Eur. Phys. J. Plus (2017) **132**: 255
DOI 10.1140/epjp/i2017-11515-8

THE EUROPEAN
PHYSICAL JOURNAL PLUS

Regular Article

Integrability, solitons, periodic and travelling waves of a generalized $(3+1)$-dimensional variable-coefficient nonlinear-wave equation in liquid with gas bubbles

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Received: 23 December 2016 / Revised: 11 April 2017
Published online: 9 June 2017 –

Abstract. Under investigation in this paper is a generalized (3+1)-dimensional varible-coefficient nonlinear-wave equation, which has been presented for nonlinear waves in liquid with gas bubbles. The bilinear form, Bäcklund transformation, Lax pair and infinitely-many conservation laws are obtained via the binary Bell polynomials. One-, two- and three-soliton solutions are generated by virtue of the Hirota method. Travelling-wave solutions are derived with the aid of the polynomial expansion method. The one-periodic wave solutions are constructed by the Hirota-Riemann method. Discussions among the soliton, periodic- and travelling-wave solutions are presented: I) the soliton velocities are related to the variable coefficients, while the soliton amplitudes are unaffected; II) the interaction between the solitons is elastic; III) there are three cases of the travelling-wave solutions, *i.e.*, the triangle-type periodical, bell-type and soliton-type travelling-wave solutions, while we notice that bell-type travelling-wave solutions can be converted into one-soliton solutions via taking suitable parameters; IV) the one-periodic waves approach to the solitary waves under some conditions and can be viewed as a superposition of overlapping solitary waves, placed one period apart.

1 Introduction

As encountered in some branches of science and engineering, such as fluid mechanics, condensed matter physics, particle physics, elastic mechanics and plasma physics [1–8], nonlinear evolution equations (NLEEs) have been used to describe some nonlinear physical phenomena and the propagation characteristics of waves [9–12]. As the exact solutions of the NLEEs can provide much physical information and more insight into the physical aspects of the problems [13–16], it is significant to derive the exact solutions of NLEEs, *e.g.*, soliton [17–20], travelling-wave [21] and periodic-wave solutions [22]. Accordingly, Bell-polynomial manipulation [23,24], Hirota bilinear method [25–28], Bäcklund transformation (BT) [29,30], polynomial expansion method [31] and Hirota-Riemann method [32–37] were proposed.

In the field of the liquid with gas bubbles, bubble-liquid mixture equations have been developed to describe the propagation of weakly nonlinear waves [38,39]. As the real nonlinear waves in a liquid with gas bubbles are multi-dimensional, a liquid with gas bubbles in the 3D case should be take consideration. Motived by that, a generalized $(3+1)$-dimensional nonlinear-wave equation [40]

$$[4u_t + 4uu_x + u_{xxx} - 4u_x]_x + 3(u_{yy} + u_{zz}) = 0, \tag{1}$$

has been proposed, where u is the wave-amplitude function of the scaled spatial coordinates x, y, z and temporal coordinate t, the subscripts x, y, z and t represent the partial derivatives.

The variable-coefficient NLEEs can describe real phenomena in the inhomogeneities of media and non-uniformities of boundaries and provide us with more properties than their constant-coefficient counterparts in some realistic physical situations [41–43]. For example, in the context of ocean waves, the temporal variability of the coefficients may be

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caused by the pressure dependence of thermal expansion coefficient of seawater coupled with the large-scale meridional variation of the oceanic temperature-salinity relation, topography of the continental shelf, changing hydrography from deep to shallow water and other dynamical conditions [44–46]. In this paper, we will investigate the generalization of eq. (1) with variable coefficients as

$$[u_t + \alpha(t) u u_x + \beta(t) u_{xxx} + \gamma(t) u_x]_x + \delta(t) u_{yy} + \varrho(t) u_{zz} = 0, \tag{2}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$ are all the real functions of t. Another constant-coefficient special case of eq. (2) has been seen [47], and another variable-coefficient version of eq. (2) has been presented [48].

This article will be organised as follows. In sect. 2, concepts and formulas about the binary Bell polynomial approach will be introduced, and the bilinear form, Bäcklund transformations, Lax pair and infinitely-many conservation laws of eq. (2) will also be obtained. The soliton, travelling-wave and periodic-wave solutions will be derived in sects. 3, 4 and 5, respectively. In sect. 6, the discussions about the soliton, travelling-wave and periodic-wave solutions will be presented analytically and graphically. Conclusions will be given in sect. 7.

2 Bilinear form, Bäcklund transformations, Lax pair and infinitely-many conservation laws of eq. (2)

In this section, Bilinear form, Bäcklund transformations, Lax pair and infinitely-many conservation laws of eq. (2) are obtained via the binary Bell polynomials.

2.1 Multi-dimensional Bell polynomials

Multi-dimensional Bell polynomials are defined as follows [47]:

$$Y_{n_1 x_1, \ldots, n_r x_r}(f) \equiv Y_{n_1, \ldots, n_r}(f_{l_1 x_1}, \ldots, f_{l_r x_r}) = e^{-f} \partial_{x_1}^{n_1} \cdots \partial_{x_r}^{n_r} e^{f}, \tag{3}$$

where $f = f(x_1, x_2, \cdots, x_n)$ be a C^∞ function, $f_{l_1 x_1, \ldots, l_r x_r} = \partial_{x_1}^{l_1} \cdots \partial_{x_r}^{l_r}$ $(0 \le l_i \le n_i,\ i = 1, 2, \ldots, r)$. Taking $n = 1$, the Bell polynomials is given by

$$Y_{nx}(f) \equiv Y_n(f_1, \ldots, f_n) = \sum \frac{n!}{s_1! \cdots s_n! (1!)^{s_1} \cdots (n!)^{s_n}} f_1^{s_1} \cdots f_n^{s_n}, \quad n = \sum_{k=1}^{n} k s_k. \tag{4}$$

The multi-dimensional binary Bell polynomials can be defined as follows:

$$\mathscr{Y}_{n_1 x_1, \ldots, n_r x_r}(v, w) = Y_{n_1, \ldots, n_r}(f)|_{f_{l_1 x_1, \ldots, l_r x_r}} = \begin{cases} v_{l_1 x_1, \ldots, l_r x_r}, & l_1 + \cdots + l_r \text{ is odd}, \\ w_{l_1 x_1, \ldots, l_r x_r}, & l_1 + \cdots + l_r \text{ is even}. \end{cases} \tag{5}$$

For example,

$$\begin{aligned} &\mathscr{Y}_x(v, w) = v_x, \qquad \mathscr{Y}_{2x}(v, w) = v_x^2 + w_{2x}, \\ &\mathscr{Y}_{3x}(v, w) = v_{3x} + 3 v_x w_{2x} + v_x^3, \qquad \mathscr{Y}_{x,t}(v, w) = v_x v_t + w_{xt}, \cdots \end{aligned} \tag{6}$$

The link between the $\mathscr{Y}$-polynomials and the Hirota bilinear operator can be given through the identity

$$\mathscr{Y}_{n_1 x_1, \ldots, n_r x_r}\left(v = \ln \frac{F}{G},\ w = \ln FG\right) = \frac{1}{FG} D_{x_1}^{n_1} \cdots D_{x_r}^{n_r} F \cdot G, \tag{7}$$

where F and G are both the functions of x and t. Taking $F = G$, the identity (7) becomes

$$\frac{1}{F^2} D_{x_1}^{n_1} \cdots D_{x_r}^{n_r} F \cdot F = \mathscr{Y}(0, q = 2 \ln F) = \begin{cases} 0, & n_1 + \cdots + n_r \text{ is odd}, \\ \mathscr{P}_{n_1 x_1, \ldots, n_r x_r}(q), & n_1 + \cdots + n_r \text{ is even}. \end{cases} \tag{8}$$

For example,

$$\begin{aligned} &\mathscr{P}_{2x}(q) = q_{2x}, \qquad \mathscr{P}_{4x}(q) = q_{4x} + 3 q_{2x}^2, \\ &\mathscr{P}_{6x}(q) = q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3, \qquad \mathscr{P}_{x,t}(q) = q_{xt}, \cdots . \end{aligned} \tag{9}$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(v,w)$ can be separated into $\mathscr{P}$-polynomials and $\mathscr{Y}$-polynomials

$$\frac{1}{FG}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G=\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(v,w)|_{v=\ln\frac{F}{G},w=\ln FG}=$$
$$\sum_{n_1+\cdots+n_r=\text{even}}\sum_{l_1=0}^{n_1}\cdots\sum_{l_r=0}^{n_r}\prod_{i=0}^{r}\binom{n_i}{l_i}p_{l_1x_1,\ldots,l_rx_r}(q)Y_{(n_1-l_1)x_1,\ldots,(n_r-l_r)x_r}(v). \tag{10}$$

The key property of the multi-dimensional Bell polynomials,

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(v)|_{v=\ln\psi}=\psi_{n_1x_1,\ldots,n_rx_r}/\psi, \tag{11}$$

implies that the binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(v,w)$ can still be linearized by means of the Hopf-Cole transformation $v=\ln\psi$, that is, $\psi=F/G$.

