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Solitons and other solutions to nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity using several different techniques

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Abstract. The (G'/G)-expansion method, the improved Sub-ODE method, the extended auxiliary equation method, the new mapping method and the Jacobi elliptic function method are applied in this paper for finding many new exact solutions including Jacobi elliptic solutions, solitary solutions, singular solitary solutions, trigonometric function solutions and other solutions to the nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity whose balance number is not positive integer. The used methods present a wider applicability for handling the nonlinear partial differential equations. A comparison of our new results with the well-known results is made. Also, we compare our results with each other yielding from these five integration tools.

1 Introduction

The exact solutions of nonlinear partial differential equations (PDEs) are very important in studying the nonlinear physical phenomena. These phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. These exact solutions can be found using many powerful methods, such as the inverse scattering method [1], the tanh function method [2–5], the Hirota bilinear transform method [6], the truncated Painlevé expansion method [7–10], the Bäcklund transform method [11,12], the exp-function method [13–17], the Jacobi elliptic function expansion method [18–20], the generalized Riccati equation method [21–23], the (G'/G)-expansion method [24–34], the $(G'/G, 1/G)$-expansion method [35–37], the Sub-ODE method [38,39], the extended auxiliary equation method [40, 41], the improved Sub-ODE method [42], the new mapping method [43], the soliton ansatz method [44–50], and so on.

The objective of this paper is to employ the (G'/G)-expansion method, the improved Sub-ODE method, the extended auxiliary equation method, the new mapping method and the Jacobi elliptic function method to construct many exact solutions including Jacobi elliptic solutions, solitons and other solutions of the following nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity [44,51–53]:

$$iq_t + aq_{xx} - bq_{xxxx} + c\left(|q|^{2m} + k_1|q|^{4m}\right)q = 0, \quad i = \sqrt{-1}, \tag{1}$$

which describes the propagation of optical pulse in a medium, and $q(x,t)$ is the slowly varying envelope of the electromagnetic field, where a, b, c are real numbers. If $b = 0$, eq. (1) reduces to the nonlinear Schrödinger equation with dual power law nonlinearity. In addition if $m = 1$, eq. (1) reduces to parabolic law nonlinearity, which has been discussed in [54] using two direct algebraic methods. The coefficient of a represents the group velocity dispersion (GVD), while the coefficient of c represents the self-phase modulation (SPM) with dual power law nonlinearity. The constant k_1 binds the two nonlinear terms and the exponent m governs the power law. Also, the coefficients of b are the fourth-order dispersion terms. Equation (1) has been discussed in [44] using the soliton ansatz method and in [53] using two methods which are different from the five methods used in this paper. Also, eq. (1) has been solved in [50]

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in the special case $m = 1$, $k_1 = 0$ using the soliton ansatz method. To our knowledge, eq. (1) has not been discussed elsewhere using the five mathematical methods mentioned above.

This paper is organized as follows: in sect. 2, we apply our proposed five methods to solve eq. (1). In sect. 3, we present some graphical representations for some solutions of eq. (1). In sect. 4, conclusions are obtained.

2 Applications

It is to be noted that eq. (1) is not integrable by the classical method of inverse scattering transform since it will fail the Painlevé test of integrability [55]. It is, however, still possible to obtain a closed 1-soliton solution to eq. (1) given by the following phase-amplitude format:

$$q(x,t) = \phi(\xi)e^{iQ(x,t)}, \tag{2}$$

where $\phi(\xi)$ is the amplitude portion which is a real function of ξ, while $Q(x,t)$ is the phase portion of the soliton. It is assumed that ξ and $Q(x,t)$ are given by

$$\xi = x - \upsilon t, \tag{3}$$

and

$$Q(x,t) = -kx + \omega t + \theta, \tag{4}$$

where υ, k, ω, θ are constants, such that υ is the velocity of the soliton, k is the frequency of the soliton, ω is the wave number and θ is a phase constant.

Substituting (2) into eq. (1) and separating the real and imaginary parts, we get

$$\mathrm{Re}: \quad -\left(\omega + ak^2 + bk^4\right)\phi + \left(a + 6bk^2\right)\phi'' - b\phi'''' + c\left(\phi^{2m+1} + k_1\phi^{4m+1}\right) = 0, \tag{5}$$

$$\mathrm{Im}: \quad -\left(\upsilon + 2ak + 4bk^3\right)\phi' + 4bk\phi''' = 0. \tag{6}$$

Differentiating (6) and substituting the resulting equation in (5), we have the nonlinear ODE:

$$a_1\phi'' + b_1\phi + c_1\left(\phi^{2m+1} + k_1\phi^{4m+1}\right) = 0, \tag{7}$$

where a_1, b_1 and c_1 are given by

$$\begin{aligned} a_1 &= 2ak + 20bk^3 - \upsilon, \\ b_1 &= -4k\left(\omega + ak^2 + bk^4\right), \\ c_1 &= 4kc. \end{aligned} \tag{8}$$

2.1 On solving eq. (1) using the (G′/G)-expansion method

According to the (G'/G)-expansion method [31–34], we assume that eq. (7) has the formal solution:

$$\phi(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^N, \tag{9}$$

where A is a constant to be determined and $G(\xi)$ satisfied the ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{10}$$

where λ and μ are constants. The power N in eq. (9) can be determined by balancing ϕ'' with ϕ^{4m+1} in eq. (7) to get $N = \frac{1}{2m}$.

