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Anisotropic stellar models admitting conformal motion

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Abstract. We address the problem of finding static and spherically symmetric anisotropic compact stars in general relativity that admit conformal motions. The study is framed in the language of $f(R)$ gravity theory in order to expose opportunity for further study in the more general theory. Exact solutions of compact stars are found under the assumption that spherically symmetric spacetimes admit conformal motion with anisotropic matter distribution in nature. In this work, two cases have been studied for the existence of such solutions: first, we consider the model given by $f(R) = R$ and then $f(R) = aR + b$. Finally, specific characteristics and physical properties have been explored analytically along with graphical representations for conformally symmetric compact stars in $f(R)$ gravity.

1 Introduction

From the time of Sir Isaac Newton our understanding of the nature of gravity has advanced but mysteries in physics still remain. The process is still continuing after the first formulation of Einstein's theory of General Relativity (GR) [1], which is one of the most successful fundamental theories of gravity in modern physics. Despite its success, many extensions of the original Einstein equations have been investigated to accommodate present observational data on both astrophysical and cosmological scales. Observational data has confirmed that the Universe is undergoing a phase of accelerated expansion. Direct support is provided by the high-precision data from Type Ia supernovae [2–4], the Cosmic Microwave Background (CMB) anisotropies [5], baryonic acoustic oscillations [6] and from gravitational lensing [7]. In particular, within the context of General Relativity, the observed cosmic acceleration of the Universe cannot be explained without resorting to additional exotic matter components (such as Dark Energy (DE) or dark mater (DM)). Until now, no consistent model of dark energy has been proposed which can yield a convincing self-consistent picture of the observed Universe, both in the field of fundamental physics and in that of astrophysics. Currently experimental tests for DE and DM are under design and it is believed that with precision instrumentation, such as the Square Kilometre Array radio telescopes, such ideas will either be confirmed or ruled out. For this reason modifications of General Relativity have been proposed and are currently receiving a great deal of attention.

Alternative gravity theories generally appear to exhibit good agreement between theory and observation, and the introduction of additional geometrical degrees of freedom may assist us in abandoning the concepts of DE or DM. Many alternative theories have been proposed such as $f(R)$ gravity [8–12], scalar-tensor theories [13], braneworld models [14], Gauss-Bonnet gravity [15,16], Galileon gravity [17] have gained much attention in recent years. Additionally, Ellis [18, 19] revived an idea proposed by Weinberg [20], that of trace-free Einstein gravity, which also goes by the name unimodular [21–23] gravity. In this framework an attempt was made to rectify the discrepancy in the value of the cosmological constant as predicted by quantum gravity in contrast with what is actually observed. Dadhich [24] has also presented strong arguments for using the Lovelock polynomial terms as the generator of the Lagrangian density. The remarkable feature of the Lovelock theory is that while it is polynomial in the Riemann tensor, Ricci tensor and

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Ricci scalar, it nevertheless results in at most second-order equations of motion. Moreover, the zero-th case corresponds to the cosmological constant while the linear case yields the familiar Einstein gravity theory. The effects of the higher curvature terms are only felt in dimensions greater than 4 so that the theory is a natural generalisation of GR to higher dimensions. Investigations into higher-dimensional gravity is strongly motivated by string theoretic ideas which are supposed to be compatible with gravity theories. It should be mentioned here that many of these modifications have provided a number of significant results in cosmology [25–30] as well as astrophysics [8–12].

In the present article we have endeavoured to frame the core problem in the language and formalism of the $f(R)$ theory although the specific cases under consideration reduce to the standard Einstein theory which contains an action principle that is linear in the Ricci scalar. The reason for this choice of formalism is that it opens the way for further investigations of more general functional forms of the Ricci scalar such as the Starobinksy model in subsequent work. In $f(R)$ gravity theories the Einstein-Hilbert action is generalized with a function of the Ricci scalar R (see the reviews [31–38]) which has also been extensively analyzed recently. Earlier interest in $f(R)$ theories occurred in connection with the early Universe [39], and a number of viable $f(R)$ models have proposed that satisfy the observable constraints —see *e.g.*, refs. [40–42]. It is noted that $f(R)$ theory was contemplated long ago by Buchdahl [43] who introduced $f(R)$ gravity using non-linear Lagrangians. One of the main reasons for increased interest in $f(R)$ gravity was motivated by inflationary scenarios, in the Starobinsky model, where $f(R) = R - \Lambda + \alpha R^2$ gravity is considered in [44]. The cosmology of generalized modified gravity models has been studied by Carroll *et al.* [45]. In this context, many authors have demonstrated interest by exploring some of their interesting properties and characteristics, such as structure formation and stability conditions of neutron stars, which were studied in [46, 47], and the gravitational collapse of spherically symmetric perfect fluids in [48].

