{\partial^2}{\partial t^2}g_{ij}(x,\lambda y,t) = \frac{\partial^2}{\partial t^2}g_{ij}(x,y,t).$$

Put

$$h'_{ij}(x,y,t) = g_{ij}(x,\lambda y,t) - g_{ij}(x,y,t) \quad \forall t \in [0,\varepsilon),$$

thus in $t = 0$, $h'_{ij}(x,y,t) = 0$, by using initial conditions, we have

$$h'_{ij}(x,y,t) = 0 \quad \forall t \in [0,\varepsilon), \quad \lambda \in \mathbb{R}, \quad \lambda > 0.$$

The third condition is obvious, FHGF metrics are smoothly dependent on x and nonzero $y \in T_xM$. Now we can consider the Finsler Ricci flow in the space of reduced Berwald metrics.

Example 2. Suppose that M is a compact Finsler manifold and $F_0 \neq 0$ is an Einstein Finsler metric which is reduced Berwald with constant Ricci scalar, equivalently $Ric(F_0) = \lambda F_0^2$ where λ is a constant. We assume that $F_t = u(t)F_0$ is a solution for the Finsler hyperbolic geometry flow,

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}F^2(x,y,t) = -2F^2\mathcal{R}ic(x,y,t) \\ F(x,y,0) = F_0, \qquad \dfrac{\partial F}{\partial t}(x,y,0) = F^0(x,y) \end{cases}. \tag{24}$$

Equations (7) and (10) imply that $Ric(F_t) = Ric(F_0)$ and

$$\mathcal{R}ic(F_t) = \frac{Ric(F_t)}{F_t^2} = \frac{\lambda}{(u(t))^2},$$

therefore by substituting it in the Finsler hyperbolic geometric flow, we have $u'' + \frac{1}{u}u'^2 = \frac{-\lambda}{u}$ and this implies $(u(t))^2 = \frac{1}{\lambda}c_1 - \lambda(t - c_2)^2$. But $u(0) = 1$ and $\frac{\partial F}{\partial t}(x, y, 0) = F^0$ so $c_1 = \lambda + \frac{1}{F_0^2}(F^0)^2$ and $c_2 = \frac{-1}{\lambda F_0}F^0$. Therefore $F^2(t) = F_0^2[((1 + \frac{1}{\lambda F_0^2}{F^0}^2)) - (\lambda t - \frac{1}{F_0}F^0)^2]$ is a solution for the hyperbolic geometric flow.

If $F_0 = 0$, then the hyperbolic geometric flow has a trivial solution.

3 Wave property of curvatures in a reduced Berwald space

The Finslerian hyperbolic geometric flow is an evolution equation on the reduced Berwald metric $g_{ij}(t, (x, y))$. The evolution for the metric implies a nonlinear wave equation for the Finsler curvature tensors. The goal in this section is to work out the evolution equations for Finsler curvatures under the Finsler hyperbolic geometric flow. More precisely, we will concentrate on obtaining the global forms of the evolutions under FHGF.

Remark 2. In this section we put $\frac{\partial}{\partial t}g^{ij} = H^{ij}$ whereas $h^{im} = -g^{mk}g^{ij}h_{kj}$.

Proposition 2. *Let F is a reduced Berwald metric and $h_{ij} = \frac{\partial}{\partial t}g_{ij}$, then we have*

a) $\frac{\partial^2}{\partial t^2}g^{ij} = -(h^{ik}h_k^j + h^{lj}h_l^i - 2g^{ik}g^{jl}Ric_{kl})$,
b) $\frac{\partial^2}{\partial t^2}\Gamma^m_{jk} = \partial_t H * \partial g + H * \partial h - g * \partial Ric$.

Proof. As $g^{ij}g_{jl} = \delta^i_l$ we find that $\frac{\partial}{\partial t}g^{ij} = -g^{ik}g^{jl}(h_{kl})$. In which case

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g^{ij} &= -g^{im}g^{kt}h_{mt}g^{jl}h_{kl} - g^{lm}g^{jt}h_{mt}g^{ik}h_{kl} - g^{ik}g^{jl}\left(\frac{\partial^2}{\partial t^2}g_{kl}\right)\\ &= -g^{im}h^k_m h^j_k - g^{lm}h^j_m h^i_l - g^{ik}g^{jl}\left(\frac{\partial^2}{\partial t^2}g_{kl}\right).\end{aligned}$$