Substituting (9) along with eq. (10) into eq. (7), collecting all the coefficients of power (G'/G) and setting them to zero, we have the following algebraic equations:

$$(G'/G)^{\frac{1}{2m}}: \quad a_1\left(\lambda^2 + 2\mu\right) + 4m^2 b_1 = 0, \tag{11}$$

$$(G'/G)^{\frac{1}{2m}-1}: \quad a_1\lambda\mu(1-m) = 0, \tag{12}$$

$$(G'/G)^{\frac{1}{2m}+1}: \quad a_1\lambda(1+m) + 2m^2 c_1 A^{2m} = 0, \tag{13}$$

$$(G'/G)^{\frac{1}{2m}-2}: \quad a_1\mu^2(1-2m) = 0, \tag{14}$$

$$(G'/G)^{\frac{1}{2m}+2}: \quad a_1(1+2m) + 4m^2 c_1 k_1 A^{4m} = 0. \tag{15}$$

On solving eqs. (11)–(15), we have the results

$$\mu = 0, \qquad \lambda^2 = \frac{-4m^2 b_1}{a_1}, \qquad A^{2m} = \frac{(1+2m)}{2\lambda k_1(1+m)}, \qquad k_1 = \frac{c_1(1+2m)}{4b_1(1+m)^2}, \tag{16}$$

where $a_1 b_1 < 0$, $\lambda k_1 > 0$ and $k_1 \neq 0$.

Now, eq. (1) has the new exact solution

$$q(x,t) = \left\{ \frac{-(1+2m)}{2k_1(1+m)} \left[\frac{A_1 e^{-\lambda\xi}}{A_0 + A_1 e^{-\lambda\xi}} \right] \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{17}$$

where A_0, A_1 are arbitrary constants. From (8) and (16), we deduce that ω and υ are given by

$$\omega = -ak^2 - bk^4 - \frac{c(1+2m)}{4k_1(1+m)^2} \quad \text{and} \quad \upsilon = 2ak + 20bk^3 + \frac{4kcm^2(1+2m)}{\lambda^2 k_1(1+m)^2}. \tag{18}$$

In particular, if $A_0 = A_1 = 1$, then we have the solitary wave solution of eq. (1) in the form

$$q(x,t) = \left\{ \frac{-(1+2m)}{4k_1(1+m)} \left[1 \pm \tanh\left(m\sqrt{\frac{-b_1}{a_1}}\xi \right) \right] \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{19}$$

while, if $A_0 = 1$, $A_1 = -1$, then we have the singular solitary wave solution of eq. (1) in the form

$$q(x,t) = \left\{ \frac{-(1+2m)}{4k_1(1+m)} \left[1 \pm \coth\left(m\sqrt{\frac{-b_1}{a_1}}\xi \right) \right] \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{20}$$

where $k_1 < 0$ and $a_1 b_1 < 0$.

Note that the solutions (17), (19) and (20) of eq. (1) are all new and not found in [44]. Douvagai *et al.* [53] solved eq. (7) using a different technique and found the exact solutions of eq. (1) where some of them are equivalent to (19) and (20). Finally, note that the above technique is not used in [44,53].

2.2 On solving eq. (1) using the improved Sub-ODE method

To this aim, we multiply eq. (7) by $\phi'(\xi)$ and integrate with respect to ξ with zero constant of integration, we get the auxiliary equation:

$$\phi'^2 = A\phi^2 + B\phi^{2m+2} + C\phi^{4m+2}, \quad m > 0. \tag{21}$$

The coefficients A, B and C are given by

$$A = \frac{-b_1}{a_1}, \qquad B = \frac{-c_1}{a_1(m+1)}, \qquad C = \frac{-c_1 k_1}{a_1(2m+1)}, \tag{22}$$

where $a_1 \neq 0$. It is well-known [42,56] that eq. (21) has many solutions. With the aid of these solutions eq. (1) has the following new solutions:

1) If $A > 0$ and $B^2 - 4AC > 0$, then we deduce that eq. (1) has the hyperbolic solutions

$$q(\xi) = \left\{ \mp\frac{2b_1(m+1)}{c_1} \left(\frac{\operatorname{sech}\left[2m\sqrt{\frac{-b_1}{a_1}}\left(\xi+\xi_0\right)\right]}{\sqrt{1 - \frac{4b_1k_1(m+1)^2}{c_1(2m+1)}} \pm \operatorname{sech}\left[2m\sqrt{\frac{-b_1}{a_1}}\left(\xi+\xi_0\right)\right]} \right) \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{23}$$

provided that $a_1 b_1 < 0$, $\frac{4b_1k_1(m+1)^2}{c_1(2m+1)} < 1$ and $b_1 c_1 < 0$ (or $b_1 c_1 > 0$).

2) If $A > 0$ and $B^2 - 4AC < 0$, then we deduce that eq. (1) has the hyperbolic solutions

$$q(\xi) = \left\{ \mp \frac{2b_1(m+1)}{c_1} \left(\frac{\operatorname{csch}\left[2m\sqrt{\frac{-b_1}{a_1}}(\xi+\xi_0)\right]}{\sqrt{\frac{4b_1k_1(m+1)^2}{c_1(2m+1)} - 1} \pm \operatorname{csch}\left[2m\sqrt{\frac{-b_1}{a_1}}(\xi+\xi_0)\right]} \right) \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{24}$$

provided that $a_1b_1 < 0$, $\frac{4b_1k_1(m+1)^2}{c_1(2m+1)} > 1$ and $b_1c_1 < 0$ (or $b_1c_1 > 0$).