Apart from the cosmological solutions, gravitational theories must be tested also at the astrophysical level, to get a more realistic picture of the gravitational fields. In this sense, strong gravitational regimes found in relativistic stars could discriminate General Relativity from its generalizations [49]. However, it should be remembered that present-day astronomical observational data suggest that neutron stars can be used to investigate possible deviations from GR and act as probes for any modified gravity model. These ideas will add impetus in the coming decade when the Square Kilometre Array experiment is fully functional. At the same time, the structure of compact stars in perturbative $f(R)$ gravity has been studied in [50–58]. Another class of interesting models on hydrostatic equilibrium of stellar structure have been investigated in the framework of $f(R)$ gravity in [59, 60]. The primary motivation for finding exact interior solutions to Einstein's field equations in $f(R)$ gravity for their static and spherically symmetric case is that they can be used to model physically relevant compact objects. With the estimate values for masses and radii from different compact objects, the recent observational prediction fails to be compatible with the standard neutron star models because densities of nuclear matter are much lower than the densities of such compact objects. This situation can theoretically be explained by the existence of two different kinds of interior pressures, namely, radial and transverse pressure. In this regard, the study of models of anisotropic stellar distributions has received great interest in GR as well as in modified theories of gravity. It is required of such models that elementary requirements are met in order to be physically viable. For example, it is expected that the energy conditions are satisfied, the interior and exterior spacetimes are smoothly matched across a suitable hypersurface and that causality is not violated. Keeping this in mind, the motivation is to study models of the anisotropic compact stars in the framework of $f(R)$ gravity by admitting non-static conformal symmetry using the diagonal tetrad method. Indeed, the model considered here corresponds to the standard Einstein theory in view of the choice of an action principle that is linear in the Ricci scalar. The consequence is that the equations of motion are of the usual second-order variety. If the more general Starobinksy model was selected, then the equations of motion would involve derivatives of the order of four. This case will be pursued in another study. At first, Maartens and Maharaj [61] have studied the static spherical distributions in charged imperfect fluids admitting conformal symmetry and now it has become a popular model to establish a natural relationship between geometry and matter source, with the help of the Einstein field equations. In the same context, Das *et al.* [62], obtained a set of solutions describing the interior of a compact star under $f(R, T)$ gravity and Ray *et al.* [63] have studied electromagnetic mass models for static spherical distribution admitting CKV.

The study of anisotropic spherically symmetric distributions has a long history. Such models are heavily under-determined and admit a large number of solutions. Bowers and Liang [64] presented a simple anisotropic model with constant density. The mass implications in anisotropic distributions were revisited by Baracco *et al.* [65] and it was shown that the well-known Buchdahl mass limit, applicable to isotropic spheres, may be violated. Ivanov [66] established mass-radius bounds for anisotropic fluids with spherical symmetry. Mathematical prescriptions, such as conformal flatness, were studied by Stewart [67] who postulated forms for the mass function to generate exact models. Non-static spheres admitting a one-parameter group of conformal motions were studied by Herrera and Ponce de Leon [68] but without specifying a linear equation of state.

Motivated by the above discussion, we study the compact stars solutions with their interesting properties and characteristics in the context of $f(R)$ theory of gravity admitting initially non-static conformal symmetry and with a specified equation of state. The paper is organized as follows: in sect. 2, we review $f(R)$ modified theories of gravity, whereas in sect. 3, the CKVs are formulated. In sect. 4, we formulate the system of gravitational field equations then, in sect. 5, we describe the star interior, using the spherically symmetric line element and CKVs. In sect. 6, we discuss various physical features of the model, such as energy conditions, equilibrium condition by using Tolman-Oppenheimer-Volkoff (TOV) equation and stability issue. Finally, in sect. 7, we conclude with a summary and discussions.

2 Basic mathematical formulation in f(R) gravity

Let us consider the following action in $f(R)$ gravity:

$$\pounds = \int \mathrm{d}x^4 \sqrt{-g}[f(R) + L_m], \tag{1}$$

where g is the determinant of the space-time metric $g_{\mu\nu}$, assuming that $8\pi G = 1$ and L_m is the matter Lagrangian. Here $f(R)$ is a general function of the Ricci scalar, R. Within the context of the metric formalism, variation of the action equation (1) with respect to $g_{\mu\nu}$ generates the field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{(\text{curv})} + T_{\mu\nu}^{(\text{matter})}, \tag{2}$$

where the term $T_{\mu\nu}^{(\text{matter})}$ represents the energy momentum tensor of the matter scaled by a factor of $1/f'(R)$ and $T_{\mu\nu}^{(\text{curv})}$ is an extra stress-energy tensor contribution by the $f(R)$ modified gravity theory and is given by

$$T_{\mu\nu}^{(\text{curv})} = \frac{1}{F(R)}\left[\frac{1}{2}g_{\mu\nu}\left(f(R) - RF(R)\right) + F(R)^{;\alpha\beta}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta}\right)\right].$$

In the above equation, $F(R) \equiv \mathrm{d}f/\mathrm{d}R$ and primes denote derivatives with respect to Ricci scalar R. Now, considering a static spherically symmetric distribution of matter, the line element may be expressed in the form

$$\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{3}$$

where the metric functions $\nu(r)$ and $\lambda(r)$ are functions of the radial coordinate, which may be utilised to compute the surface gravitational redshift and the mass function, respectively. From the metric it is possible to construct asymptotically flat spacetimes when $\nu(r) \to 0$ and $\lambda(r) \to 0$ as $r \to \infty$.