2.2 Bilinear form of eq. (2)

In order to obtain the linearizable representation of eq. (2), we introduce

$$u=cq_{2x}, \tag{12}$$

where c is a real nonzero constant. Substituting expression (12) into eq. (2) and integrating once with respect to x, we get

$$E(q)=q_{x,t}+\frac{1}{2}c\alpha(t)q_{2x}^2+\beta(t)q_{4x}+\gamma(t)q_{2x}+\delta(t)q_{2y}+\varrho(t)q_{2z}=0. \tag{13}$$

Setting $c=6$, $\alpha(t)=\beta(t)$ and using (9), we can obtain the connection between eq. (13) and the $\mathscr{P}$-polynomials, as

$$E(q)=\mathscr{P}_{x,t}(q)+\beta(t)\mathscr{P}_{4x}(q)+\gamma(t)\mathscr{P}_{2x}(q)+\delta(t)\mathscr{P}_{2y}(q)+\varrho(t)\mathscr{P}_{2z}(q)=0. \tag{14}$$

Finally, through property (8) and setting $q(x,y,z,t)=2\ln f(x,y,z,t)$, eq. (14) gives the bilinear form of eq. (2) as follows:

$$[D_xD_t+\beta(t)D_x^4+\gamma(t)D_x^2+\delta(t)D_y^2+\varrho(t)D_z^2]f\cdot f=0, \tag{15}$$

2.3 Bäcklund transformations of eq. (2)

In order to seek the Bäcklund transformations of eq. (2), we suppose $\tilde{q}=2\ln\tilde{f}$ be another solution of eq. (14), then

$$\begin{aligned}E(\tilde{q})-E(q)=&[\mathscr{P}_{x,t}(\tilde{q})+\beta(t)\mathscr{P}_{4x}(\tilde{q})+\gamma(t)\mathscr{P}_{2x}(\tilde{q})+\delta(t)\mathscr{P}_{2y}(\tilde{q})+\varrho(t)\mathscr{P}_{2z}(\tilde{q})]\\&-[\mathscr{P}_{x,t}(q)+\beta(t)\mathscr{P}_{4x}(q)+\gamma(t)\mathscr{P}_{2x}(q)+\delta(t)\mathscr{P}_{2y}(q)+\varrho(t)\mathscr{P}_{2z}(q)]=0.\end{aligned} \tag{16}$$

Those conditions can be regarded as an ansatz for a bilinear Bäcklund transformation and may produce the required transformation under the appropriate additional constraints. To find such constraints, we introduce some new auxiliary variables

$$W=\ln(f\tilde{f}),\qquad V=\ln\frac{\tilde{f}}{f},\qquad \tilde{q}=2\ln\tilde{f}=W+V,\qquad q=2\ln f=W-V, \tag{17}$$

then rewrite the condition (16) as

$$\begin{aligned}E(\tilde{q})-E(q)=&[\mathscr{P}_{x,t}(W+V)-\mathscr{P}_{x,t}(W-V)]+\beta(t)[\mathscr{P}_{4x}(W+V)-\mathscr{P}_{4x}(W-V)]\\&+\gamma(t)[\mathscr{P}_{2x}(W+V)-\mathscr{P}_{2x}(W-V)]+\delta(t)[\mathscr{P}_{2y}(W+V)-\mathscr{P}_{2y}(W-V)]\\&+\varrho(t)[\mathscr{P}_{2z}(W+V)-\mathscr{P}_{2z}(W-V)]\\=&\,2V_{x,t}+2\beta(t)(V_{4x}+6W_{2x}V_{2x})+2\gamma(t)V_{2x}+2\delta(t)V_{2y}+2\varrho(t)V_{2z}=0.\end{aligned} \tag{18}$$

For the sake of expressing eq. (18) as $\mathscr{Y}$-polynomials, we choose

$$\begin{aligned}\mathscr{Y}_{2x}(V,W)+\varpi\mathscr{Y}_z(V,W)&=\lambda,\\3\varpi^2\beta(t)&=\varrho(t),\end{aligned} \tag{19}$$

in which ϖ is an undetermined constant, and λ is an arbitrary parameter. On account of eq. (19), eq. (18) can be rewritten as

$$\partial_x\{\mathscr{Y}_t(V,W)+\beta(t)\mathscr{Y}_{3x}(V,W)+[3\lambda\beta(t)+\gamma(t)]\mathscr{Y}_x(V,W)-3\varpi\beta(t)\mathscr{Y}_{x,z}(V,W)\}+\partial_y[\delta(t)\mathscr{Y}_y(V,W)]=0. \quad (20)$$

With the aid of formula (10), eq. (7) can be transformed into a bilinear form as follows:

$$\begin{aligned}&(D_x^2+\varpi D_z-\lambda)f\cdot g=0,\\&\{\partial_x[D_t+\beta(t)D_x^3+(3\lambda\beta(t)+\gamma(t))D_x-3\varpi\beta(t)D_xD_z]+\partial_y[\delta(t)D_y]\}f\cdot g=0,\end{aligned} \quad (21)$$

where $\varpi^2=\frac{\varrho(t)}{3\beta(t)}$, then g is another solution of eq. (15). The system (21) is called a Bäcklund transformation of the generalized $(3+1)$-dimensional variable-coefficient nonlinear wave equation (2).

2.4 Lax pair and infinitely-many conservation laws of eq. (2)

With the help of (11) and the Cole-Hopf transformation $V=\ln\psi$, (10) read as follows:

$$\begin{aligned}\mathscr{Y}_x(V,W)&=\frac{\psi_x}{\psi},\\\mathscr{Y}_y(V,W)&=\frac{\psi_y}{\psi},\\\mathscr{Y}_z(V,W)&=\frac{\psi_z}{\psi},\\\mathscr{Y}_t(V,W)&=\frac{\psi_t}{\psi},\\\mathscr{Y}_{2x}(V,W)&=q_{2x}+\frac{\psi_{2x}}{\psi},\\\mathscr{Y}_{3x}(V,W)&=\frac{\psi_{3x}}{\psi}+\frac{3q_{2x}\psi_x}{\psi},\\\mathscr{Y}_{2x,y}(V,W)&=\frac{q_{2x}\psi_y}{\psi}+\frac{2q_{x,y}\psi_x}{\psi}+\frac{\psi_{2x,y}}{\psi}.\end{aligned} \quad (22)$$

Via formula (22), eqs. (19) and (20) can be linearized into the following Lax system:

$$\begin{aligned}&\psi_{2x}+\varpi\psi_z+(u_{2x}-\lambda)\psi=0,\\&\partial_x\{\psi_x+\beta(t)(\psi_{3x}+3u_{2x}\psi_x)+[3\lambda\beta(t)+\gamma(t)]\psi_x-3\varpi\beta(t)(u_{x,t}\psi+\psi_{x,t})\}+\partial_y[\delta(t)\psi_y]=0.\end{aligned} \quad (23)$$

In order to derive infinitely-many conservation laws of eq. (2), we shall decompose (19) and (20) into the x-, y- and z-derivative as follows:

$$\begin{aligned}&V_x^2+W_{2x}+\varpi V_z-\lambda=0,\\&\partial_x\{\beta(t)(V_{3x}+3V_xW_{2x}+V_x^3)+[3\lambda\beta(t)+\gamma(t)]V_x-3\varpi\beta(t)V_xV_z\}\\&+\partial_y[\delta(t)V_y]+\partial_z[-3\varpi\lambda\beta(t)+3\varpi\beta(t)V_x^2+\varrho(t)V_z]+\partial_t(V_x)=0.\end{aligned} \quad (24)$$

By introducing a new potential function,

$$\eta=(\tilde{q_x}-q_x)/2, \quad (25)$$

there follows, from relations (17), that

$$V_x=\eta,\qquad V_y=\partial_x^{-1}(\eta_y),\qquad V_z=\partial_x^{-1}(\eta_z),\qquad W_x=q_x+\eta. \quad (26)$$

Substituting (26) into (24), we decompose (18) into a Riccati-type equation

$$q_{2x}+\eta_x+\eta^2+\varpi\partial_x^{-1}\eta_z-\lambda=0, \quad (27)$$

which is a new potential equation with respect to q and a divergence-type equation

$$\begin{aligned}&\partial_x[\beta(t)\eta_{2x}+6\lambda\beta(t)\eta-2\beta(t)\eta^3-6\varpi\beta(t)\eta\partial_x^{-1}\eta_z+\gamma(t)\eta]+\partial_y[\delta(t)\partial_x^{-1}\eta_y]\\&+\partial_z[-3\varpi\lambda\beta(t)+3\varpi\beta(t)\eta^2+\varrho(t)\partial_x^{-1}\eta_z]+\eta_t=0,\end{aligned} \quad (28)$$

Letting $\lambda = \varepsilon^2$ and making use of transformation $\eta = \tilde{\eta} + \varepsilon$, eqs. (27) and (28) can be rewritten as

$$q_{2x} + \tilde{\eta}_x + \tilde{\eta}^2 + 2\tilde{\eta}\varepsilon + \varpi\partial_x^{-1}\tilde{\eta}_z = 0, \tag{29}$$

$$\begin{aligned}&\partial_x[\beta(t)\tilde{\eta}_{2x} + 4\beta(t)\varepsilon^3 - 2\beta(t)\tilde{\eta}^3 - 6\beta(t)\varepsilon\tilde{\eta}^2 - 6\varpi\beta(t)\tilde{\eta}\partial_x^{-1}\tilde{\eta}_z - 6\varpi\beta(t)\varepsilon\partial_x^{-1}\tilde{\eta}_z \\ &+ \gamma(t)(\tilde{\eta}+\varepsilon)] + \partial_y[\delta(t)\partial_x^{-1}\tilde{\eta}_y] + \partial_z[3\varpi\beta(t)\tilde{\eta}^2 + 6\varpi\beta(t)\varepsilon\tilde{\eta} + \varrho(t)\partial_x^{-1}\tilde{\eta}_z] + \tilde{\eta}_t = 0.\end{aligned} \tag{30}$$