3) If $A > 0$, $B^2 - 4AC = 0$ and $B < 0$, then we deduce that eq. (1) has the dark soliton solutions

$$q(\xi) = \left\{ \frac{-(2m+1)}{4k_1(m+1)} \left[1 \pm \tanh\left(m\sqrt{\frac{-b_1}{a_1}}(\xi+\xi_0) \right) \right] \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{25}$$

and the singular soliton solutions

$$q(\xi) = \left\{ \frac{-(2m+1)}{4k_1(m+1)} \left[1 \pm \coth\left(m\sqrt{\frac{-b_1}{a_1}}(\xi+\xi_0) \right) \right] \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{26}$$

provided that $a_1b_1 < 0$, $k_1 < 0$, $k_1 = \frac{c_1(2m+1)}{4b_1(m+1)^2}$ and $b_1c_1 < 0$. It is easy to show that $\omega = \frac{c_1(2m+1)}{4k_1(m+1)^2} - (ak^2 + bk^4)$.

Note that the solutions (25) and (26) are equivalent to the solutions (19) and (20), respectively, if $\xi_0 = 0$.

4) If $A < 0$ and $B^2 - 4AC > 0$, then we deduce that eq. (1) has the trigonometric solutions

$$q(\xi) = \left\{ \mp \frac{2b_1(m+1)}{c_1} \left(\frac{\sec\left[2m\sqrt{\frac{b_1}{a_1}}(\xi+\xi_0)\right]}{\sqrt{1 - \frac{4b_1k_1(m+1)^2}{c_1(2m+1)}} \pm \sec\left[2m\sqrt{\frac{b_1}{a_1}}(\xi+\xi_0)\right]} \right) \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{27}$$

and

$$q(\xi) = \left\{ \mp \frac{2b_1(m+1)}{c_1} \left(\frac{\csc\left[2m\sqrt{\frac{b_1}{a_1}}(\xi+\xi_0)\right]}{\sqrt{1 - \frac{4b_1k_1(m+1)^2}{c_1(2m+1)}} \pm \csc\left[2m\sqrt{\frac{b_1}{a_1}}(\xi+\xi_0)\right]} \right) \right\}^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{28}$$

provided that $a_1b_1 < 0$, $\frac{4b_1k_1(m+1)^2}{c_1(2m+1)} < 1$ and $b_1c_1 < 0$ (or $b_1c_1 > 0$).

5) If $A = 0$ and $B \neq 0$, then we deduce that eq. (1) has the rational solution

$$q(\xi) = \left[\frac{-4a_1(m+1)(2m+1)}{c_1m^2(2m+1)\xi^2 + 4a_1k_1(m+1)^2} \right]^{\frac{1}{2m}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{29}$$

where $c_1 \neq 0$ and $b_1 = 0$. Thus, from (8), we have $\omega = -(ak^2 + bk^4)$.

Note that the exact solutions (23)–(29) of eq. (1) are all new and not found in [44,53] or elsewhere.

2.3 On solving eq. (1) when m = 1 using an extended auxiliary equation method

To this aim, we set $m = 1$ in eq. (7) and multiply the resulting equation by $\phi'(\xi)$ and integrate with respect to ξ, we get the extended auxiliary equation

$$\phi'^2 = c_0 + c_2\phi^2 + c_4\phi^4 + c_6\phi^6, \tag{30}$$

where

$$c_0 = \frac{2\epsilon_1}{a_1}, \qquad c_2 = \frac{-b_1}{a_1}, \qquad c_4 = \frac{-c_1}{2a_1}, \qquad c_6 = \frac{-c_1k_1}{3a_1}, \tag{31}$$

where $a_1 \neq 0$ and ϵ_1 is the constant of integration. It is well known [40,41,57] that eq. (30) has many Jacobi elliptic function solutions. With the aid of these solutions eq. (1) has the following Jacobi elliptic solutions:

1) If $c_0 = \frac{c_4^3(M_1^2-1)}{32c_6^2M_1^2}$, $c_2 = \frac{c_4^2(5M_1^2-1)}{16c_6M_1^2}$, $0 < M_1 < 1$, $c_6 > 0$, then we have

$$q(\xi) = \left\{ \frac{-3}{8k_1} \left[1 \pm \operatorname{sn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1M_1^2}} \right) \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{32}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{1}{M_1\,\mathrm{sn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1M_1^2}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{33}$$

where $k_1=\frac{3c_1(5M_1^2-1)}{64b_1M_1^2}<0$, $\epsilon_1=\frac{-9c_1(M_1^2-1)}{512k_1^2M_1^2}$ and $a_1c_1>0$.

If $M_1=1$, then sn = tanh, and we have the same solutions (25) and (26) when $m=1$, respectively.

2) If $c_0=\frac{c_4^3(1-M_1^2)}{32c_6^2}$, $c_2=\frac{c_4^2(5-M_1^2)}{16c_6}$, $0<M_1<1$, $c_6>0$, then we have

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm M_1\,\mathrm{sn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{34}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{1}{\mathrm{sn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{35}$$

where $k_1=\frac{3c_1(5-M_1^2)}{64b_1}<0$, $\epsilon_1=\frac{-9c_1(1-M_1^2)}{512k_1^2}$ and $a_1c_1>0$.