To obtain some particular strange star models we assume that the material composition filling the interior of the compact object is anisotropic in nature and accordingly we express the energy-momentum tensor in the form

$$T_{\alpha\beta}^{m} = (\rho + p_t)\,u_\alpha u_\beta - p_t g_{\alpha\beta} + (p_r - p_t)\,v_\alpha v_\beta, \tag{4}$$

where ρ depicts energy density, p_r is the radial pressure, p_t to be the tangential pressure, respectively. In the expression u_α is the four-velocity of the fluid and v_α corresponds to the radial four vector. Note that with the assumption of an anisotropic particle pressure, the field equations may be solved by an infinite number of metrics. Additional restrictions based on physical grounds must be introduced to model realistic objects. Customarily an equation of state relating the pressure and the density may be imposed, however, this approach runs into sever mathematical problems making the equation of pressure isotropy impractical to solve. In this work, we choose to utilise a geometric prescription. New classes of anisotropic star solutions admitting conformal motions are sought and investigated.

3 Conformal Killing vector

To establish a natural relationship between geometry and matter through the Einstein's GR, and searching for an exact solutions by convincing approach we use a non-static conformal symmetry. Following the idea a work has recently been considered in [62]. For instance, one may specify the vector ξ as the generator of this conformal symmetry, and the metric g is conformally mapped onto itself along ξ so that the following relationship,

$$\mathcal{L}_\xi g_{ij} = \psi g_{ij}, \tag{5}$$

is obeyed. Here $\mathcal{L}$ represents the Lie derivative operator of the metric tensor whereas ψ is the conformal Killing vector.

As for the choice of the vector, ξ generates the conformal symmetry within the framework of the standard GR, then the metric g is conformally mapped onto itself along ξ. According to Böhmer *et al.* [69], for the choice of static metric, neither ξ nor ψ need to be static. It should be noted that if $\psi = 0$, then eq. (6) gives the Killing vector, if $\psi = \text{const.}$, then it yields a homothetic vector and, if $\psi = \psi(x,t)$, then it gives conformal vectors. Furthermore, the spacetime becomes asymptotically flat when $\psi = 0$, which also implies that the Weyl tensor will vanish. All such conformally flat spacetimes have been found. They are either the Schwarzschild interior solution in the case of no expansion or the Stephani [70] stars if expansion is occurring. Accumulated, all of these properties of CKV have effectively provided a comprehensive picture of the underlying spacetime geometry. Here eq. (5) implies that

$$\mathcal{L}_\xi g_{ik} = \xi_{i;k} + \xi_{k;i} = \psi g_{ik}, \tag{6}$$

with $\xi_i = g_{ik}\xi^k$. From eqs. (4) and (7), the following expressions [69] are obtained:

$$\xi^1 \nu' = \psi, \quad \xi^4 = \text{const.}, \quad \xi^1 = \frac{\psi r}{2} \quad \text{and} \quad \xi^1 \lambda' + 2\xi^1_{,1} = \psi,$$

where 1 and 4 stand for spatial and temporal coordinates r and t, respectively. Then eq. (6) provides the following relationship, by using the metric (2):

$$e^\nu = C_0^2 r^2, \tag{7}$$

$$e^\lambda = \left[\frac{C}{\psi}\right]^2, \tag{8}$$

$$\xi^i = C_1 \delta_4^i + \left[\frac{\psi r}{2}\right] \delta_1^i, \tag{9}$$

where C, C_0 and C_1 are all integration constants.

4 The field equations

The tetrad matrix for the metric (4) is defined as

$$e^i_\mu = \text{diag}\left[\sqrt{C_0^2 r^2}, e^{\frac{\lambda}{2}}, r, r\sin\theta\right]. \tag{10}$$

Therefore, the Ricci scalar is determined as

$$R(r) = -\frac{2}{r^2} + e^{-\lambda(r)}\left(\frac{6}{r^2} - \frac{3\lambda'(r)}{r}\right). \tag{11}$$

Using the spacetime metric given by eq. (4), the modified field equation (2) generates the equations

$$\rho = -e^{-\lambda}F'' + e^{-\lambda}\left(\frac{\lambda'}{2} - \frac{2}{r}\right)F' + \frac{e^{-\lambda}}{r^2}\left(\frac{\nu'' r^2}{2} + \frac{\nu'^2 r^2}{4} - \frac{\nu'\lambda' r^2}{4} + \nu' r\right)F - \frac{1}{2}f, \tag{12}$$

$$p_r = e^{-\lambda}\left(\frac{\nu'}{2} + \frac{2}{r}\right)F' - \frac{e^{-\lambda}}{r^2}\left(\frac{\nu'' r^2}{2} + \frac{\nu'^2 r^2}{4} - \frac{\nu'\lambda' r^2}{4} - \lambda' r\right)F + \frac{1}{2}f, \tag{13}$$

$$p_t = -e^{-\lambda}F'' + e^{-\lambda}\left(\frac{\nu'}{2} - \frac{\lambda'}{2} + \frac{1}{r}\right)F' - \frac{e^{-\lambda}}{r^2}\left(\frac{\nu' r}{2} - \frac{\lambda' r}{2} - e^\lambda + 1\right)F + \frac{1}{2}f, \tag{14}$$

governing the evolution of the fluid. In the following we focus on several physically relevant $f(R)$ gravity models and seek interior solutions of the Einstein field equations with conformal Killing vector, that can model compact objects. The field equations of $f(R)$ modified theories are highly non-trivial to solve, so the entire calculations are highly dependent on the equation ansatz. We choose to solve the continuity equation for decoding the system of equations for our purpose.