By inserting the expansion,

$$\tilde{\eta} = \sum_{n=1}^{\infty} \mathscr{I}_n(q, q_x, \ldots)\varepsilon^{-n}, \tag{31}$$

into eq. (29) and equating the coefficients for the same powers of ε, we can get

$$q_{2x} + \sum_{n=1}^{\infty} \mathscr{I}_{n,x}\varepsilon^{-n} + \varpi\partial_x^{-1}\sum_{n=1}^{\infty} \mathscr{I}_{n,z}\varepsilon^{-n} + 2\mathscr{I}_1 + 2\sum_{n=1}^{\infty} \mathscr{I}_{n+1}\varepsilon^{-n} + \sum_{n=1}^{\infty}\left(\sum_{n=1}^{n-1} \mathscr{I}_i\mathscr{I}_{n-i}\varepsilon^{-n}\right) = 0, \tag{32}$$

then we can explicitly derive the recursion relationship of the conserved densities $\mathscr{I}$ as follows:

$$\begin{aligned}\mathscr{I}_1 &= -\frac{1}{2}q_{2x} = -\frac{1}{12}u,\\ \mathscr{I}_2 &= \frac{1}{4}q_{3x} + \frac{1}{4}\varpi q_{x,z} = \frac{1}{24}(u_x + \varpi\partial_x^{-1}u_z),\\ &\vdots\\ \mathscr{I}_{n+1} &= -\frac{1}{2}\left(\mathscr{I}_{n,x} + \varpi\partial_x^{-1}\mathscr{I}_{n,z} + \sum_{i=1}^{n-1}\mathscr{I}_i\mathscr{I}_{n-i}\right), \quad (n = 2, 3, \cdots).\end{aligned} \tag{33}$$

In addition, substituting expansion (31) into (30), we have

$$\begin{aligned}&\partial_x\left[\beta(t)\sum_{n=1}^{\infty}\mathscr{I}_{n,2x}\varepsilon^{-n} + 4\beta(t)\varepsilon^3 - 2\beta(t)\sum_{n=1}^{\infty}\left(\sum_{k_1+k_2+k_3=n}\mathscr{I}_{k_1}\mathscr{I}_{k_2}\mathscr{I}_{k_3}\varepsilon^{-n}\right)\right.\\ &- 6\beta(t)\varepsilon\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n-1}\mathscr{I}_i\mathscr{I}_{n-i}\varepsilon^{-n}\right) - 6\varpi\beta(t)\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)\partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{I}_{n,z}\varepsilon^{-n}\right)\\ &\left.- 6\varpi\beta(t)\varepsilon\partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{I}_{n,z}\varepsilon^{-n}\right) + \gamma(t)\left(\varepsilon + \sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)\right] + \partial_y\left[\delta(t)\partial_x^{-1}\left(\sum_{n=1}^{\infty}\mathscr{I}_{n,y}\varepsilon^{-n}\right)\right]\\ &+ \partial_z\left\{3\varpi\beta(t)\left[\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n-1}\mathscr{I}_i\mathscr{I}_{n-i}\varepsilon^{-n}\right) + 2\left(\mathscr{I}_1 + \sum_{n=1}^{\infty}\mathscr{I}_{n+1}\varepsilon^{-n}\right)\right] + \varrho(t)\left(\partial_x^{-1}\sum_{n=1}^{\infty}\mathscr{I}_{n,z}\varepsilon^{-n}\right)\right\}\\ &+ \sum_{n=1}^{\infty}\mathscr{I}_{n,t}\varepsilon^{-n} = 0.\end{aligned} \tag{34}$$

While the first fluxes $\mathscr{X}_n$'s are given by

$$\begin{aligned}\mathscr{X}_1 &= \beta(t)\mathscr{I}_{1,2x} + \gamma(t)\mathscr{I}_1 - 6\beta(t)\mathscr{I}_1^2 - 6\beta(t)\varpi\partial_x^{-1}\mathscr{I}_{2,z},\\ \mathscr{X}_2 &= \beta(t)\mathscr{I}_{2,2x} + \gamma(t)\mathscr{I}_2 - 12\beta(t)\mathscr{I}_1\mathscr{I}_2 - 6\beta(t)\varpi\partial_x^{-1}\mathscr{I}_{3,z} - 6\beta(t)\varpi\mathscr{I}_1\partial_x^{-1}\mathscr{I}_{1,z},\\ &\vdots\end{aligned} \tag{35}$$

$$\begin{aligned}\mathscr{X}_n &= \beta(t)\left(\mathscr{I}_{n,2x} - 6\sum_{i=1}^{n}\mathscr{I}_i\mathscr{I}_{n-i+1} - 2\sum_{k_1+k_2+k_3=n}\mathscr{I}_{k_1}\mathscr{I}_{k_2}\mathscr{I}_{k_3}\right)\\ &\quad - 6\beta(t)\varpi\left(\partial_x^{-1}\mathscr{I}_{n+1,z} + \sum_{k=1}^{n}\mathscr{I}_k\partial_x^{-1}\mathscr{I}_{n-k,y}\right) + \gamma(t)\mathscr{I}_n, \quad (n = 3, 4, \cdots),\end{aligned} \tag{36}$$

the second fluxes $\mathscr{A}_n$'s are given by

$$\begin{aligned}
\mathscr{A}_1 &= \delta(t)\partial_x^{-1}\mathscr{I}_{1,y} = -\frac{1}{12}\delta(t)\partial_x^{-1}u_y,\\
\mathscr{A}_2 &= \delta(t)\partial_x^{-1}\mathscr{I}_{2,y} = \frac{1}{24}\delta(t)(u_y + \varpi\partial_{2x}^{-1}u_{y,z}),\\
&\vdots\\
\mathscr{A}_{n+1} &= \delta(t)\partial_x^{-1}\mathscr{I}_{n,y}, \quad (n = 2, 3, \cdots),
\end{aligned} \tag{37}$$

where ∂_{2x}^{-1} means integrating with respect to x twice, and the third fluxes $\mathscr{Z}_n$'s are given by

$$\begin{aligned}
\mathscr{Z}_1 &= 6\beta(t)\varpi\mathscr{I}_2 + \varrho(t)\partial_x^{-1}\mathscr{I}_{1,z},\\
\mathscr{Z}_2 &= 3\beta(t)\varpi\mathscr{I}_1^2 + 6\beta(t)\varpi\mathscr{I}_3 + \varrho(t)\partial_x^{-1}\mathscr{I}_{2,z},\\
&\vdots\\
\mathscr{Z}_{n+1} &= 3\beta(t)\varpi\sum_{i=1}^{n-1}\mathscr{I}_i\mathscr{I}_{n-i} + 6\beta(t)\varpi\mathscr{I}_{n+1} + \varrho(t)\partial_x^{-1}\mathscr{I}_{n,z} \quad (n = 2, 3, \cdots).
\end{aligned} \tag{38}$$

The conversed densities $\mathscr{I}$ and three fluxes $\mathscr{X}, \mathscr{A}, \mathscr{Z}$ provide us the infinitely-many conservation laws

$$\mathscr{I}_{n,t} + \mathscr{X}_{n,x} + \mathscr{A}_{n,y} + \mathscr{Z}_{n,z} = 0, \quad (n = 1, 2, \ldots). \tag{39}$$

3 Soliton solutions of eq. (2)

Soliton solutions of eq. (2) can be derived by expanding $f(x, y, z, t)$ as

$$f(x, y, z, t) = 1 + \varepsilon f_1(x, y, z, t) + \varepsilon^2 f_2(x, y, z, t) + \varepsilon^3 f_3(x, y, z, t) + \ldots, \tag{40}$$

where $f_v(x, y, z, t)$'s $(v = 1, 2, 3, \ldots)$ are the real functions of x, y, z and t.

3.1 One-soliton solutions

Truncating expressions (40) as

$$f(x, y, z, t) = 1 + \varepsilon f_1(x, y, z, t), \tag{41}$$

setting $\varepsilon = 1$ and substituting expression (41) into eq. (15), we obtain the one-soliton solutions for eq. (2) as

$$u = 12(\ln f)_{2x} = \frac{12k_1^2 e^{\theta_1}}{(1 + e^{\theta_1})^2}, \tag{42}$$

with

$$\begin{aligned}
\omega_1(t) &= \int \frac{-k_1^4\beta(t) - k_1^2\gamma(t) - l_1^2\delta(t) - p_1^2\varrho(t)}{k_1}\,\mathrm{d}t,\\
\theta_1 &= k_1x + l_1y + p_1z + \omega_1(t) + \eta_1,
\end{aligned} \tag{43}$$

where k_1, l_1, p_1 and η_1 are all complex constants.

3.2 Two-soliton solutions

Truncating expressions (40) as

$$f(x, y, z, t) = 1 + \varepsilon f_1(x, y, z, t) + \varepsilon^2 f_2(x, y, z, t), \tag{44}$$

substituting expression (44) into eq. (15) and setting $\varepsilon = 1$, we can get the two-soliton solutions for eq. (2) as

$$u = \frac{12[k_1^2 e^{\theta_1}(1+e^{\theta_2})(1+A_{12}e^{\theta_2}) + 2k_1k_2(A_{12}-1)e^{\theta_1+\theta_2} + k_2^2 e^{\theta_2}(1+e^{\theta_1})(1+A_{12}e^{\theta_1})]}{(1+e^{\theta_1}+e^{\theta_2}+A_{12}e^{\theta_1+\theta_2})^2}, \tag{45}$$

where

$$\begin{aligned}
\omega_i(t) &= \int \frac{-k_i^4\beta(t) + k_i^2(-\gamma(t)) - l_i^2\delta(t) - p_i^2\varrho(t)}{k_i}\,\mathrm{d}t \quad (i=1,2),\\
\theta_i &= k_i x + l_i y + p_i z + \omega_i(t) + \eta_i \quad (i=1,2),\\
A_{12} &= 1 + 12k_1^3k_2^3\beta(t)/[-3k_2^2k_1^4\beta(t) - 6k_2^3k_1^3\beta(t) - 3k_2^4k_1^2\beta(t) + k_1^2l_2^2\delta(t)\\
&\quad - 2k_2k_1l_1l_2\delta(t) + k_2^2l_1^2\delta(t) + \varrho(t)\,(k_1p_2 - k_2p_1)^2],
\end{aligned} \tag{46}$$

with k_i, l_i, p_i and η_i are all the complex constants.