If $M_1=1$, then sn = tanh, and we have the same solutions (25) and (26) when $m=1$ respectively, while if $M_1=0$, then sn = sin, and we have the solution

$$q(\xi)=\left(\frac{-3}{8k_1}\right)^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{36}$$

or the trigonometric solutions

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\csc\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{37}$$

where $k_1=\frac{15c_1}{64b_1}<0$, $\epsilon_1=\frac{-9c_1}{512k_1^2}$ and $a_1c_1>0$.

3) If $c_0=\frac{c_4^3}{32c_6^2M_1^2}$, $c_2=\frac{c_4^2(4M_1^2+1)}{16c_6M_1^2}$, $0<M_1<1$, $c_6<0$, then we have

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm M_1\,\mathrm{cn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1M_1^2}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{38}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{\sqrt{1-M_1^2}\,\mathrm{sn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1M_1^2}}\right)}{\mathrm{dn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1M_1^2}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{39}$$

where $k_1=\frac{3c_1(4M_1^2+1)}{64b_1M_1^2}<0$, $\epsilon_1=\frac{-9c_1}{512k_1^2M_1^2}$ and $a_1c_1<0$.

If $M_1=1$, then cn = sech, and we have the bright soliton solutions

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\mathrm{sech}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{40}$$

and the same solution (36), where $k_1=\frac{15c_1}{64b_1}<0$, $\epsilon_1=\frac{-9c_1}{512k_1^2}$ and $a_1c_1<0$.

4) If $c_0=\frac{c_4^3M_1^2}{32c_6^2(M_1^2-1)}$, $c_2=\frac{c_4^2(5M_1^2-4)}{16c_6(M_1^2-1)}$, $0<M_1<1$, $c_6<0$, then we have

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{\mathrm{dn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1\left(1-M_1^2\right)}}\right)}{\sqrt{1-M_1^2}}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{41}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{1}{\mathrm{dn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1\left(1-M_1^2\right)}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{42}$$

where $k_1=\frac{3c_1(5M_1^2-4)}{64b_1(M_1^2-1)}<0$, $\epsilon_1=\frac{-9c_1M_1^2}{512k_1^2(M_1^2-1)}$ and $a_1c_1<0$.

If $M_1=0$, then $\mathrm{dn}=1$, and we have the solution

$$q(\xi)=\left(\frac{-3}{4k_1}\right)^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{43}$$

where $k_1=\frac{3c_1}{16b_1}<0$, $\epsilon_1=0$.

5) If $c_0=\frac{c_4^3}{32c_6^2(1-M_1^2)}$, $c_2=\frac{c_4^2(4M_1^2-5)}{16c_6(M_1^2-1)}$, $0<M_1<1$, $c_6>0$, then we have

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{1}{\mathrm{cn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1\left(1-M_1^2\right)}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{44}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{\mathrm{dn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1\left(1-M_1^2\right)}}\right)}{\sqrt{1-M_1^2}\,\mathrm{sn}\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1\left(1-M_1^2\right)}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{45}$$

where $k_1=\frac{3c_1(4M_1^2-5)}{64b_1(M_1^2-1)}<0$, $\epsilon_1=\frac{-9c_1}{512k_1^2(1-M_1^2)}$ and $a_1c_1>0$.

If $M_1=0$, then $\mathrm{dn}=1$, $\mathrm{cn}=\cos$, $\mathrm{sn}=\sin$ and we have the trigonometric solutions

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\sec\left(\xi\sqrt{\frac{-3c_1}{16a_1k_1}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{46}$$

and the same solution (37), where $k_1=\frac{15c_1}{64b_1}<0$, $\epsilon_1=\frac{-9c_1}{512k_1^2}$ and $a_1c_1>0$.

6) If $c_0=\frac{c_4^3M_1^2}{32c_6^2}$, $c_2=\frac{c_4^2(M_1^2+4)}{16c_6}$, $0<M_1<1$, $c_6<0$, then we have

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\mathrm{dn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1}}\right)\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{47}$$

or

$$q(\xi)=\left\{\frac{-3}{8k_1}\left[1\pm\frac{\sqrt{1-M_1^2}}{\mathrm{dn}\left(\xi\sqrt{\frac{3c_1}{16a_1k_1}}\right)}\right]\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{48}$$

where $k_1=\frac{3c_1(M_1^2+4)}{64b_1}<0$, $\epsilon_1=\frac{-9c_1M_1^2}{512k_1^2}$ and $a_1c_1<0$.

If $M_1=1$, then $\mathrm{dn}=\mathrm{sech}$, and we have the same solutions (40) and (36), respectively, while if $M_1=0$, then we have the same solution (43).

Note that our solutions (32)–(48) of eq. (1) are new and not found in [44,53] or elsewhere.