5 Solutions

As a first step in our analysis we consider solutions for $f(R) = R$ and $f(R) = aR + b$, where a and b are purely constant quantities. Note that this prescription is essentially the general relativity case without and with a cosmological constant. We further assume that the standard matter for a fluid distribution obeys the equation of state,

$$p_r = \omega\rho, \tag{15}$$

where the state parameter w is a constant and $0 < w < 1$ for causality. That is the speed of sound remains subluminal provided that $0 \leq \frac{\mathrm{d}p}{\mathrm{d}\rho} < 1$. Now we determine the solutions under the above conditions.

5.1 Case I: f(R) = R

5.1.1 When $p_r = p_t = p$

Indeed, by using eqs. (10)–(14), we obtain the solution of this ordinary differential equation in the form

$$\rho(r) = \frac{1}{r^2} + \frac{3B}{2}\,, \tag{16}$$

$$p(r) = \frac{1}{2r^2} - \frac{3B}{2}\,, \tag{17}$$

while the conformal factor, Ricci scalar and mass functions are given by

$$\psi = C\left[\frac{1}{2} - \frac{B}{2}r^2\right]^{\frac{1}{2}}, \tag{18}$$

$$R(r) = -6B + \frac{1}{r^2}\,, \tag{19}$$

$$e^{-\lambda(r)} = \frac{1}{2} - \frac{B}{2}r^2, \tag{20}$$

where B is an integration constant. In this case, the velocity of sound v_s^2 is given by

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{1}{2}\,. \tag{21}$$

On account of the assumption of isotropic particle pressure, we have not invoked the relationship $p = \omega\rho$ so as not to overdetermine the system. However, the solution above does indeed exhibit an equation of state of the form $p = p(\rho)$, consistent with perfect fluid distributions. Observe that setting $B = 0$ gives the Saslaw *et al.* [71] isothermal fluid model with both the energy density and the pressure obeying the inverse square law and, obviously, the barotropic linear equation of state $p = \omega\rho$. The fact that the metric potential λ is not constant in this model may be attributed to the presence of the conformal symmetry.

5.1.2 When $p_r \neq p_t$

Using eqs. (10)–(15), we obtain the solution when matter distribution is anisotropic in nature and is given by

$$\rho(r) = -\frac{6D}{\omega}r^{\frac{-(6+\omega)}{(3+\omega)}} + \frac{2}{(3+\omega)}r^{-2}, \tag{22}$$

$$p_r = -6Dr^{\frac{-(6+\omega)}{(3+\omega)}} + \frac{2\omega}{(3+\omega)}r^{-2} \tag{23}$$

and

$$p_t = \frac{-\omega^2 - 8\omega - 3}{2\omega(3+\omega)}r^{-2} - \frac{6D}{\omega}r^{\frac{-(6+\omega)}{(3+\omega)}} + 3Dr^{-3}. \tag{24}$$

The other set of expressions is given by

$$\psi = C\left[\frac{1+\omega}{3+\omega} + Dr^{\frac{\omega}{3+\omega}}\right]^{\frac{1}{2}}, \tag{25}$$

$$e^{-\lambda(r)} = \frac{1+\omega}{3+\omega} + Dr^{\frac{\omega}{3+\omega}}, \tag{26}$$

$$R(r) = -\frac{2}{r^2} + \frac{6(1+\omega)}{(3+\omega)r^2} + 6Dr^{-3} + \frac{3(3+\omega)D}{\omega}r^{\frac{-(6+\omega)}{(3+\omega)}}\,, \tag{27}$$

where D is integration constant. The sound velocity, v_s^2, is determined as

$$\frac{\mathrm{d}p_t}{\mathrm{d}\rho} = \frac{\frac{\omega^2+8\omega+3}{2\omega(3+\omega)r^3} + \frac{6D(6+\omega)}{\omega(3+\omega)}r^{\frac{-2\omega-9}{3+\omega}} - 9Dr^{-4}}{\frac{6D(6+\omega)}{\omega(3+\omega)}r^{\frac{-2\omega-9}{3+\omega}} - \frac{4\omega}{(3+\omega)r^3}}\,. \tag{28}$$

Note that in view of the latitude afforded by the assumption of pressure anisotropy the field equations were augmented by the equation of state $p = \omega\rho$.

5.2 Case II: $f(R) = aR + b$

Now we are going to consider the second case, where the effects of the cosmological constant enter the picture through the constant b.