3.3 Three-soliton solutions

Truncating expressions (40) as

$$f(x,y,z,t) = 1 + \varepsilon f_1(x,y,z,t) + \varepsilon^2 f_2(x,y,z,t) + \varepsilon^3 f_3(x,y,z,t), \tag{47}$$

substituting expression (47) into eq. (15) and setting $\varepsilon = 1$, we can derive the three-soliton solutions for eq. (2) as

$$\begin{aligned}
u &= 12((A_{12}e^{\theta_1+\theta_2} + A_{123}e^{\theta_1+\theta_2+\theta_3} + A_{13}e^{\theta_1+\theta_3} + A_{23}e^{\theta_2+\theta_3} + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + 1)\\
&\quad \times (A_{12}e^{\theta_1+\theta_2}(k_1+k_2)^2 + A_{123}e^{\theta_1+\theta_2+\theta_3}(k_1+k_2+k_3)^2 + A_{13}e^{\theta_1+\theta_3}(k_1+k_3)^2\\
&\quad + A_{23}e^{\theta_2+\theta_3}(k_2+k_3)^2 + e^{\theta_1}k_1^2 + e^{\theta_2}k_2^2 + e^{\theta_3}k_3^2) - (A_{12}e^{\theta_1+\theta_2}(k_1+k_2)\\
&\quad + A_{123}e^{\theta_1+\theta_2+\theta_3}(k_1+k_2+k_3) + A_{13}e^{\theta_1+\theta_3}(k_1+k_3) + A_{23}e^{\theta_2+\theta_3}(k_2+k_3) + e^{\theta_1}k_1 + e^{\theta_2}k_2\\
&\quad + e^{\theta_3}k_3)^2)/(A_{12}e^{\theta_1+\theta_2} + A_{123}e^{\theta_1+\theta_2+\theta_3} + A_{13}e^{\theta_1+\theta_3} + A_{23}e^{\theta_2+\theta_3} + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + 1)^2,
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\omega_i(t) &= \int \frac{-k_i^4\beta(t) + k_i^2(-\gamma(t)) - l_i^2\delta(t) - \vartheta_i^2\varrho(t)}{k_i}\,\mathrm{d}t \quad (i=1,2,3),\\
\theta_i &= k_i x + l_i y + p_i z + \omega_i(t) + \eta_i \quad (i=1,2,3),\\
A_{ij} &= 1 + 12k_i^3k_j^3\beta(t)/[-3k_j^2k_i^4\beta(t) - 6k_j^3k_i^3\beta(t) - 3k_j^4k_i^2\beta(t) + k_i^2l_j^2\delta(t)\\
&\quad - 2k_jk_il_il_j\delta(t) + k_j^2l_i^2\delta(t) + \varrho(t)\,(k_ip_j - k_jp_i)^2], \quad (i=1,2;\ j=2,3.)
\end{aligned} \tag{49}$$

$$\begin{aligned}
A_{123} &= [3k_2k_3((-A_{12}+A_{13}+A_{23})k_2 + (A_{12}-A_{13}+A_{23})k_3)\beta(t)k_1^4\\
&\quad - 6k_2k_3((A_{12}+A_{13}+A_{23})k_2^2 - 2(A_{12}+A_{13}-A_{23})k_3k_2\\
&\quad + (A_{12}+A_{13}+A_{23})k_3^2)\beta(t)k_1^3 - (3(A_{12}-A_{13}-A_{23})k_3\beta(t)k_2^4\\
&\quad - 12(A_{12}-A_{13}+A_{23})k_3^2\beta(t)k_2^3 + 12(A_{12}-A_{13}-A_{23})k_3^3\beta(t)k_2^2\\
&\quad - (A_{12}-A_{13}+A_{23})(3\beta(t)k_3^4 - l_3^2\delta(t) - p_3^2\varrho(t))k_2\\
&\quad - (A_{12}-A_{13}-A_{23})k_3(\varrho(t)p_2^2 + l_2^2\delta(t)))k_1^2 + (3(A_{12}+A_{13}-A_{23})k_3^2\beta(t)k_2^4\\
&\quad - 6(A_{12}+A_{12}+A_{23})k_3^3\beta(t)k_2^3 + (A_{12}+A_{13}-A_{23})(3\beta(t)k_3^4 - l_3^2\delta(t) - p_3^2\varrho(t))k_2^2\\
&\quad + 2k_3(((-A_{12}+A_{13}+A_{23})l_1l_2 + ((A_{12}-A_{13}+A_{23})l_1 + (A_{12}+A_{13}-A_{23})l_2)l_3)\delta(t)\\
&\quad + ((-A_{12}+A_{13}+A_{23})p_1p_2 + ((A_{12}-A_{13}+A_{23})p_1 + (A_{12}+A_{13}-A_{23})p_2)p_3)\varrho(t))k_2\\
&\quad - (A_{12}+A_{13}-A_{23})k_3^2(\varrho(t)p_2^2 + l_2^2\delta(t)))k_1 + k_2k_3((A_{12}-A_{13}-A_{23})k_2\\
&\quad - (A_{12}-A_{13}+A_{23})k_3)(\varrho(t)p_1^2 + l_1^2\delta(t))]/[3k_2k_3(k_2+k_3)\beta(t)k_1^4\\
&\quad + 6k_2k_3(k_2+k_3)^2\beta(t)k_1^3 + (3k_3\beta(t)k_2^4 + 12k_3^2\beta(t)k_2^3 + 12k_3^3\beta(t)k_2^2\\
&\quad + (3\beta(t)k_3^4 - l_3^2\delta(t) - p_3^2\varrho(t))k_2 - k_3(\varrho(t)p_2^2 + l_2^2\delta(t)))k_1^2 + (3k_3^2\beta(t)k_2^4 + 6k_3^3\beta(t)k_2^3\\
&\quad + (3\beta(t)k_3^4 - l_3^2\delta(t) - p_3^2\varrho(t))k_2^2 + 2k_3((l_1l_2 + (l_1+l_2)l_3)\delta(t)\\
&\quad + (p_1p_2 + (p_1+p_2)p_3)\varrho(t))k_2 - k_3^2(\varrho(t)_2^2 + l_2^2\delta(t)))k_1 - k_2k_3(k_2+k_3)(\varrho(t)p_1^2 + l_1^2\delta(t))],
\end{aligned} \tag{50}$$

with k_i, l_i, p_i and η_i as complex constants.

4 Travelling-wave solutions of eq. (2)

In this section, with the aid of the polynomial expansion method, we would like to construct the travelling wave solutions of eq. (2).

4.1 Polynomial-expansion method

For a given $(3+1)$-dimensional NLEE

$$F(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}, u_{xy}, u_{xz}, u_{yz}, u_{xt}, u_{yt}, u_{zt}, \cdots) = 0, \tag{51}$$

we seek its travelling-wave solutions in the following form [49]:

$$u(x,y,z,t) = U(\xi), \qquad \xi = kx + ly + pz + \omega(t), \tag{52}$$

where $\omega(t)$ is a function of t, and k, l, p are all free constants.

Substituting eq. (52) into eq. (51), eq. (51) can be converted into a nonlinear ordinary differential equation (ODE)

$$F\left(U, \frac{\mathrm{d}U}{\mathrm{d}\xi}, \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, \frac{\mathrm{d}^3U}{\mathrm{d}\xi^3}, \cdots\right) = 0. \tag{53}$$

We assume the solutions of eq. (53) have the following form:

$$U(\xi) = a_0(t) + \sum_{i=1}^{n} a_i(t) Z(\xi)^i + \sum_{i=1}^{n} b_i(t) Z(\xi)^{-i}, \tag{54}$$

with

$$\frac{\mathrm{d}Z}{\mathrm{d}\xi} = Z^2 + AZ + B, \tag{55}$$

where $a_i(t)$, $b_i(t)$ $(i = 1, 2, \cdots, n)$ are functions of t, and A, B are free constants. The constant parameter n is a positive integer determined by the equilibrium of the nonlinear terms and the highest-order linear term of eq. (53).

Accordingly, eq. (55) have several types of solutions as

1) when $A = 0$, $B = 0$,

$$Z(\xi) = -\frac{1}{\xi}\,; \tag{56}$$

2) when $A \neq 0$, $B = 0$,

$$Z(\xi) = \frac{A}{C_0 e^{-A\xi} - 1}\,, \tag{57}$$

where C_0 is an integrating constant;

3) when $A = 0$, $B > 0$,

$$Z(\xi) = \begin{cases} \sqrt{B}\tan(\sqrt{B}\xi), \\ -\sqrt{B}\cot(\sqrt{B}\xi); \end{cases} \tag{58}$$

4) when $A = 0$, $B < 0$,

$$Z(\xi) = \begin{cases} -\sqrt{-B}\tanh(\sqrt{-B}\xi), \\ \sqrt{-B}\coth(\sqrt{-B}\xi); \end{cases} \tag{59}$$

5) when $A \neq 0$, $B \neq 0$,

$$Z(\xi) = \frac{\theta_1 - C_1\theta_2 e^{(\theta_1-\theta_2)\xi}}{1 - C_1 e^{(\theta_1-\theta_2)\xi}}\,, \tag{60}$$

with

$$\theta_1 = \frac{-A + \sqrt{A^2 - 4B}}{2}\,, \qquad \theta_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}\,, \tag{61}$$

where C_1 is an integrating constant.