2.4 On solving eq. (1) when m = 1 using the new mapping method

To this aim, we follow sect. 2.3, and rewrite eq. (30) in the form

$$\phi'^2=r+p\phi^2+\frac{q}{2}\phi^4+\frac{s}{3}\phi^6,\tag{49}$$

where

$$r = \frac{2\epsilon_2}{a_1}, \qquad p = \frac{-b_1}{a_1}, \qquad q = \frac{-c_1}{a_1}, \qquad s = \frac{-c_1 k_1}{a_1}, \tag{50}$$

where $a_1 \neq 0$ and ϵ_2 is the constant of integration. It is well known [43,58] that eq. (49) has many exact solutions. With the aid of these solutions eq. (1) has the following exact solutions:

1) If $p < 0$, $q > 0$, $s = \frac{3q^2}{16p}$, $r = \frac{16p^2}{27q}$, then we have

$$q(\xi) = \left\{ \frac{-1}{k_1} \left[\frac{\tanh^2\left(\epsilon\sqrt{\frac{b_1}{3a_1}}\xi\right)}{3 + \tanh^2\left(\epsilon\sqrt{\frac{b_1}{3a_1}}\xi\right)} \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{51}$$

and

$$q(\xi) = \left\{ \frac{-1}{k_1} \left[\frac{\coth^2\left(\epsilon\sqrt{\frac{b_1}{3a_1}}\xi\right)}{3 + \coth^2\left(\epsilon\sqrt{\frac{b_1}{3a_1}}\xi\right)} \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{52}$$

where $a_1 b_1 > 0$, $k_1 = \frac{3c_1}{16b_1} < 0$, $\epsilon_2 = \frac{-b_1}{18k_1}$.

2) If $p > 0$, $q < 0$, $s = \frac{3q^2}{16p}$, $r = \frac{16p^2}{27q}$, then we have

$$q(\xi) = \left\{ \frac{1}{k_1} \left[\frac{\tan^2\left(\epsilon\sqrt{\frac{-b_1}{3a_1}}\xi\right)}{3 - \tan^2\left(\epsilon\sqrt{\frac{-b_1}{3a_1}}\xi\right)} \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{53}$$

and

$$q(\xi) = \left\{ \frac{1}{k_1} \left[\frac{\cot^2\left(\epsilon\sqrt{\frac{-b_1}{3a_1}}\xi\right)}{3 - \cot^2\left(\epsilon\sqrt{\frac{-b_1}{3a_1}}\xi\right)} \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{54}$$

where $a_1 b_1 < 0$, $k_1 = \frac{3c_1}{16b_1} > 0$, $b_1 c_1 > 0$, $\epsilon_2 = \frac{-b_1}{18k_1}$.

3) If $p > 0$, $s = \frac{3q^2}{16p}$, $r = 0$, then we have

$$q(\xi) = \left\{ \frac{-3}{8k_1} \left[1 + \tanh\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right) \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{55}$$

and

$$q(\xi) = \left\{ \frac{-3}{8k_1} \left[1 + \coth\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right) \right] \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{56}$$

where $a_1 b_1 < 0$ and $k_1 = \frac{3c_1}{16b_1} < 0$, which are equivalent to (25) and (26) respectively, when $m = 1$, and $\epsilon_2 = 0$, while $\epsilon = \pm 1$.

4) If $p > 0$ and $r = 0$, then we have

$$q(\xi) = \left\{ \frac{-6b_1 \operatorname{sech}^2\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}{3c_1 - 4b_1 k_1 \left[1 + \epsilon\tanh\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)\right]^2} \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{57}$$

and

$$q(\xi) = \left\{ \frac{6b_1 \operatorname{csch}^2\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}{3c_1 - 4b_1 k_1 \left[1 + \epsilon\coth\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)\right]^2} \right\}^{\frac{1}{2}} \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{58}$$

where $a_1 b_1 < 0$ and $\epsilon_2 = 0$, while $\epsilon = \pm 1$.

5) If $p>0$, $s>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-6b_1\operatorname{sech}^2\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}{3c_1-4\epsilon\sqrt{3c_1b_1k_1}\tanh\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{59}$$

and

$$q(\xi)=\left\{\frac{6b_1\operatorname{csch}^2\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}{3c_1-4\epsilon\sqrt{3c_1b_1k_1}\coth\left(\sqrt{\frac{-b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{60}$$

where $a_1b_1<0$, $c_1b_1k_1>0$, $\epsilon_2=0$, $\epsilon=\pm1$.

6) If $p<0$, $s>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-6b_1\sec^2\left(\sqrt{\frac{b_1}{a_1}}\xi\right)}{3c_1-4\epsilon\sqrt{-3c_1b_1k_1}\tan\left(\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{61}$$

and

$$q(\xi)=\left\{\frac{-6b_1\csc^2\left(\sqrt{\frac{b_1}{a_1}}\xi\right)}{3c_1-4\epsilon\sqrt{-3c_1b_1k_1}\cot\left(\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{62}$$

where $a_1b_1>0$, $c_1b_1k_1<0$, $\epsilon_2=0$, $\epsilon=\pm1$.

7) If $p>0$, $M=9q^2-48ps>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-12b_1}{\epsilon\sqrt{9c_1^2-48b_1c_1k_1}\cosh\left(2\sqrt{\frac{-b_1}{a_1}}\xi\right)+3c_1}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{63}$$

and

$$q(\xi)=\left\{\frac{-12\epsilon b_1\operatorname{sech}\left(2\sqrt{\frac{-b_1}{a_1}}\xi\right)}{\sqrt{9c_1^2-48b_1c_1k_1}+3\epsilon c_1\operatorname{sech}\left(2\sqrt{\frac{-b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{64}$$

where $a_1b_1<0$, $9c_1^2>48b_1c_1k_1$, $\epsilon_2=0$, $\epsilon=\pm1$. We note that the solution (64) is equivalent to (23) if $m=1$, $\xi_0=0$.