5.2.1 When $p_r = p_t = p$

Using eqs. (10)–(14) and assuming the isotropic condition provides the following general solution:

$$\rho(r) = \frac{a}{2r^2} - 3aE - \frac{ab}{2}, \tag{29}$$

$$p(r) = 3aE + \frac{a}{2r^2} + \frac{ab}{2}, \tag{30}$$

and the conformal factor, Ricci scalar and mass functions are given by

$$\psi = C\left[\frac{1}{2} + Er^2\right]^{\frac{1}{2}}, \tag{31}$$

$$R(r) = 12E + \frac{1}{r^2}, \tag{32}$$

$$e^{-\lambda(r)} = \frac{1}{2} + Er^2, \tag{33}$$

where E is an integration constant. The sound velocity, v_s^2, is then obtained as

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = 1, \tag{34}$$

which is the extreme value for the sound speed ordinarily associated with a stiff fluid.

5.2.2 When $p_r \neq p_t$

Using eqs. (10)–(14), and eq. (15) to solve the differential equations gives

$$\rho(r) = \frac{b(1+\omega)}{\omega} + \frac{2a}{(3+\omega)r^2} + \frac{3aH}{\omega} r^{\frac{-3(1+\omega)}{\omega}}, \tag{35}$$

$$p_r = b(1+\omega) + \frac{2a\omega}{(3+\omega)r^2} + 3aHr^{\frac{-3(1+\omega)}{\omega}}, \tag{36}$$

$$p_t = \frac{a(1+\omega)}{(3+\omega)r^2} - \frac{b}{2\omega} + H\left(2a - \frac{3}{\omega}\right) r^{\frac{-3(1+\omega)}{\omega}}, \tag{37}$$

with other functions given by

$$\psi = C\left[\frac{1+\omega}{3+\omega} - \frac{br^2}{6a} + Hr^{\frac{-(3+\omega)}{\omega}}\right]^{\frac{1}{2}}, \tag{38}$$

$$R(r) = \frac{4\omega}{(3+\omega)r^2} - \frac{b(1+2\omega)}{a\omega} + \frac{3\omega-9}{\omega} Hr^{\frac{-3(1+\omega)}{\omega}}, \tag{39}$$

and

$$e^{-\lambda(r)} = \frac{1+\omega}{3+\omega} - \frac{br^2}{6a} + Hr^{\frac{-(3+\omega)}{\omega}}. \tag{40}$$

The sound velocity, v_s^2, can be found as

$$\frac{\mathrm{d}p_t}{\mathrm{d}\rho} = \frac{\frac{2a(1+\omega)}{(3+\omega)r^3} + \frac{3(1+\omega)}{\omega}(2a - \frac{3}{\omega})r^{\frac{-3-4\omega}{\omega}}}{\frac{4a}{(3+\omega)r^3} + \frac{9aH(1+\omega)}{\omega} r^{\frac{-3-4\omega}{\omega}}}, \tag{41}$$

where H is an integration constant.

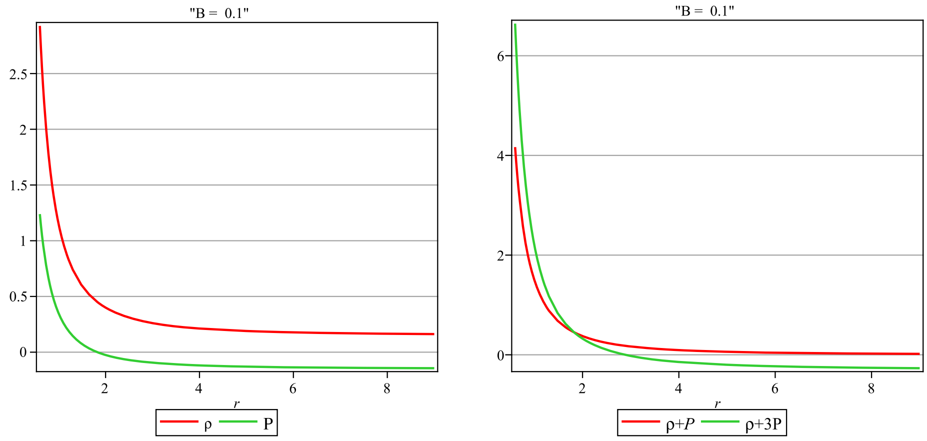

Fig. 1. Variation of density and pressure with energy conditions of the first model: We have plotted the energy density and pressures, respectively, in the left panel whereas, in the right panel, we have drawn the behavior of different energy conditions when the constant term "$B = 0.8$" assuming a positive value. Here we see that the null energy condition (NEC) is satisfied but the strong energy condition (SEC) is violated for the isotropic case.