4.2 Travelling-wave solutions

According to the travelling-wave transformation (52), eq. (2) can be rewrite as the following ODE form:

$$k^4\beta(t)U^{(4)} + [k\omega'(t) + k^2\gamma(t) + l^2\delta(t) + p^2\varrho(t)]U'' + k^2\beta(t)U'^2 + k^2\beta(t)UU'' = 0. \tag{62}$$

Based on the definition n of eq. (54), we derive that $n = 2$ in eq. (62). Then eq. (54) can be reduced to

$$U(\xi) = a_0(t) + a_1(t)Z(\xi) + a_2(t)Z(\xi)^2 + \frac{b_1(t)}{Z(\xi)} + \frac{b_2(t)}{Z(\xi)^2}\,, \tag{63}$$

substituting the expression (63) into eq. (62) and using eq. (55), algebraic equation of $Z(\xi)$ can be derived. Then equating the coefficients of the $Z(\xi)$ power series at all levels to zero, we get the following results:

Case 1)

$$\begin{aligned} &A \neq 0, \qquad B = b_1(t) = b_2(t) = 0, \qquad a_0(t) = -k^2A^2, \qquad a_1(t) = -12k^2A,\\ &a_2(t) = -12k^2, \qquad \omega(\mathrm{t}) = \int \frac{-k^2\gamma(t) - l^2\delta(t) - p^2\varrho(t)}{k}\,\mathrm{d}t. \end{aligned} \tag{64}$$

Case 2)

$$\begin{aligned} &B \neq 0, \qquad A = a_1(t) = b_1(t) = 0, \qquad a_0(t) = -8k^2B, \qquad a_2(t) = -12k^2,\\ &b_2(t) = -12k^2B^2, \qquad \omega(\mathrm{t}) = \int \frac{-k^2\gamma(t) - l^2\delta(t) - p^2\varrho(t)}{k}\,\mathrm{d}t. \end{aligned} \tag{65}$$

Combing Case 1), Case 2) with eq. (56)–(60), we obtain the following three types of travelling-wave solutions of eq. (2):

I) when $A = 0$, $B > 0$,

$$u_1 = -8k^2B - 12k^2B\cot^2(\sqrt{B}\xi) - 12k^2B\tan^2(\sqrt{B}\xi); \tag{66}$$

II) when $A \neq 0$, $B = 0$,

$$u_2 = -k^2A^2 - \frac{12k^2A^2(1+\Theta)}{\Theta^2}\,, \tag{67}$$

with

$$\Theta = C_0e^{-A\xi} - 1; \tag{68}$$

III) when $A = 0$, $B < 0$,

$$u_3 = -8k^2B + 12k^2B\coth^2(\sqrt{-B}\xi) + 12k^2B\tanh^2(\sqrt{-B}\xi); \tag{69}$$

where $\xi = kx + ly + pz + \int \frac{-k^2\gamma(t)-l^2\delta(t)-p^2\varrho(t)}{k}\,\mathrm{d}t$, k, l, p and C_0 are all free constants.

5 Periodic-wave solutions of eq. (2)

In this section, with the help of the Hirota-Riemann method [47], we will construct the one-periodic wave solutions of eq. (2).

5.1 Hirota-Riemann method for NLEEs

The multi-dimensional Riemann theta function is defined as follows:

$$\vartheta(\xi,\tau) = \sum_{n\in Z^N} e^{\pi i\langle n\tau,n\rangle + 2\pi i\langle \xi,n\rangle}, \tag{70}$$

where the integer value vector $n = (n_1,\ldots,n_N)^T \in Z^N$ and complex phase variables $\xi = (\xi_1,\ldots,\xi_N)^T \in C^N$. Furthermore, for two vectors $f = (f_1,\ldots,f_N)^T$ and $g = (g_1,\ldots,g_N)^T$, their inner product is defined by

$$\langle f,g\rangle = f_1g_1 + f_2g_2 + \cdots + f_Ng_N. \tag{71}$$

The equality of $-i\tau = -i(\tau_{ij})$ is a positive definite and real-valued symmetric $N \times N$ matrix, which is called the period matrix of the theta function. We consider the entries τ_{ij} of the period matrix as free parameters of the theta function (70). Equation (70) can converge to a real-valued function for an arbitrary vector $\xi \in C^N$ under the bilinear conditions. The periodic-wave solutions can be constructed by an algebro-geometric method [47]. The matrix τ is usually derived by a compact Riemann surface $\mathscr{R}$ of genus $n \in N$. In this paper, the real-valued theta function (70) can be obtained through taking the matrix τ to be a pure imaginary matrix.

5.2 One-periodic wave solutions

To construct the one-periodic wave solutions of eq. (2), a more generalized form of the bilinear equation should be introduced. Suppose that eq. (2) satisfies the asymptotic condition $u \to u_0$ when $|\xi| \to 0$, we can find that the periodic-wave solutions of eq. (2) follow the constraint

$$u = u_0(t) + 12\partial_x^2 \ln \vartheta(\xi, \tau), \tag{72}$$

where $u_0(t)$ is a function with respect to t of eq. (2), and phase variable ξ is of the form $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T, \xi_i = K_i x + L_i y + P_i z + \mu_i(t) + \epsilon_i$, $(i = 1, 2, \ldots, N)$.

Substituting the expression (72) into eq. (2), a new bilinear form can be obtained as follows:

$$[D_x D_t + \beta(t) D_x^4 + u_0(t)\beta(t) D_x^2 + \gamma(t) D_x^2 + \delta(t) D_y^2 + \varrho(t) D_z^2 + c(t)]\vartheta(\xi, \tau) \cdot \vartheta(\xi, \tau) = 0, \tag{73}$$

where $c(t)$ is a function with respect to t.

Setting (70) with $N = 1$, then (70) is reduced to the following form

$$\vartheta(\xi, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2 \tau + 2\pi i n \xi}, \tag{74}$$

where $\xi = Kx + Ly + Pz + \mu(t) + \epsilon$, τ is a complex constant and meet the condition $\mathrm{Im}(\tau) > 0$. Substituting (74) into (73), we have

$$\begin{aligned}
\mathscr{B}(D_x, D_y, D_z, D_t)\vartheta(\xi, \tau) \cdot \vartheta(\xi, \tau) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathscr{B}(D_x, D_y, D_z, D_t) e^{\pi i m^2 \tau + 2\pi i m \xi} \cdot e^{\pi i n^2 \tau + 2\pi i n \xi} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathscr{B}(2i\pi(n-m)K, 2i\pi(n-m)L, 2i\pi(n-m)P, 2i\pi(n-m)\mu_t) \\
&\quad e^{\pi i(m^2+n^2)\tau + 2\pi i(m+n)\xi} \overset{m'=m+n}{=} \sum_{m'=-\infty}^{+\infty} \tilde{\mathscr{B}}(m') e^{2\pi i m' \xi},
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
\tilde{\mathscr{B}}(m') &= \sum_{n=-\infty}^{+\infty} \mathscr{B}(2i\pi(2n-m')K, 2i\pi(2n-m')L, 2i\pi(2n-m')P, 2i\pi(2n-m')\mu_t) e^{\pi i[n^2+(n-m')^2]\tau} \\
&\overset{n=n'+1}{=} \sum_{n'=-\infty}^{+\infty} \mathscr{B}(2i\pi[2n'-(m'-2)]K, 2i\pi[2n'-(m'-2)]L, 2i\pi[2n'-(m'-2)]P, \\
&\quad 2i\pi[2n'-(m'-2)]\mu_t) e^{\pi i[n'^2+(n'-(m'-2))^2]\tau} \cdot e^{2\pi i(m'-1)\tau} = \\
&\quad \tilde{\mathscr{B}}(m'-2) e^{2\pi i(m'-1)\tau} = \cdots = \begin{cases} \tilde{\mathscr{B}}(0) e^{\pi i m' \tau}, & m' \text{ is even}, \\ \tilde{\mathscr{B}}(1) e^{\pi i(m'+1)\tau}, & m' \text{ is odd}, \end{cases} \quad m', n' \in Z,
\end{aligned} \tag{76}$$

from which we find that $\tilde{\mathscr{B}}(m')$ for $m' \in Z$ are completely dominated by $\tilde{\mathscr{B}}(0)$ and $\tilde{\mathscr{B}}(1)$. If $\tilde{\mathscr{B}}(0) = \tilde{\mathscr{B}}(1) = 0$, then it follows that $\tilde{\mathscr{B}}(m') = 0$, and thus (72) is an exact solution of the bilinear form (73), *i.e.*, $\mathscr{B}(D_x, D_y, D_z, D_t)\vartheta(\xi, \tau) \cdot \vartheta(\xi, \tau) = 0$.