Proposition 3. *If F_t is a smooth one-parameter family of reduced Berwald metrics on a manifold M, then the curvature tensor R^i_{jkl} evolves by*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R^i_{jkl} &= \partial_t\partial H * \partial g + \partial H * \partial h + \partial g * \partial Ric + \partial_t H * \partial^2 g + H * \partial^2 h\\ &\quad + g * \partial^2 Ric + (\partial_t H * \partial g + H * \partial h - g * \partial Ric)(g * \partial g)\\ &\quad + (H * \partial g + g * \partial h)(H * \partial g + g * \partial h),\end{aligned}$$

where $\frac{\partial}{\partial t}g^{ij} = H^{ij}$.

Proof. We compute

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R^i_{jkl} &= \frac{1}{3}\frac{\partial^2}{\partial t^2}\frac{\partial^3 G^i}{\partial x^l\partial y^j\partial y^k} + l \longleftrightarrow k + 2l \longleftrightarrow j\\ &\quad + \frac{1}{3}\left(\frac{\partial^2}{\partial t^2}\frac{\partial^2 G^m}{\partial y^j\partial y^l}\right)\left(\frac{\partial^2 G^i}{\partial y^m\partial y^k}\right) - \frac{2}{3}k \longleftrightarrow j + \frac{1}{3}k \longleftrightarrow l\\ &\quad + \frac{1}{3}\left(\frac{\partial^2}{\partial t^2}\frac{\partial^2 G^i}{\partial y^m\partial y^k}\right)\left(\frac{\partial^2 G^m}{\partial y^j\partial y^l}\right) - \frac{2}{3}k \longleftrightarrow j + \frac{1}{3}k \longleftrightarrow l\\ &\quad + \frac{2}{3}\left(\frac{\partial}{\partial t}\frac{\partial^2 G^m}{\partial y^j\partial y^l}\right)\left(\frac{\partial}{\partial t}\frac{\partial^2 G^i}{\partial y^m\partial y^k}\right) - \frac{4}{3}k \longleftrightarrow j + \frac{2}{3}l \longleftrightarrow k\\ &= I + II + III.\end{aligned}$$

We have

$$\begin{aligned}
I &= \frac{1}{4}\frac{\partial^2 H^{ic}}{\partial x^l \partial t}\left(2\frac{\partial g_{jc}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^c} + 2\frac{\partial g_{kc}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^c}\right)\\
&+ \frac{1}{2}\frac{\partial H^{ic}}{\partial x^l}\left(2\frac{\partial h_{jc}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^c} + 2\frac{\partial h_{kc}}{\partial x^j} - \frac{\partial h_{kj}}{\partial x^c}\right)\\
&+ \frac{\partial}{\partial x^l} g^{ic}\left(-\frac{\partial}{\partial x^k} Ric_{jc} + \frac{1}{2}\frac{\partial}{\partial x^c} Ric_{jk} - \frac{\partial}{\partial x^j} Ric_{kc} + \frac{1}{2}\frac{\partial}{\partial x^c} Ric_{kj}\right)\\
&+ \frac{1}{4}\frac{\partial}{\partial t} H^{ic}\left(2\frac{\partial^2 g_{jc}}{\partial x^l \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^l \partial x^c} + 2\frac{\partial^2 g_{kc}}{\partial x^l \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^c}\right)\\
&+ \frac{1}{2} H^{ic}\left(2\frac{\partial^2 h_{jc}}{\partial x^l \partial x^k} - \frac{\partial^2 h_{jk}}{\partial x^l \partial x^c} + 2\frac{\partial^2 h_{kc}}{\partial x^l \partial x^j} - \frac{\partial^2 h_{kj}}{\partial x^l \partial x^c}\right)\\
&+ g^{ic}\left(-\frac{\partial^2 Ric_{jc}}{\partial x^l \partial x^k} + \frac{1}{2}\frac{\partial^2 Ric_{jk}}{\partial x^l \partial x^c} - \frac{\partial^2 Ric_{kc}}{\partial x^l \partial x^j} + \frac{1}{2}\frac{\partial^2 Ric_{kj}}{\partial x^l \partial x^c}\right).
\end{aligned}$$