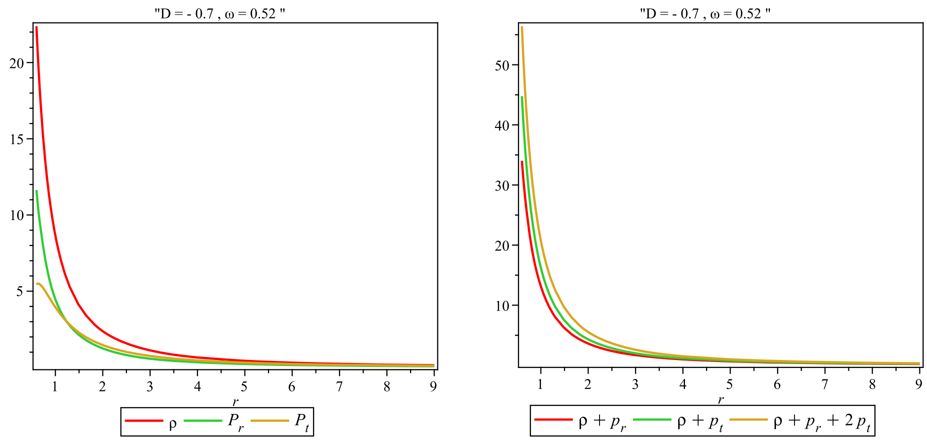

Fig. 2. Variation of density and pressure with energy conditions of the second model: In the left panel the functions of energy density and pressures have been plotted with the radial coordinate and in the right panel we have plotted all the energy conditions inside the star when $f(R) = R$ and $p_r \neq p_t$. The parametric values taken for the graphs are shown in the legend.

6 Physical features of the model

6.1 Energy conditions

For a physically acceptable model, we are going to verify whether the anisotropic fluid sphere satisfies all the energy conditions or not, namely: i) null energy condition (NEC); ii) weak energy condition (WEC); and iii) strong energy condition (SEC), at all points in the interior the star. We therefore attempt to write down the following inequalities as follows:

$$\text{NEC:} \quad \rho(r) + p_r \geq 0, \quad \text{and} \quad \rho(r) + p_r \geq 0, \tag{42}$$

$$\text{WEC:} \quad \rho \geq 0, \quad \rho(r) + p_r \geq 0, \quad \text{and} \quad \rho(r) + p_r \geq 0, \tag{43}$$

$$\text{SEC:} \quad \rho(r) + p_r \geq 0, \quad \text{and} \quad \rho(r) + p_r + 2p_t \geq 0, \tag{44}$$

In figs. 1, 2, 3 and 4 we plot variation of all the energy conditions inside the compact objects for $w = 0.52$, with different values of constant parameters. In the case of isotropic pressure when $f(R) = R$, plotted in fig. 2, it is evident that the null energy condition (NEC) is satisfied but the strong energy condition (SEC) is violated for our parametric choice; however, for other cases, the energy conditions are valid inside the compact star.

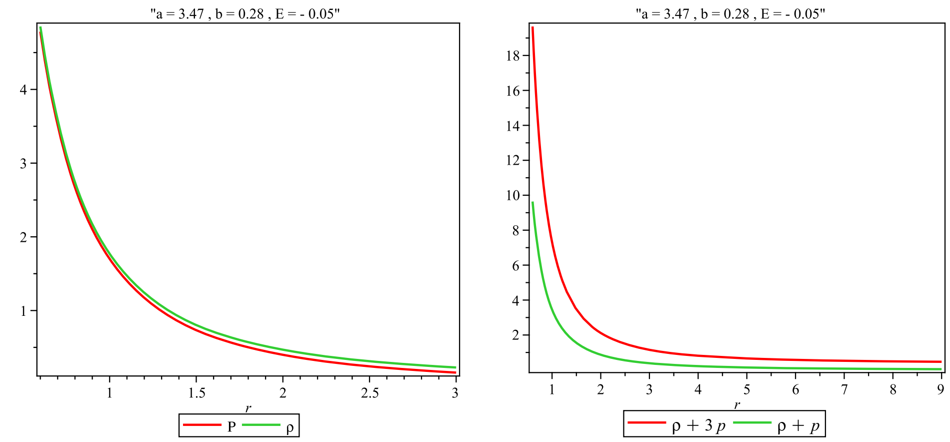

Fig. 3. Plots for variation of density and pressure with energy conditions when $f(R) = aR + b$: In the left panel the functions of energy density and pressures have been plotted with the radial coordinate and, in the right panel, we have plotted all the energy conditions inside the star for the isotropic model of the fluid sphere. The parameter values taken for the graphs are shown in the legend.

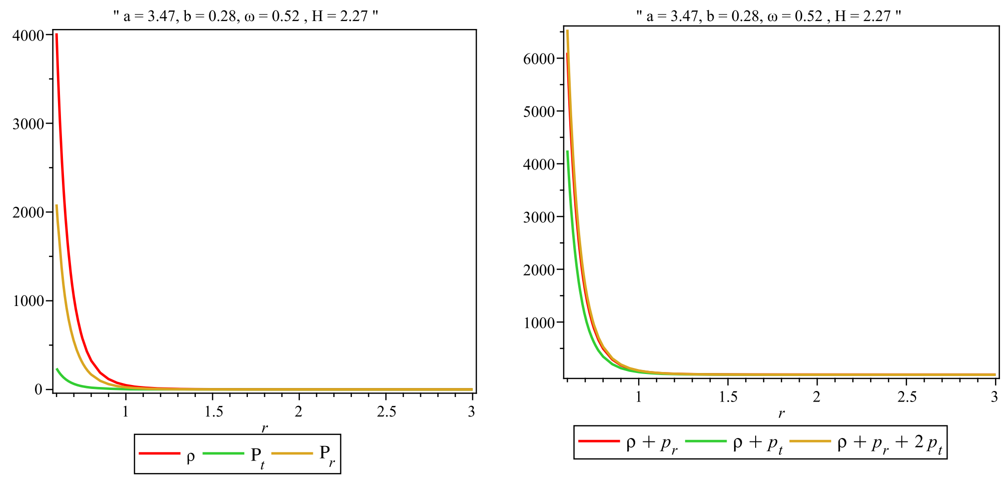

Fig. 4. Radial variation of energy density and pressures (left panel); all the energy conditions inside the star for anisotropic model of the fluid sphere for the $f(R) = aR + b$ solution (right panel).