Noticing the specific form of the bilinear form (73), the one-periodic wave solutions can be derived by

$$\begin{aligned}\tilde{\mathscr{B}}(0) &= \sum_{n=-\infty}^{+\infty} \mathscr{B}(4n\pi iK, 4n\pi iL, 4n\pi iP, 4n\pi i\mu_t)e^{2n^2\pi i\tau}\\ &= \sum_{n=-\infty}^{+\infty} [-16\pi^2 n^2 K\mu_t + 256\beta(t)\pi^4 n^4 K^4 + 16u_0(t)\beta(t)\pi^2 n^2 K^2 - 16\gamma(t)\pi^2 n^2 K^2\\ &\quad - 16\delta(t)\pi^2 n^2 L^2 - 16\varrho(t)\pi^2 n^2 P^2 + c(t)]e^{2i\pi n^2\tau} = 0,\end{aligned} \tag{77}$$

$$\begin{aligned}\tilde{\mathscr{B}}(1) &= \sum_{n=-\infty}^{+\infty} \mathscr{B}(2i\pi(2n-1)K, 2i\pi(2n-1)L, 2i\pi(2n-1)P, 2i\pi(2n-1)\mu_t)e^{\pi i(2n^2-2n+1)\tau}\\ &= \sum_{n=-\infty}^{+\infty} [-4\pi^2(2n-1)^2 K\mu_t + 16\beta(t)\pi^4(2n-1)^4 K^4 + 4u_0(t)\beta(t)\pi^2(2n-1)^2 K^2\\ &\quad - 4\gamma(t)\pi^2(2n-1)^2 K^2 - 4\delta(t)\pi^2(2n-1)^2 L^2 - 16\varrho(t)\pi^2(2n-1)^2 P^2 + c(t)]\\ &\quad \times e^{\pi i(2n^2-2n+1)\tau} = 0.\end{aligned} \tag{78}$$

By introducing the following notations:

$$\begin{aligned}&\Lambda = e^{\pi i\tau},\\ &a_{11}(t) = -\sum_{n=-\infty}^{+\infty} 16\pi^2 n^2 K\Lambda^{2n^2}, \quad a_{12}(t) = \sum_{n=-\infty}^{+\infty} \Lambda^{2n^2},\\ &a_{21}(t) = -\sum_{n=-\infty}^{+\infty} 4\pi^2(2n-1)^2 K\Lambda^{2n^2-2n+1}, \qquad a_{22}(t) = \sum_{n=-\infty}^{+\infty} \Lambda^{2n^2-2n+1},\\ &b_I(t) = \sum_{n=-\infty}^{+\infty} [-256\beta(t)\pi^4 n^4 K^4 - 16u_0(t)\beta(t)\pi^2 n^2 K^2 + 16\gamma(t)\pi^2 n^2 K^2\\ &\qquad + 16\delta(t)\pi^2 n^2 L^2 + 16\varrho(t)\pi^2 n^2 P^2]\Lambda^{2n^2},\\ &b_{II}(t) = \sum_{n=-\infty}^{+\infty} [-16\beta(t)\pi^4(2n-1)^4 K^4 - 4u_0(t)\beta(t)\pi^2(2n-1)^2 K^2 + 4\gamma(t)\pi^2(2n-1)^2 K^2\\ &\qquad + 4\delta(t)\pi^2(2n-1)^2 L^2 + 4\varrho(t)\pi^2(2n-1)^2 P^2]\Lambda^{2n^2-2n+1},\end{aligned} \tag{79}$$

eqs. (77) and (78) can be simplified to a linear system about the μ_t and $c(t)$ as

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} \mu_t \\ c(t) \end{pmatrix} = \begin{pmatrix} b_I(t) \\ b_{II}(t) \end{pmatrix}. \tag{80}$$

Solving the eq. (80), the one-periodic wave solutions of eq. (2) can be derived as

$$u = u_0(t) + 12\partial_x^2 \ln\vartheta(\xi,\tau), \tag{81}$$

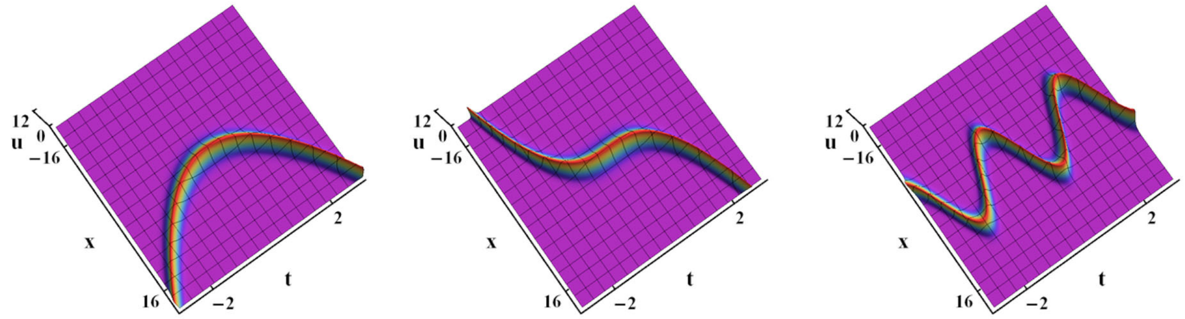
where $u_0(t)$ is a function with respect to t and the parameters K, L, P, τ and ϵ are arbitrary.

6 Discussion

In this section, we will discuss the properties of the solitary, periodic and travelling waves analytically and graphically.

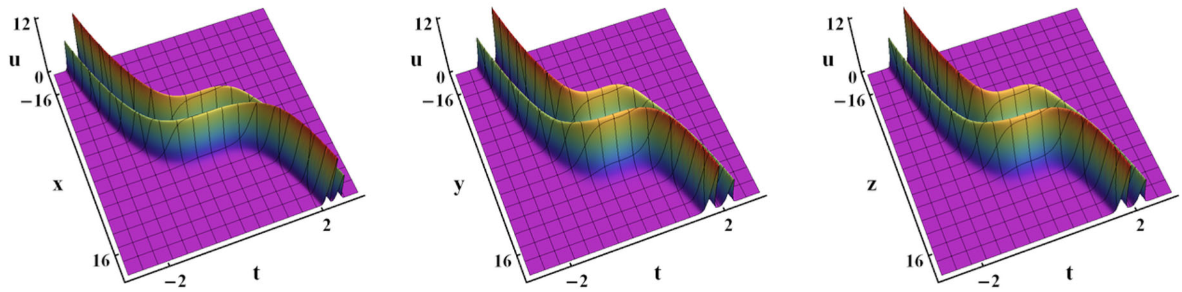
6.1 Soliton solutions

From eq. (42), we notice that the amplitude of the one-soliton solutions is $12k_1^2$, which means the amplitude of the one solitons is not related to the variable coefficients $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$. Then, we will investigate the velocity of



(a) (b) (c)

Fig. 1. One-soliton solutions via (42) with $k_1 = \frac{3}{2}$, $l_1 = p_1 = 1$ and $y = z = 0$: (a) $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t$; (b) $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t^2$; (c) $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = 4\cos\frac{5t}{2}$.



(a) (b) (c)

Fig. 2. Two-soliton solutions via (45) with $k_1 = \frac{3}{2}$, $k_2 = 2$, $l_1 = l_2 = 1$ and $p_1 = p_2 = 1$, when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t^2$: (a) $y = z = 0$; (b) $x = z = 0$; (c) $x = y = 0$.

the one solitons by characteristic line equation of $u(x, y, z, t)$ as

$$k_1 x + l_1 y + p_1 z + \omega_1(t) + \eta_1 = C_1, \tag{82}$$

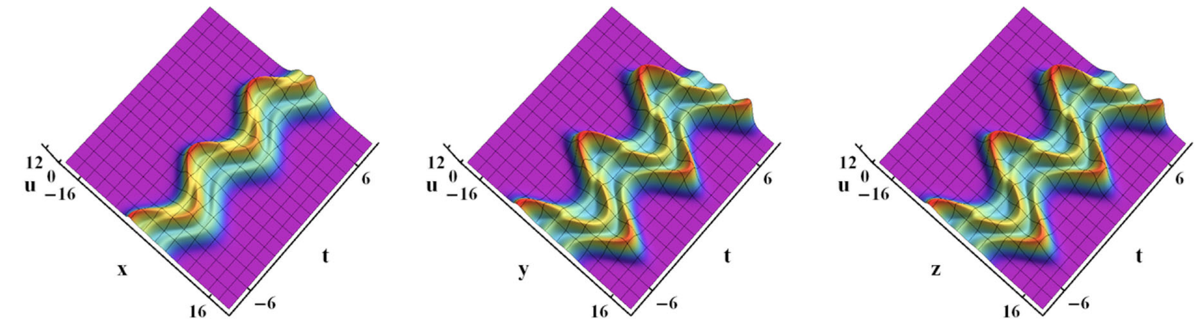
where C_1 is a free real constant. Differentiating the eq. (82) with respect to t, we derive the velocity of the one solitons as

$$\begin{aligned} V_x &= \frac{k_1^4\beta(t) + k_1^2\gamma(t) + l_1^2\delta(t) + p_1^2\varrho(t)}{k_1^2}\,, \\ V_y &= \frac{k_1^4\beta(t) + k_1^2\gamma(t) + l_1^2\delta(t) + p_1^2\varrho(t)}{k_1 l_1}\,, \\ V_z &= \frac{k_1^4\beta(t) + k_1^2\gamma(t) + l_1^2\delta(t) + p_1^2\varrho(t)}{k_1 p_1}\,. \end{aligned} \tag{83}$$

Hence, we find that the velocity of the one-soliton solutions is determined by the parameters k_1, l_1, p_1 and the variable coefficients $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$.