The second and third terms are

$$\begin{aligned}
II &= \frac{1}{2} g^{ic}\left(\frac{\partial g_{kc}}{\partial x^m} + \frac{\partial g_{mc}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^c}\right)\left[\frac{1}{2}\frac{\partial H^{mc}}{\partial t}\left(\frac{\partial g_{jc}}{\partial x^l} + \frac{\partial g_{lc}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^c}\right)\right.\\
&\left. + H^{mc}\left(\frac{\partial h_{jc}}{\partial x^l} + \frac{\partial h_{lc}}{\partial x^j} - \frac{\partial h_{jl}}{\partial x^c}\right) - g^{mc}\left(\frac{\partial Ric_{jc}}{\partial x^l} + \frac{\partial Ric_{lc}}{\partial x^j} - \frac{\partial Ric_{jl}}{\partial x^c}\right)\right]\\
III &= \left[\frac{1}{2} H^{mc}\left(\frac{\partial g_{jc}}{\partial x^l} + \frac{\partial g_{lc}}{\partial x^j} - \frac{\partial g_{lj}}{\partial x^c}\right) + \frac{1}{2} g^{mc}\left(\frac{\partial h_{jc}}{\partial x^l} + \frac{\partial h_{lc}}{\partial x^j} - \frac{\partial h_{lj}}{\partial x^c}\right)\right]\\
&\times \left[\frac{1}{2} H^{ic}\left(\frac{\partial g_{kc}}{\partial x^m} + \frac{\partial g_{mc}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^c}\right) + \frac{1}{2} g^{ic}\left(\frac{\partial h_{kc}}{\partial x^m} + \frac{\partial h_{mc}}{\partial x^k} - \frac{\partial h_{km}}{\partial x^c}\right)\right],
\end{aligned}$$

we can rewrite the formula as

$$\begin{aligned}
I &= \partial_t \partial H * \partial g + \partial H * \partial h + \partial g * \partial Ric + \partial_t H * \partial^2 g + H * \partial^2 h + g * \partial^2 Ric\\
II &= (\partial_t H * \partial g + H * \partial h - g * \partial Ric)(g * \partial g)\\
III &= (H * \partial g + g * \partial h)(H * \partial g + g * \partial h).
\end{aligned}$$

Proposition 4. *Under the Finsler hyperbolic geometric flow, the evolution of the curvature tensor R^i_{jk} is given by*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R^i_{jk} &= (\partial_t H * (\partial g) y + H * (\partial h) y - g * (\partial Ric) y)(g * \partial g)\\
&+ (\partial_t H * \partial g + H * \partial h - g * \partial Ric)(g * (\partial g) y)\\
&+ (H * \partial g + g * \partial h)(H * (\partial g) y + g * (\partial h) y).
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R^i_{jk} &= \left(\frac{\partial^2}{\partial t^2}\frac{\partial G^m}{\partial y^k}\right)\left(\frac{\partial^2 G^i}{\partial y^m \partial y^j}\right) + \left(\frac{\partial^2}{\partial t^2}\frac{\partial^2 G^i}{\partial y^m \partial y^j}\right)\left(\frac{\partial G^m}{\partial y^k}\right)\\
&+ 2\left(\frac{\partial}{\partial t}\frac{\partial^2 G^i}{\partial y^m \partial y^j}\right)\left(\frac{\partial}{\partial t}\frac{\partial G^m}{\partial y^k}\right) - k \longleftrightarrow j\\
&= I + II + III.
\end{aligned}$$

We have

$$\begin{aligned}
I &= \left[\frac{1}{4}\frac{\partial H^{mc}}{\partial t}\left(\left(2\frac{\partial g_{kc}}{\partial x^b} - \frac{\partial g_{kb}}{\partial x^c}\right) y^b + \left(2\frac{\partial g_{ac}}{\partial x^k} - \frac{\partial g_{ak}}{\partial x^c}\right) y^a\right)\right.\\
&+ \frac{1}{2} H^{mc}\left(\left(2\frac{\partial h_{kc}}{\partial x^b} - \frac{\partial h_{kb}}{\partial x^c}\right) y^b + \left(2\frac{\partial h_{ac}}{\partial x^k} - \frac{\partial h_{ak}}{\partial x^c}\right) y^a\right)\\
&\left. - g^{mc}\left[\left(\frac{\partial}{\partial x^b} Ric_{kc} - \frac{1}{2}\frac{\partial}{\partial x^c} Ric_{kb}\right) y^b + \left(\frac{\partial}{\partial x^k} Ric_{ac} - \frac{1}{2}\frac{\partial}{\partial x^c} Ric_{aj}\right) y^a\right]\right]\\
&\times \left(\frac{1}{2} g^{ic}\left(\frac{\partial}{\partial x^m} g_{jc} + \frac{\partial}{\partial x^j} g_{mc} - \frac{\partial}{\partial x^c} g_{jm}\right)\right),
\end{aligned}$$