6.2 TOV equation

The success of this model lies in its stability under different forces, namely, gravitational, hydrostatic and anisotropic forces. To examine these, we consider the generalized Tolman-Oppenheimer-Volkov (TOV) equation for anisotropic fluid distribution given by [72, 73]

$$-\frac{M_G(r)(\rho + p_r)}{r^2} e^{\frac{\lambda-\nu}{2}} - \frac{\mathrm{d}p_r}{\mathrm{d}r} + \frac{2}{r}(p_t - p_r) = 0, \tag{45}$$

where the effective gravitational mass, $M_G(r)$, is defined by

$$M_G(r) = \frac{1}{2} r e^{\frac{\mu-\nu}{2}} \nu'. \tag{46}$$

Substituting eq. (46) into eq. (45), we obtain the simple expression

$$-\frac{\nu'}{2}(\rho + p_r) - \frac{\mathrm{d}p_r}{\mathrm{d}r} + \frac{2}{r}(p_t - p_r) = 0. \tag{47}$$

Summarising, this expression is arranged in terms of gravitational mass and hence gives the equilibrium condition for the compact star, involving the gravitational, hydrostatic and anisotropic forces for stellar objects. This equation can be expressed in a more compact form as

$$F_g + F_h + F_a = 0, \tag{48}$$

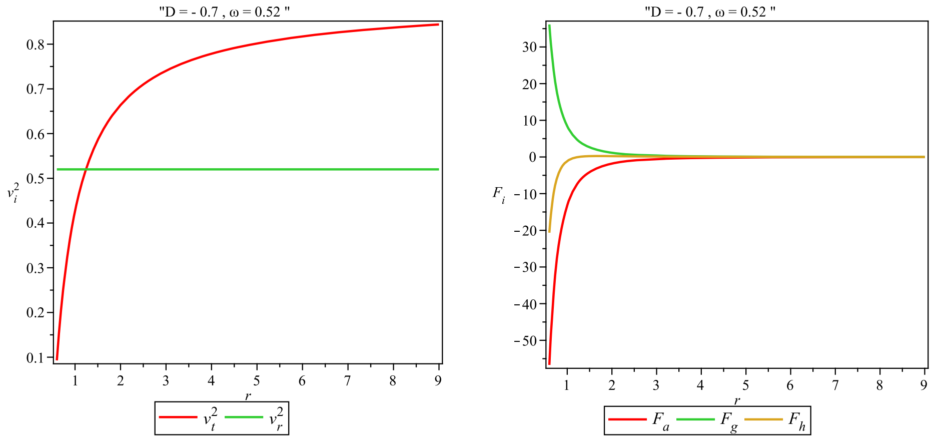

Fig. 5. Illustrative plot of square of sound speed (left panel), and variation of different forces (right panel) so that the system is subjected to the equilibrium position for the $f(R) = R$ solution. The parameter values taken for the graphs are shown in the legend.

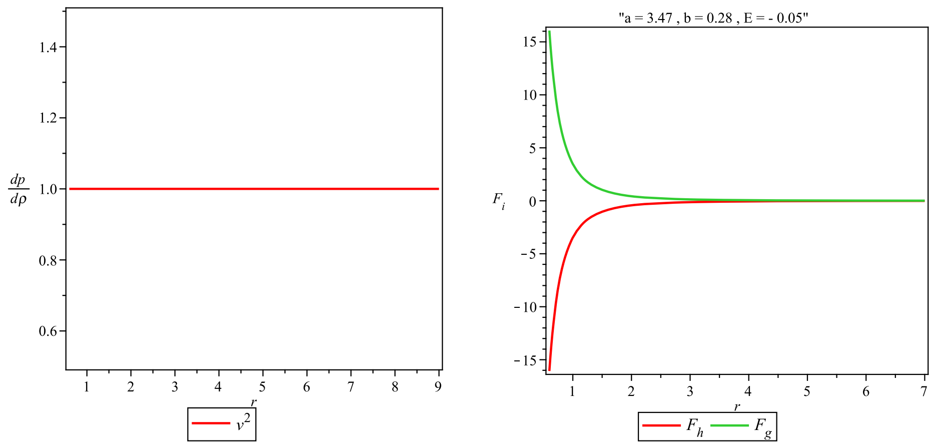

Fig. 6. Illustrative plot of square of sound speed (left panel), and variation of different forces (right panel), so that the system is subjected to the equilibrium position for the $f(R) = aR + b$ solution. The parameter values taken for the graphs are shown in the legend.