8) If $p>0$, $q<0$, $s<0$, $M=9q^2-48ps>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-4b_1\operatorname{sech}^2\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right)}{2c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}-c_1\left(\sqrt{1-\frac{16b_1k_1}{3c_1}}-1\right)\operatorname{sech}^2\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{65}$$

and

$$q(\xi)=\left\{\frac{-4b_1\operatorname{csch}^2\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right)}{2c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}+c_1\left(\sqrt{1-\frac{16b_1k_1}{3c_1}}+1\right)\operatorname{csch}^2\left(\epsilon\sqrt{\frac{-b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{66}$$

where $a_1b_1<0$, $\frac{16b_1k_1}{3c_1}<1$ and $b_1c_1<0$, $\epsilon_2=0$.

9) If $p<0$, $q>0$, $s<0$, $M>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{4b_1\sec^2\left(\epsilon\sqrt{\frac{b_1}{a_1}}\xi\right)}{2c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}-c_1\left(\sqrt{1-\frac{16b_1k_1}{3c_1}}+1\right)\sec^2\left(\epsilon\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right],\tag{67}$$

and

$$q(\xi)=\left\{\frac{-4b_1\csc^2\left(\epsilon\sqrt{\frac{b_1}{a_1}}\xi\right)}{2c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}-c_1\left(\sqrt{1-\frac{16b_1k_1}{3c_1}}-1\right)\csc^2\left(\epsilon\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{68}$$

where $a_1b_1>0$, $\frac{16b_1k_1}{3c_1}<1$, $b_1c_1<0$, $\epsilon_2=0$.

10) If $p<0$, $M>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-4b_1}{c_1\left[\epsilon\sqrt{1-\frac{16b_1k_1}{3c_1}}\cos\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)+1\right]}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{69}$$

and

$$q(\xi)=\left\{\frac{-4b_1}{c_1\left[\epsilon\sqrt{1-\frac{16b_1k_1}{3c_1}}\sin\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)+1\right]}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{70}$$

where $a_1b_1>0$, $\frac{16b_1k_1}{3c_1}<1$, $b_1c_1<0$, $\epsilon_2=0$.

11) If $p<0$, $M>0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-4\epsilon b_1\sec\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)}{c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}+\epsilon c_1\sec\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{71}$$

and

$$q(\xi)=\left\{\frac{-4\epsilon b_1\csc\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)}{c_1\sqrt{1-\frac{16b_1k_1}{3c_1}}+\epsilon c_1\csc\left(2\sqrt{\frac{b_1}{a_1}}\xi\right)}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{72}$$

where $a_1b_1>0$, $\frac{16b_1k_1}{3c_1}<1$, $b_1c_1<0$, (or $b_1c_1>0$), $\epsilon_2=0$, $\epsilon=\pm1$.

Note that the solutions (71) and (72) are in agreement with (27) and (28), respectively, when $m=1$ and $\xi_0=0$.

12) If $p>0$, $M=9q^2-48ps<0$ and $r=0$, then we have

$$q(\xi)=\left\{\frac{-4b_1}{\epsilon c_1\sqrt{\frac{16b_1k_1}{3c_1}-1}\sinh\left(2\sqrt{\frac{-b_1}{a_1}}\xi\right)+c_1}\right\}^{\frac{1}{2}}\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{73}$$

where $a_1b_1<0$, $\frac{16b_1k_1}{3c_1}>1$, $b_1c_1<0$, $\epsilon_2=0$, $\epsilon=\pm1$.

Note that the solutions (51)–(73) of eq. (1) are all new and not found in [44,53] or elsewhere.

2.5 On solving eq. (1) when $m=1$ and $k_1=0$ using the Jacobi elliptic equation

Biswas *et al.* [50] have solved eq. (1) in the special case $m=1$ and $k_1=0$, using the soliton ansatz method and have obtained only 1-soliton solution. In this section, we obtain many exact solutions using the solutions of the Jacobi elliptic equation.

Following sect. 2.4, and setting $k_1=0$ in eq. (49), we have the Jacobi elliptic equation [59,60]:

$$\phi'^2(\xi)=r+p\phi^2(\xi)+\frac{q}{2}\phi^4(\xi), \tag{74}$$

where $r=\frac{2\varepsilon_2}{a_1}$, $p=\frac{-b_1}{a_1}$, $q=\frac{-c_1}{a_1}$, $a_1\neq0$.

It is well known [19,20,59,60] that eq. (74) has the following solutions:

1) If $p=1$, $q=-2$, $r=0$, then $\phi(\xi)=\mathrm{sech}(\xi)$. In this case, eq. (1) has the bright soliton solution

$$q(x,t)=\mathrm{sech}\left(x-\upsilon t\right)\exp\left[i\left(-kx+\omega t+\theta\right)\right], \tag{75}$$

where

$$\upsilon = 2k\left(a - c + 10bk^2\right), \qquad \omega = \frac{c}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = 0.$$

2) If $p = -2$, $q = 2$, $r = 1$, then $\phi(\xi) = \tanh(\xi)$. In this case, eq. (1) has the dark soliton solution

$$q(x,t) = \tanh(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{76}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = c - ak^2 - bk^4, \qquad \varepsilon_2 = -kc.$$

3) If $p = -(M_1^2 + 1)$, $q = 2M_1^2$, $r = 1$, $0 < M_1 < 1$, then $\phi(\xi) = \mathrm{sn}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution

$$q(x,t) = \mathrm{sn}(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{77}$$

where

$$\upsilon = 2k\left(a + \frac{c}{M_1^2} + 10bk^2\right), \qquad \omega = \frac{c(M_1^2 + 1)}{2M_1^2} - ak^2 - bk^4, \qquad \varepsilon_2 = \frac{-kc}{M_1^2}.$$

If $M_1 = 1$, then $\mathrm{sn}(\xi) = \tanh(\xi)$, and we get the same dark soliton solution (76).