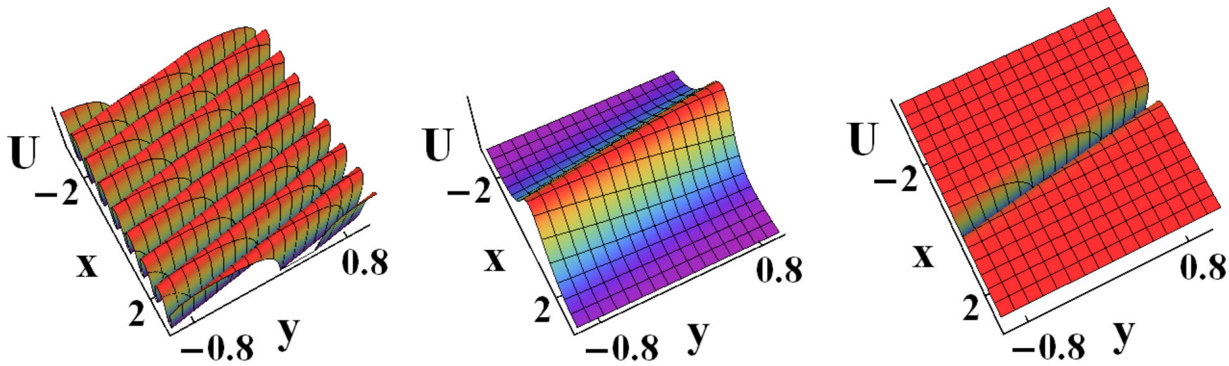
Figures 1 present the propagation of the one soliton when the variable coefficients $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$ are of different types of functions. when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t$, a parabolic-shape one soliton is obtained, as shown in fig. 1(a); when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t^2$, a cubic-shape one soliton is obtained, as shown in fig. 1(b); when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = 4\cos(\frac{5t}{2})$, a periodic-shape one soliton is obtained, as shown in fig. 1(c). From which we discover that $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$ could influence the velocities of the solitons, but the amplitudes of the solitons do not change during the propagation, it is consistent with analytical analysis above.

Figures 2 present the interaction between the two solitons when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t^2$, there appear two cubic-shape solitons. From which we find that the amplitudes of the two solitons remain unchanged after the interaction, except for a phase shift. Figure 3 exhibit the three solitons when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = \frac{1}{2}\cos(t)$, there appear three periodic-shape solitons and propagate in the same direction all the way. From which we observe that the characteristics of fig. 3 are similar to fig. 2, the soliton amplitudes remain unchanged after each interaction which indicates the interaction among the three solitons is elastic.



(*a*) (*b*) (*c*)

Fig. 3. Three-soliton solutions via (48) with $k_1 = \frac{3}{2}$, $k_2 = 2$, $k_3 = 1$, $l_1 = l_2 = l_3 = 1$ and $p_1 = p_2 = p_3 = 1$, when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = \frac{1}{2}\cos t$: (a) $y = z = 0$; (b) $x = z = 0$; (c) $x = y = 0$.



(*a*) (*b*) (*c*)

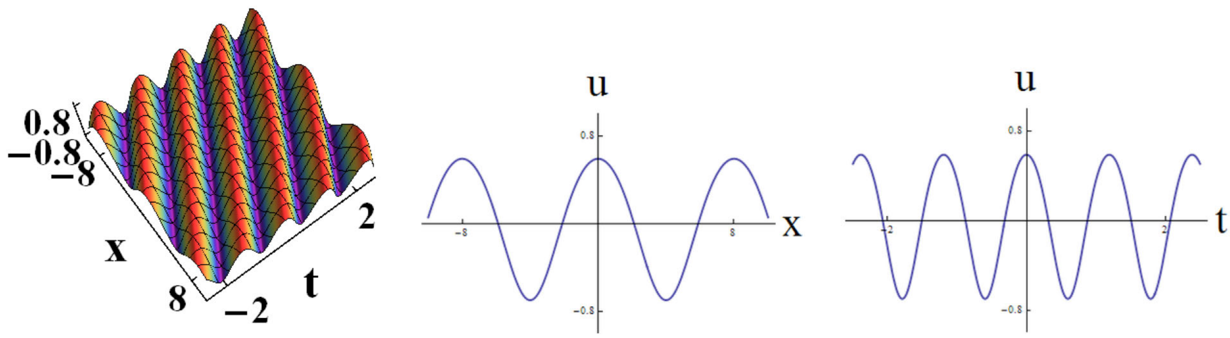
Fig. 4. Travelling-wave solutions with $k = 2$, $l = 1$, $p = 3$ and $\gamma(t) = \delta(t) = \varrho(t) = t$: (a) via (66) at $B = 1$; (b) via (67) at $A = C_0 = -1$; (c) via (69) at $B = -1$.

6.2 Travelling-wave solutions

Figure 4 exhibit three different travelling waves in the x-y plane of eq. (2), by taking certain parameters, namely, the triangle-type periodical, bell-type, soliton-type travelling waves [49]. We notice that bell-type travelling-wave solitons (67) can be converted into one-soliton solutions when $A = C_0 = -1$, but there exist singular points in the travelling-wave solutions (66) and (69).

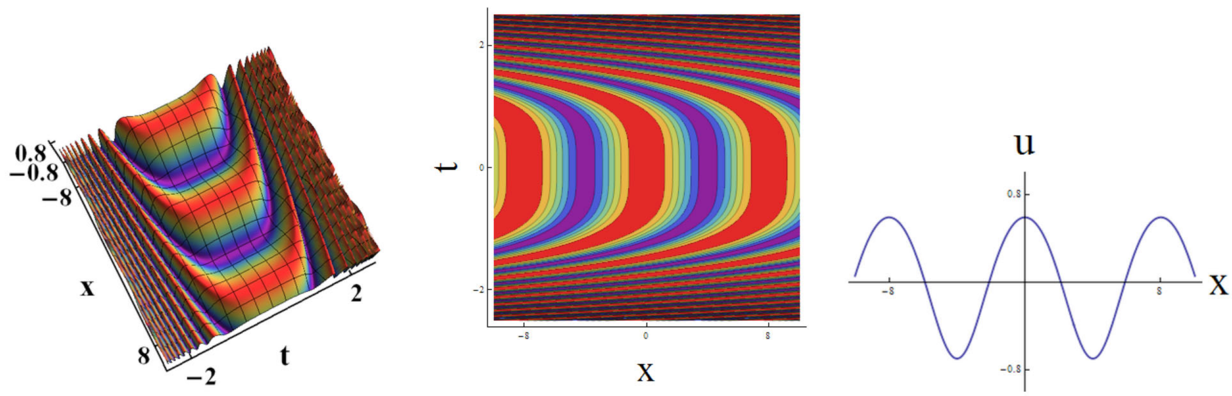
6.3 One-periodic wave solutions

Figure 5 present the propagation of the one-periodic waves with constant coefficients, viewed as a superposition of overlapping solitary waves, placed one period apart. Figures 6 and 7 show the propagation of one-periodic waves when the variable coefficients $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$ are selected as cubic and sine functions, respectively. When the cubic function case, one-periodic waves with u-shape is obtained along the x-direction. When the sine function case, one-periodic waves with wave-shape is derived along the x-direction. From which we find that variable coefficients $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\varrho(t)$ can affect the structure of the one-periodic waves, but not influence the period characteristics along the x-direction.



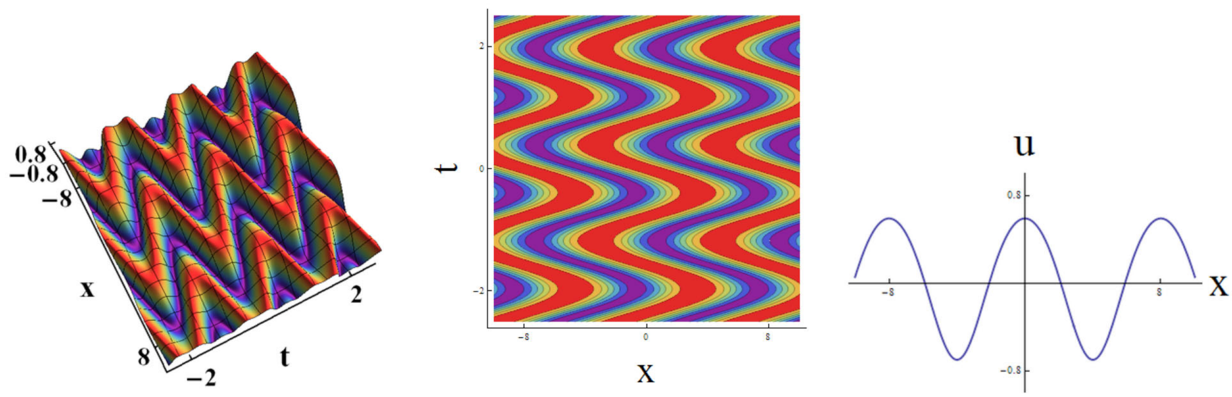
(a) (b) (c)

Fig. 5. One-periodic wave solutions via (81) with $u_0(t) = 0$, $k = l = \frac{1}{8}$, $p = \frac{1}{6}$, $\tau = i$ and $\epsilon = 0$, when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = 1$: (a) perspective view of the wave when $y = z = 0$; (b) wave propagation pattern of the wave along the x-axis; (c) wave propagation pattern of the wave along the t-axis.



(a) (b) (c)

Fig. 6. One-periodic wave solutions via (81) with $u_0(t) = 0$, $k = l = \frac{1}{8}$, $p = \frac{1}{6}$, $\tau = i$ and $\epsilon = 0$, when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = t^3$: (a) perspective view of the wave when $y = z = 0$; (b) overhead view of the wave, with contour plot shown; (c) wave propagation pattern of the wave along the x-axis.



(a) (b) (c)

Fig. 7. One-periodic wave solutions via (81) with $u_0(t) = 0$, $k = l = \frac{1}{8}$, $p = \frac{1}{6}$, $\tau = i$ and $\epsilon = 0$, when $\beta(t) = \gamma(t) = \delta(t) = \varrho(t) = 2\cos(4t)$: (a) perspective view of the wave when $y = z = 0$; (b) overhead view of the wave, with contour plot shown; (c) wave propagation pattern of the wave along the x-axis.