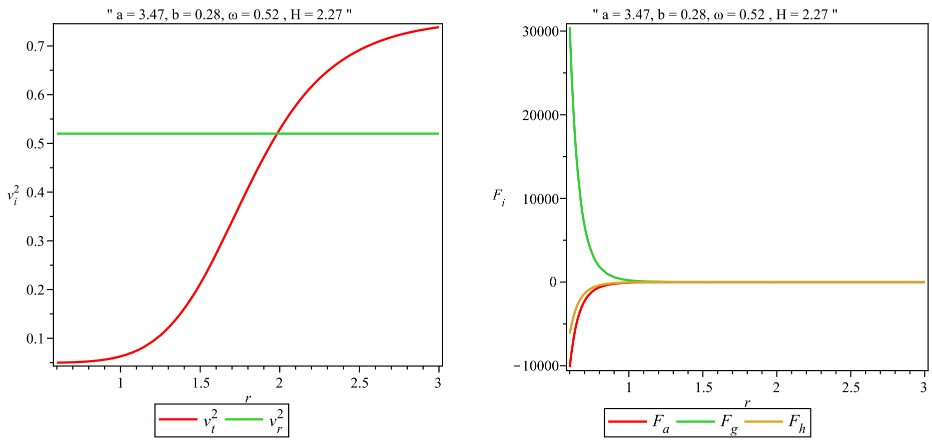

Fig. 7. In the left panel, the variation of the radial and transverse sound speed have been plotted, whereas, in the right panel, the equilibrium condition concerning TOV equation has been considered for the anisotropic $f(R) = aR + b$ solution. The parameter values taken for the graphs are shown in the legend.

where $F_g = -\frac{\nu'}{2}(\rho + p_r)$, $F_h = -\frac{\mathrm{d}p_r}{\mathrm{d}r}$ and $F_a = \frac{2}{r}(p_t - p_r)$ represent the gravitational, hydrostatic and anisotropic forces, respectively. In order to illustrate this qualitatively, we use the graphical representation for both cases, $f(R) = R$ and $f(R) = aR + b$, which are shown in figs. 5, 6 and 7 (right panel) by assigning the value $\omega = 0.58$. In spite of this model, we see that in every cases F_h and F_a take negative values, while F_g is positive. As a result, it is clear that the gravitational force is counterbalanced by the combined effect of hydrostatic and anisotropic forces to hold the system in a static equilibrium. Models to explain the static equilibrium in depth have been extensively studied by Rahaman *et al.* [74] and Rani and Jawad [75].

6.3 Stability analysis

We are interested in checking the sound speed, using the concept of Herrera's cracking (or overtuning) [76], which lies within the range $0 < v_{si}^2 \leq 1$, *i.e.*, according to this procedure the radial and transverse velocity of sound lies within the proposed range. We have shown graphically the radial and transverse velocity of sound for both cases, $f(R) = R$ and $f(R) = aR + b$, which are shown in figs. 5, 6 and 7 (left panel) and observe that these parameters satisfy the inequalities $0 < v_{sr}^2 \leq 1$ and $0 < v_{st}^2 \leq 1$ everywhere within the stellar object, except for the isotropic case when $f(R) = R$.

7 Conclusion

In this work, we have studied analytical solutions for compact stellar objects with a general static interior source in the framework of $f(R)$ gravity satisfying the conformal Killing vectors equations. The investigation has been performed for two specific cases: when $f(R) = R$ and $f(R) = aR + b$. Further the stars are assumed to be anisotropic in their internal structure. In order to get exact solutions we have considered a systematic approach by assuming the spherical symmetry and interior of the dense star admitting non-static conformal symmetry.

For a simplest form of the fluid sphere, we have studied the two specific arguments for both the isotropic ($p_r = p_t$) as well as the non-isotropic ($p_r \neq p_t$) cases. Interestingly, we have identified that, among the two cases (Case I, sect. 5.1, and Case II, sect. 5.2), only the solution for the isotropic condition $p_r = p_t = p$, when $f(R) = R$, is not physically valid because the behavior of different energy conditions does not attribute the regularity conditions at the interior of the star. Furthermore, it has been found that our solution satisfies all the energy conditions and that pressures are positive and finite throughout the interior of the stars, which is needed for physically possible configurations. For a stable configuration, we have shown that the generalized TOV equation, which describes the equilibrium condition for an anisotropic fluid subject to different forces, *viz.* gravitational force (F_g), hydrostatic force (F_h) and anisotropic force (F_a) in figs. 5, 6 and 7 (left panel). Another interesting result of this paper is related to checking the stability of our model by adapting Herrera's cracking concept [76], and it has been found that the radial and transverse speed of sound lies within the limit of $(0, 1]$, as is shown in figs. 5, 6 and 7 (right panel). Hence, we conclude that our solution might have astrophysical relevance by this theory through fine tuning. Therefore, it would be an interesting task to verify our solution with sample data for more satisfactory features in the realm of physical reality, which will be our next venture in this line of study. Moreover, we propose to consider more general functional forms of f to go beyond the standard model of general relativity, as considered in this work.

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