the second term is

$$\begin{aligned} II = &\left[\frac{1}{2}\frac{\partial H^{ic}}{\partial t}\left(\frac{\partial g_{jc}}{\partial x^m}+\frac{\partial g_{mc}}{\partial x^j}-\frac{\partial g_{jm}}{\partial x^c}\right)+H^{ic}\left(\frac{\partial h_{jc}}{\partial x^m}+\frac{\partial h_{mc}}{\partial x^j}-\frac{\partial h_{jm}}{\partial x^c}\right)\right.\\ &\left.-\,g^{ic}\left(\frac{\partial}{\partial x^m}Ric_{jc}+\frac{\partial}{\partial x^j}Ric_{mc}-\frac{\partial}{\partial x^c}Ric_{jm}\right)\right]\\ &\times\left[\frac{1}{4}g^{mc}\left(\left(2\frac{\partial g_{kc}}{\partial x^b}-\frac{\partial g_{kb}}{\partial x^c}\right)y^b+\left(2\frac{\partial g_{ac}}{\partial x^k}-\frac{\partial g_{ak}}{\partial x^c}\right)y^a\right)\right] \end{aligned}$$

and the third term is

$$\begin{aligned} III = &\left(\frac{1}{2}H^{ic}\left(\frac{\partial g_{jc}}{\partial x^m}+\frac{\partial g_{mc}}{\partial x^j}-\frac{\partial g_{jm}}{\partial x^c}\right)+\frac{1}{2}g^{ic}\left(\frac{\partial h_{jc}}{\partial x^m}+\frac{\partial h_{mc}}{\partial x^j}-\frac{\partial h_{jm}}{\partial x^c}\right)\right)\\ &\times\left\{\frac{1}{4}H^{mc}\left[\left(2\frac{\partial g_{jc}}{\partial x^b}-\frac{\partial g_{jb}}{\partial x^c}\right)y^b+\left(2\frac{\partial g_{ac}}{\partial x^j}-\frac{\partial g_{aj}}{\partial x^c}\right)y^a\right]\right.\\ &\left.+\,\frac{1}{4}g^{mc}\left[\left(2\frac{\partial h_{jc}}{\partial x^b}-\frac{\partial h_{jb}}{\partial x^c}\right)y^b+\left(2\frac{\partial h_{ac}}{\partial x^j}-\frac{\partial h_{aj}}{\partial x^c}\right)y^a\right]\right\}. \end{aligned}$$

We can rewrite the formula as

$$\begin{aligned} I &= (\partial_t H*(\partial g)y+H*(\partial h)y-g*(\partial Ric)y)(g*\partial g)\\ II &= (\partial_t H*\partial g+H*\partial h-g*\partial Ric)(g*(\partial g)y)\\ III &= (H*\partial g+g*\partial h)(H*(\partial g)y+g*(\partial h)y). \end{aligned}$$

4 Discussions

Hyperbolic partial differential equations have been used to describe the wave phenomena in nature. In this paper, we introduced a new kind of hyperbolic geometric flows, the Finsler hyperbolic geometric flow, to illustrate the wave character of the Finsler metrics, which also implies the wave property of the curvature. The FHGF possesses very interesting geometric properties and dynamical behaviour. So far there is great success of elliptic and parabolic equations applied to mathematics and physics, but by now very few results on the applications of hyperbolic PDEs are known. We believe that the Finsler hyperbolic geometric flow is a new tool to study geometric problems and physical application. In the future we will study several fundamental problems, for examples, solitons, stability, long-time existence, formation of singularities, as well as the general Finsler case and the physical and geometrical applications.

We know that if we want to study the global properties of FHGF, then it is important to find curvature conditions that are preserved under the evolution. How to develop such techniques? For instance, suppose M is compact Finsler manifold and let $F(t)$, $t \in [0,t)$ be a solution to FHGF on M and consider $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{R}ic$; can we claim that non-negative isometric curvature is preserved?

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