4) If $p = -(M_1^2 + 1)$, $q = 2$, $r = M_1^2$, $0 < M_1 < 1$, then $\phi(\xi) = \mathrm{ns}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution

$$q(x,t) = \mathrm{ns}(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{78}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = \frac{c\left(M_1^2 + 1\right)}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = -kcM_1^2.$$

If $M_1 = 0$, then $\mathrm{ns}(\xi) = \csc(\xi)$, and we get the trigonometric solution

$$q(x,t) = \csc(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{79}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = \frac{c}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = 0.$$

If $M_1 = 1$, then $\mathrm{ns}(\xi) = \coth(\xi)$, and we get the singular soliton solution

$$q(x,t) = \coth(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{80}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = c - ak^2 - bk^4, \qquad \varepsilon_2 = -kc.$$

5) If $p = 2M_1^2 - 1$, $q = -2M_1^2$, $r = 1 - M_1^2$, $0 < M_1 < 1$, then $\phi(\xi) = \mathrm{cn}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution

$$q(x,t) = \mathrm{cn}(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{81}$$

where

$$\upsilon = 2k\left(a - \frac{c}{M_1^2} + 10bk^2\right), \qquad \omega = \frac{c(2M_1^2 - 1)}{2M_1^2} - ak^2 - bk^4, \qquad \varepsilon_2 = \frac{kc(1 - M_1^2)}{M_1^2}.$$

If $M_1 = 1$, then $\mathrm{cn}(\xi) = \mathrm{sech}(\xi)$, and we get the same bright soliton solution (75).

6) If $p = 2M_1^2 - 1$, $q = 2(1 - M_1^2)$, $r = -M_1^2$, $0 < M_1 < 1$, then $\phi(\xi) = \mathrm{nc}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution:

$$q(x,t) = \mathrm{nc}(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{82}$$

where

$$\upsilon = 2k\left(a + \frac{c}{1 - M_1^2} + 10bk^2\right), \qquad \omega = \frac{-c(2M_1^2 - 1)}{2(1 - M_1^2)} - ak^2 - bk^4, \qquad \varepsilon_2 = \frac{kcM_1^2}{1 - M_1^2}.$$

If $M_1 = 0$, then $\mathrm{nc}(\xi) = \sec(\xi)$, and eq. (1) has the trigonometric solution

$$q(x,t) = \sec(x - \upsilon t)\exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{83}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = \frac{c}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = 0.$$

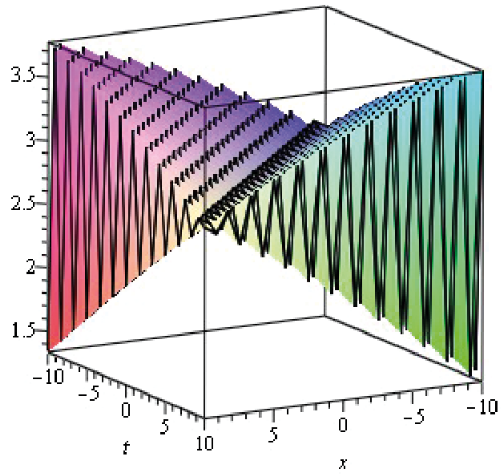

Fig. 1. Plot solution $|q(x,t)|$ of (32) with $b_1 = -1$, $c_1 = 1$, $k_1 = -3/64$.

7) If $p=2-M_1^2$, $q=2$, $r=1-M_1^2$, $0 < M_1 < 1$, then $\phi(\xi)=\mathrm{cs}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution

$$q(x,t) = \mathrm{cs}\,(x - \upsilon t) \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{84}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = \frac{-c(2-M_1^2)}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = -kc\left(1 - M_1^2\right).$$

If $M_1 = 0$, then $\mathrm{cs}(\xi) = \cot(\xi)$, and eq. (1) has the trigonometric solution

$$q(x,t) = \cot\,(x - \upsilon t) \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{85}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = -c - ak^2 - bk^4, \qquad \varepsilon_2 = -kc.$$

If $M_1 = 1$, then $\mathrm{cs}(\xi) = \mathrm{csch}(\xi)$, and eq. (1) has the singular soliton solution

$$q(x,t) = \mathrm{csch}\,(x - \upsilon t) \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{86}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = -\frac{c}{2} - ak^2 - bk^4, \qquad \varepsilon_2 = 0.$$

Note that the solution (86) is equivalent to the solution (58) if we set $k_1 = 0$ in (58).