In this section, the relations between the one-periodic wave solutions (81) and the one-soliton solutions (42) of eq. (2) will be discussed. Firstly, we expand a_{11}, a_{12}, a_{21}, a_{22}, $b_I(t)$, and $b_{II}(t)$ of eqs. (79) as

$$
\begin{aligned}
a_{11}(t) &= -32K\pi^2(\Lambda^2 + 4\Lambda^8 + \cdots + n^2\Lambda^{2n^2} + \cdots),\\
a_{12}(t) &= 1 + 2(\Lambda^2 + \Lambda^8 + \cdots + \Lambda^{2n^2} + \cdots),\\
a_{21}(t) &= -8K\pi^2(\Lambda + 9\Lambda^5 + \cdots + (2n-1)^2\Lambda^{2n^2-2n+1} + \cdots),\\
a_{22}(t) &= 2(\Lambda + \Lambda^5 + \cdots + \Lambda^{2n^2-2n+1} + \cdots),\\
b_I(t) &= 2[-256\beta(t)K^4\pi^4 - 256u_0(t)\beta(t)K^4\pi^4 + 16\gamma(t)K^2\pi^2 + 16\delta(t)L^2\pi^2 + 16\varrho(t)P^2\pi^2]\Lambda^2\\
&\quad + \cdots + 2[-256\beta(t)K^4\pi^4n^4 - 256u_0(t)\beta(t)K^4\pi^4n^4 + 16\gamma(t)K^2\pi^2n^2\\
&\quad + 16\delta(t)L^2\pi^2n^2 + 16\varrho(t)P^2\pi^2n^2]\Lambda^{2n^2} + \cdots,\\
b_{II}(t) &= 2[-16\beta(t)K^4\pi^4 - 16u_0(t)\beta(t)K^4\pi^4 + 4\gamma(t)K^2\pi^2 + 4\delta(t)L^2\pi^2 + 4\varrho(t)P^2\pi^2]\Lambda + \cdots\\
&\quad + 2[-16\beta(t)K^4\pi^4(2n-1)^4 - 16u_0(t)\beta(t)K^4\pi^4(2n-1)^4 + 4\gamma(t)K^2\pi^2(2n-1)^2\\
&\quad + 4\delta(t)L^2\pi^2(2n-1)^2 + 4\varrho(t)P^2\pi^2(2n-1)^2]\Lambda^{2n^2-2n+1} + \cdots.
\end{aligned}
\tag{84}
$$

Accordingly, we can rewrite the eq. (79) into power series as follows:

$$
\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} = A_0(t) + A_1(t)\Lambda + A_2(t)\Lambda^2 + \cdots, \tag{85}
$$

$$
\begin{pmatrix} \mu_t(t) \\ c(t) \end{pmatrix} = X_0(t) + X_1(t)\Lambda + X_2(t)\Lambda^2 + \cdots, \tag{86}
$$

$$
\begin{pmatrix} b_I(t) \\ b_{II}(t) \end{pmatrix} = B_0(t) + B_1(t)\Lambda + B_2(t)\Lambda^2 + \cdots, \tag{87}
$$

where

$$
\begin{aligned}
&A_0(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1(t) = \begin{pmatrix} 0 & 0 \\ -8K\pi^2 & 2 \end{pmatrix}, \qquad A_2(t) = \begin{pmatrix} -32K\pi^2 & 2 \\ 0 & 0 \end{pmatrix},\\
&A_3(t) = A_4(t) = 0, \quad A_5(t) = \begin{pmatrix} 0 & 0 \\ -72K\pi^2 & 2 \end{pmatrix}, \cdots, B_0(t) = 0, \qquad B_3(t) = B_4(t) = 0,\\
&B_1(t) = \begin{pmatrix} 0 \\ 2\left(-16\pi^4K^4\beta(t) - 16\pi^4K^4\beta(t)u_0(t) + 4\pi^2K^2\gamma(t) + 4\pi^2L^2\delta(t) + 4\pi^2P^2\varrho(t)\right) \end{pmatrix},\\
&B_2(t) = \begin{pmatrix} 2\left(-256\pi^4K^4\beta(t) - 256\pi^4K^4\beta(t)u_0(t) + 16\pi^2K^2\gamma(t) + 16\pi^2L^2\delta(t) + 16\pi^2P^2\varrho(t)\right) \\ 0 \end{pmatrix},\\
&B_5(t) = \begin{pmatrix} 0 \\ 2\left(-1296\pi^4K^4\beta(t) - 1296\pi^4K^4\beta(t)u_0(t) + 36\pi^2K^2\gamma(t) + 36\pi^2L^2\delta(t) + 36\pi^2P^2\varrho(t)\right) \end{pmatrix},\\
&\quad\cdots.
\end{aligned}
\tag{88}
$$

Then, we can derive

$$
\begin{aligned}
&X_0(t) = \begin{pmatrix} \dfrac{2B_0^{[1]}(t) - B_1^{[2]}(t)}{8K\pi^2} \\ B_0^{[1]}(t) \end{pmatrix}, \qquad X_1(t) = \begin{pmatrix} \dfrac{2B_1^{[1]}(t) - [B_2(t) - A_2(t)X_0(t)]^{[2]}}{8K\pi^2} \\ B_1^{[1]}(t) \end{pmatrix},\\
&X_n(t) = \begin{pmatrix} \dfrac{2[B_{n+1}(t) - \Sigma_{j=2}^{n}A_j(t)X_{n-j}(t)]^{[1]} - [B_{n+1}(t) - \Sigma_{j=2}^{n+1}A_j(t)X_{n-j+1}(t)]^{[2]}}{8K\pi^2} \\ [B_{n+1}(t) - \Sigma_{j=2}^{n}A_j(t)X_{n-j}(t)]^{[1]} \end{pmatrix}, \cdots,
\end{aligned}
\tag{89}
$$

where $n \geq 2$, $n \in N$, and Δ^κ ($\kappa = 1, 2$) denotes the κ-th elements of the two-dimensional vector Δ.

From eq. (89), we have

$$X_0(t)=\begin{pmatrix}4K^3\pi^2\beta(t)-K\gamma(t)-L^2K^{-1}\delta(t)-P^2K^{-1}\varrho(t)+4K^3\pi^2\beta(t)u_0(t)\\ 0\end{pmatrix},\qquad X_1(t)=0,$$

$$X_2(t)=\begin{pmatrix}-8\left(-4\pi^2K^3\beta(t)\left(u_0(t)+1\right)+L^2\delta(t)K^{-1}+P^2\varrho(t)K^{-1}+K\gamma(t)\right)\\ -32\pi^2\left(-4\pi^2K^4\beta(t)\left(u_0(t)+1\right)+K^2\gamma(t)+L^2\delta(t)+P^2\varrho(t)\right)\end{pmatrix},\cdots. \tag{90}$$

Substituting eq. (90) into the system (86) and setting $\Lambda\longrightarrow 0$, we can obtain

$$c(t)\longrightarrow 0,$$
$$\mu(t)\longrightarrow\int 4K^3\pi^2\beta(t)-K\gamma(t)-L^2K^{-1}\delta(t)-P^2K^{-1}\varrho(t)+4K^3\pi^2\beta(t)u_0(t)\mathrm{d}_t. \tag{91}$$

By assuming

$$u_0(t)=0,\qquad K=\frac{k_1}{2i\pi},\qquad L=\frac{l_1}{2i\pi},\qquad P=\frac{p_1}{2i\pi},\qquad \epsilon=\frac{\eta_1-i\pi\tau}{2i\pi}, \tag{92}$$

we have

$$2i\pi\xi=2i\pi[\int 4K^3\pi^2\beta(t)-K\gamma(t)-L^2K^{-1}\delta(t)-P^2K^{-1}\varrho(t)+4K^3\pi^2\beta(t)u_0(t)\mathrm{d}_t$$
$$Kx+Ly+Pz+\epsilon]=k_1x+l_1y+p_1z+\omega_1(t)+\eta_1-\pi i\tau=\theta_1-\pi i\tau. \tag{93}$$

Combining eq. (74) and (93), we further obtain

$$\vartheta(\xi,\tau)=\sum_{n=-\infty}^{+\infty}e^{\pi in^2\tau+2\pi in\xi}=1+(e^{2\pi i\xi}+e^{-2\pi i\xi})\Lambda+\cdots=1+e^{\theta_1}+e^{-\theta_1}\Lambda^2+\cdots$$
$$\overset{\Lambda\to 0}{=}1+e^{\theta_1}. \tag{94}$$

Hence, according to the results of analysis, we conclude that the one-periodic wave solutions (81) approach to the one-soliton solutions (42) when the amplitude $\Lambda\to 0$.

7 Conclusions

In this paper, a generalized $(3+1)$-dimensional variable-coefficients nonlinear-wave equation, *i.e.*, eq. (2), has been investigated, which has been presented for nonlinear waves in liquid with gas bubbles. The bilinear form (15), Bäcklund transformation (21), Lax pair (23) and infinitely-many conservation laws (39) are obtained via the binary Bell polynomials. One-, two- and three-soliton solutions are generated by virtue of the bilinear form (15) and the Hirota method, *i.e.* (42), (45)and (48). Travelling-wave solutions (66), (67) and (69) are derived with the aid of the polynomial expansion method, and the one-periodic wave solutions (81) are constructed by the Hirota-Riemann method. In addition, we find that the soliton velocities are related to the variable coefficients $\beta(t),\gamma(t),\delta(t)$ and $\varrho(t)$, but the soliton amplitudes are independent to that, and the interactions between the solitons are elastic. We also notice that the bell-type travelling-wave solitons (67) can be converted to one-soliton solutions (42) when $A=C_0=-1$, and that the one-periodic waves (81) approach to the one-solitary waves when the amplitude $\Lambda\to 0$ and can be viewed as a superposition of overlapping solitary waves, placed one period apart.

We express our sincere thanks to all the members of our discussion group for their valuable comments. This work was supported by the National Natural Science Foundation of China (grant number 11272023); the Fundamental Research Funds for the Central Universities under Grant No. 50100002016105010.

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