8) If $p=2-M_1^2$, $q=2(1-M_1^2)$, $r=1$, $0<M_1<1$, then $\phi(\xi)=\mathrm{sc}(\xi)$. In this case, eq. (1) has the Jacobi elliptic solution

$$q(x,t) = \mathrm{sc}\,(x - \upsilon t) \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{87}$$

where

$$\upsilon = 2k\left(a + \frac{c}{1-M_1^2} + 10bk^2\right), \qquad \omega = \frac{-c(2-M_1^2)}{2(1-M_1^2)} - ak^2 - bk^4, \qquad \varepsilon_2 = \frac{-kc}{1-M_1^2}.$$

If $M_1 = 0$, then $\mathrm{sc}(\xi) = \tan(\xi)$, and eq. (1) has the trigonometric solution

$$q(x,t) = \tan\,(x - \upsilon t) \exp\left[i\left(-kx + \omega t + \theta\right)\right], \tag{88}$$

where

$$\upsilon = 2k\left(a + c + 10bk^2\right), \qquad \omega = -c - ak^2 - bk^4, \qquad \varepsilon_2 = -kc.$$

There are other Jacobi elliptic solutions which are omitted here for simplicity. Note that the solutions (75)–(88) are new and not found in [44,50,53] or elsewhere.

3 Some graphical representations of some solutions

In this section, we present graphs of the Jacobi elliptic function solutions of the original equation (1). Let us now examine figs. 1–4 as they illustrate some of our solutions obtained in this article. To this aim, we select some special values of the parameters obtained, for example, in some of the Jacobi elliptic function solutions, (32), (38), (45) and (47), of the nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity (1) with $\upsilon = a_1 = 1$, $M_1 = 1/2$, $-10 < x, t < 10$, respectively. For the reader's convenience the graphical representations of these solutions are shown in figs. 1–4.

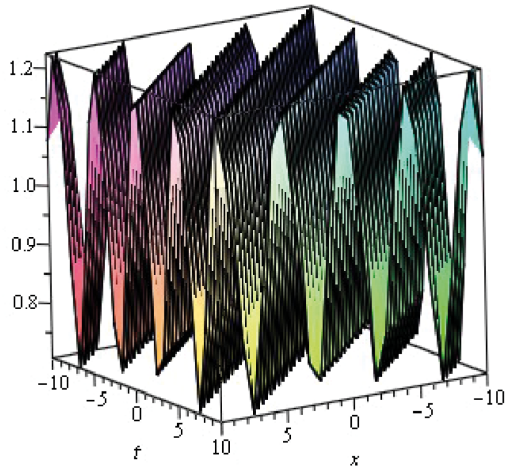

Fig. 2. Plot solution $|q(x,t)|$ of (38) with $b_1 = 1$, $c_1 = -1$, $k_1 = -3/8$.

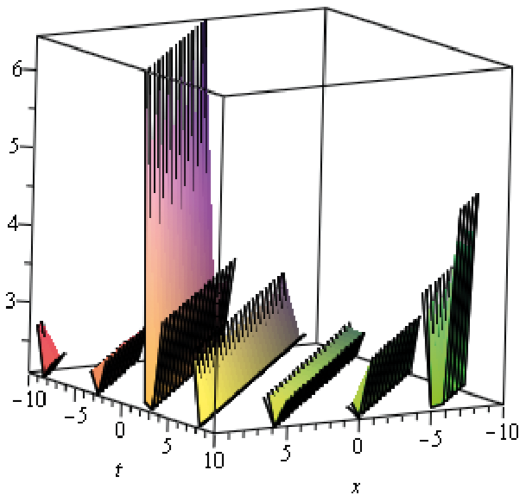

Fig. 3. Plot solution $|q(x,t)|$ of (45) with $b_1 = -1$, $c_1 = 1$, $k_1 = -11/64$.

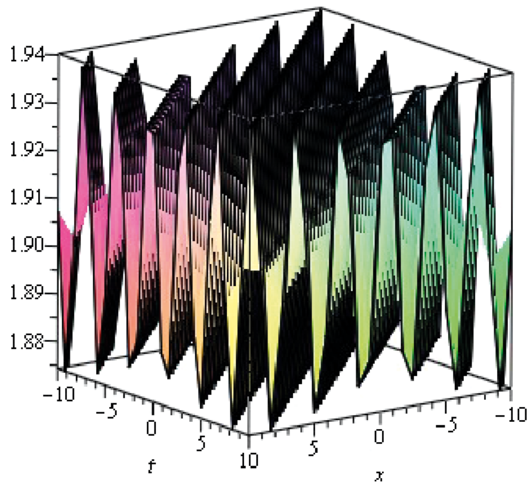

Fig. 4. Plot solution $|q(x,t)|$ of (47) with $b_1 = 1$, $c_1 = -1$, $k_1 = -51/256$.

4 Conclusions

Biswas *et al.* [44] have solved eq. (1) using the soliton ansatz method and found only the 1-soliton solution. Recently, Douvagai *et al.* [53] have solved eq. (1) using two methods which are different from the five methods used in this paper. Here, we used five different mathematical techniques namely, the (G'/G)-expansion method, the improved Sub-ODE method, the extended auxiliary equation method, the new mapping method and the Jacobi elliptic method for constructing many new exact solutions of the nonlinear Schrödinger equation with fourth-order dispersion and dual power law nonlinearity (1). These exact solutions include Jacobi elliptic solutions, hyperbolic function solutions, trigonometric function solutions and rational function solutions. The special case of eq. (1) when $m = 1$ and $k_1 = 0$, has been discussed in [50] using the soliton ansatz method where the 1-soliton solution has been found, while in sect. 2.5, we have found many new solutions of this special case using the Jacobi elliptic equation (74). Comparing our results in this paper with the well-known results of [44, 50–53], we conclude that our results are new and not found elsewhere. Finally, our results in this paper have been checked with the aid of the Maple by putting them back into the original equation (1).

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